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QUESTIONS OF THE STATISTICAL THEORY OF RADAR (VOL II)

P. A. Bakut, et al

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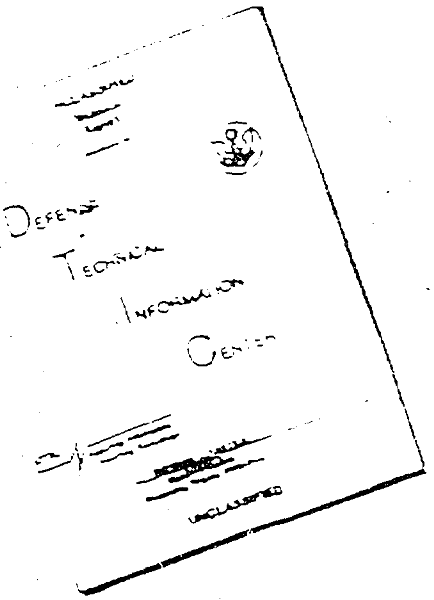

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QUESTIONS OF THE STATISTICAL THEORY OF RADAR
(VOL. II)

By

P. A. Bakut and I. A. Bol'shakov, et al

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pod obshchey redaktsiyey
professora G. P. Tartakovskogo

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Moskva — 1964

Pages 1-1079

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

* <u>ye</u> initially, after vowels, and after ъ, ь; <u>e</u> elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

FOLLOWING ARE THE CORRESPONDING RUSSIAN AND ENGLISH DESIGNATIONS OF THE TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csch
arc sin	$\sin^{-1}$
arc cos	$\cos^{-1}$
arc tg	$\tan^{-1}$
arc ctg	$\cot^{-1}$
arc sec	$\sec^{-1}$
arc cosec	$\csc^{-1}$
arc sh	$\sinh^{-1}$
arc ch	$\cosh^{-1}$
arc th	$\tanh^{-1}$
arc cth	$\coth^{-1}$
arc sch	sech^{-1}
arc csch	csch^{-1}
rot	curl
lg	log

~~This~~ second volume is dedicated to the theory of radar measurements and questions of target resolution.

There is developed a general theory of radar measurements, containing an analysis of tracking and nontracking measuring systems both in linear approximation, and also taking into account their nonlinearity, and also synthesis of optimum systems of measurement of separate time-variable parameters of motion of targets and sets of them.

On the basis of this theory there is conducted analysis and synthesis of range finding systems, systems of measurement of velocity and goniometrical systems. There are investigated cases of reception of both coherent, and also incoherent signals.

In examining questions of target resolution there are investigated possibilities of resolution of signals reflected from them, and there are found receivers optimum from this point of view, and also optimum systems of resolution in regimes of detection and measurement of coordinates.

In the course of development of theory there are revealed laws governing radar measurements and resolution of targets.

The book is intended for scientists and engineers studying questions of radar, and also for post graduates and students of corresponding specialities. Many questions of the developed general theory are also of interest to persons studying theoretical problems in all regions based on the theory of statistical solutions, in particular, in the area of automatic control.

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Questions of the Statistical Theory of
Radar, Vol. II, "Soviet Radio," Moscow,
1964.
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CHAPTER VI

GENERAL LAWS GOVERNING RADAR MEASUREMENTS

§ 6.1. Introduction

In the preceding chapters we considered questions of radar detection of targets. With this chapter we start a presentation of questions of radar measurements to which the subsequent chapters (7-12), too, are devoted. Before studying concrete schemes for construction of radars in a regime of measurement of coordinates of targets, dividing them by the form of measured coordinates (parameters of motion) of targets, we shall turn in this chapter to certain general laws, peculiar to radar measurements. The fact is that it is possible in fairly general form to theoretically give a basis for methods of analysis and synthesis of the most diverse radar meters. With certain, not too limiting, assumptions it is also possible to obtain basic performance characteristics of any meters and general functional circuits of their optimum construction. Being interested in general laws governing radar measurements, we shall consider two large groups of questions, namely: analysis of meters whose circuit is known, and synthesis of meters ensuring the best possible performance characteristics for measurement.

In order to grasp the domain of applicability of subsequent results, it is necessary first of all to deal with the question of the interrelation between the regime of detection studied above and the regime of measurement. Ultimately, any radar is intended for measurement of various parameters of motion of targets. Therefore, measurement is a basic operation of radar and should start from the very beginning of work of the system.

However, under conditions when the very fact of the presence of a signal

reflected from target still has not been fixed, consistent carrying out of this principle in theory leads to difficulties both methodological and mathematical. Therefore, everywhere in radars there is implicitly assumed the presence of the devices of detection and lock-in, which were analyzed in Chapters 3-5. These devices in one way or another produce the initial determination of parameters with precision, allowing us to pass to precision measurement (tracking). From what follows it will follow that in the case of application of tracking meters for this transition we need lock-in for all coordinates with accuracy of the width of discrimination curves of the meters, comprising the radar set.

It is important to indicate that in reality establishment of the fact of the presence of the target for finite time of lock-in is possible only with a finite probability, differing from unity. Everywhere below we shall consider that if lock-in occurred, a signal from the target indeed exists, and all qualities of the meters will be calculated on this assumption.

We shall also discuss the statistical character of the studied problem. Coordinates of objects, measured by radars, are included ("coded") in parameters of the signal reflected from the target or radiated by the target. Thus, the range of a target is usually coded in the time delay of modulation; velocity in the Doppler frequency shift; angular coordinates in modulation introduced by the receiving antenna. In Chapter 1 we indicated that a reflected radar signal has a random character. Various interferences of radar also have a fluctuating nature. These circumstances already compel us to describe the process of measurement of coordinates in statistical categories. However, in the problem of measurement it is impossible to do without one more aspect of the problem — change of coordinates in time. Actually, in practical conditions interesting us the located object is always moving, and the necessity of measurement appears, essentially, when we do not exactly know parameters of this motion. Here, it is possible to suggest two methods of description of the varying parameters:

1) variation of parameters are considered statistically, given the distribution of probabilities of their values at different moments of time or more limited statistical evidence;

2) unknown functions describing motion are considered determined, and the available information is presented in the form of a series of mathematical conditions (limitations), imposed, for instance, on the derivatives of the

functions with respect to time.

Both approaches have their advantages and disadvantages. A disadvantage of the nonstatistical approach is the necessity to judge the quality of measurement by certain of the worst cases, which, possibly, will never be observed in practice. A basic deficiency of the statistical approach is the so-called "a priori difficulty" in determining statistical properties of parameters. The fact is that their variation is determined sometimes by a great number of factors, including psychological (for instance, behavior of the operator or pilot). Determination of statistical properties, implying mass trials, is difficult even in idealized conditions, and is sometimes simply impossible in the absence in the past of similar systems or conditions of their use. However, it is possible to give a series of no less convincing arguments in favor of the statistical approach:

1) there exist a whole series of applications where statistics can be considered fully given. An example can be the case of a Doppler meter of groundspeed of an aircraft, where stationary variations of speed during a large number of flights can be statistically studied in sufficient detail;

2) satisfactory results in the sense of accuracy of measurement are given by meters, built only with qualitatively correct allowance for statistical properties of the parameter, inasmuch as namely the qualitative aspect determines the meter's functional scheme. Quantitatively circuit elements can be regulated in the process of tuning and testing.

For meters it is possible to consider the statistical approach more consistent.

Subsequently we will use both methods of describing variation of parameters, leaning in most cases toward the statistical. Practically, during designing of each new system of concrete assignment there always exists a considerable amount of information and physical considerations, albeit not rigorous, which are permissibly interpreted as statistical properties of the measured quantity.

Thus, the problem of radar measurement of coordinates has a statistical character both due to the random character of the mixture of the input signal and interferences, and also due to changes of measured quantities, unpredictable in general. In the course of the following presentation it is frequently necessary, however, to isolate an ensemble of parameters from the ensemble of the mixture of the signal and interferences. The latter, in turn, are sometimes conveniently divided into

subensembles.

During consideration of general questions of analysis and synthesis of radar meters in this chapter there is adopted the following order of presentation.

First of all in § 6.2 there is conducted analysis of the most wide-spread tracking meters; there are introduced into consideration necessary characteristics of their elements; there is developed a method of investigation of the accuracy of measurements; and there are given results of analysis of accuracy.

In § 6.3 there are investigated these same meters under the action of intense interferences, leading to breakoff of tracking. In practice there are also applied nontracking meters; questions of their analysis are the contents of § 6.4.

Remaining sections of the chapter are devoted to questions of synthesis of meters. In § 6.5 there are discussed possible criteria and methods of synthesis. There are presented propositions of the general theory of statistical solutions and its separate branches necessary for problems of measurements. Further, in § 6.6, having for subsequent considerations very great importance, there are expounded methods, and there are presented general results of statistical synthesis of optimum meters, based on close approximation of the introduced probability characteristics.

The next two sections are devoted to synthesis of separate parts of an optimum meter — discriminators (§ 6.7) and smoothing circuits (§ 6.8), with different statistical properties of input signals and measured quantities. The physical nature of measured parameters, here, is not made concrete; these questions are saved for subsequent chapters.

In § 6.9 there is given a certain expansion of the method of synthesis in the direction of parameters with Markov variation. In the same place there is given a series of results of synthesis for a limited knowledge of the statistics. The conclusion (§ 6.10) gives a brief survey of methods and results and contains the formulation of a series of new problems.

§ 6.2. General Analysis of Tracking Meters

At present there exists a large number of various forms of radar measuring devices. High accuracy of measurement is usually attained by automatic-tracking meters, built on the principle of a servo system. Meters of this type will be studied in the present section.

6.2.1. Basic Features of Circuit Construction and Components of Errors of Measurement

In the general form tracking meters can be represented by the schematic of Fig. 6.1. On it there are marked two basic elements — element for separation of the signal of mismatch between current and measured values of a parameter, usually called a discriminator, and amplifying-smoothing circuits together with drive units. Sometimes in the circuit there is also provided insertion through an adder of a certain additional voltage $\lambda_B(t)$, intended for compensation of clearly known components of the measured quantity. This can be carried out by timers, transducers of coordinates of the actual meter, by output data of roug' means of preliminary measurement, and so forth. Numerous examples of range finder. goniometers and speedometers, which can be reduced to the scheme of Fig. 6.1, are available in the literature, and also will be considered below in Chapters 7-11. Here we shall only indicate that the division of the whole circuit into two basic elements, to the second of which there belongs all the basic inertia, in the vast majority of cases can be carried out, at least in principle.

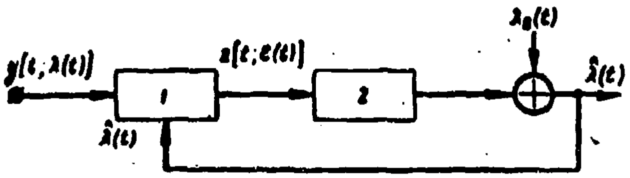

Fig. 6.1. General schematic of a tracking meter: 1) discriminator; 2) smoothing circuits and drive unit.

From the given description of the circuit of a meter it may be concluded that such types of automatic systems have very wide use in different areas of technology. Diverse control systems for industrial and military purposes, power systems, and so forth, include closed control circuits, separate elements of which are connected by communication circuits. During transmission of information in communication circuits there is produced its encoding and decoding in conditions of noises. If to investigation of such systems we bring the apparatus of the classical theory of automatic control, the devices for transmission of information usually are considered crudely simplified in the form of inertialess (sometimes linear) elements with additively imposed noises. For radar, however, as for many other contemporary areas, consideration of devices for processing data carried by a high-frequency signal (in this case, of discriminators) turns out to be absolutely obligatory. Here, this is no less important than to analyze smoothing and drive circuits. In this is the peculiarity of the subseqeunt analysis.

Tracked parameters in radar meters are angular coordinates, distance and speed,

concretely coded in time and in frequency shifts and other parameters of signal modulation. Input and output signals of a discriminator most frequently have an electrical nature, i.e., are certain voltages. Therefore, it is reasonable to introduce a binary system of designations, which will also be convenient in the theory of optimization of systems. Let us assume that $y(t; \lambda(t))$ and $z(t; \varepsilon(t))$ – input and output voltages of the discriminator; $\lambda(t)$ and $\hat{\lambda}(t)$ are the input and output values of the tracked parameter, and $\varepsilon(t) = \lambda(t) - \hat{\lambda}(t)$ – current mismatch (error). By $y(t; \lambda(t))$ we understand the whole mixture of useful signal and interferences at the meter input; an analogous mixture, obtained after processing in the discriminator, is implied by $z(t; \varepsilon(t))$.

In general to the discriminator input there proceed several input mixtures $y_i(t; \lambda(t))$ $(i = 1, 2, \ldots)$, which, naturally, changes neither the further methods of analysis nor its results.

It is important to indicate that between $y_i(t; \lambda(t))$ $(i = 1, 2, \ldots)$ and $z(t; \varepsilon(t))$, on the one hand, and their parameters $\lambda(t)$ and $\varepsilon(t)$, on the other, there does not exist a one-to-one correspondence. Thus $z(t; \varepsilon(t))$ is proportional to $\varepsilon(t)$ only on the average (over the ensemble of input signals), and in a narrow range of values of ε statistical properties of fluctuations in $z(t; \varepsilon(t))$ to a great degree depend on the intensity and form of the input interference included in mixtures $y_i(t; \lambda(t))$. The output value, i.e., result of measurement can be characterized by function $\hat{\lambda}(t)$.

The rest of this paragraph will be devoted to a basic characteristic of the described meters – accuracy of measurement. In this connection one should separate different components of measurement error.

In the first place fluctuating error is important. It appears due to internal noises of the system (noises of transmitters, receivers, antennas), external interferences of varied origin, most frequently additively joined with the useful input signal (but not with the measured quantity), and also due to fluctuations of the reflected radar signal. Furthermore, the measured variable can be coded in the signal in sum with a certain spurious component. We have in mind, for instance, the phenomenon of random shifts of the center of reflection of an extended radar target in angular coordinates and distance, distortion of the beam and dispersed propagation of radio waves in the atmosphere, etc. Inasmuch as the goal is measurement of a certain undistorted value of a coordinate, these factors introduce

additional error, which can have both a fluctuating, and also a constant component.

The second component of meter error is dynamic error due to changes of the measured magnitude itself, both random and those which are regular, but not compensated inside the meter.

We also distinguish systematic error. This is the usually somewhat uncompensated constant component of error, explained by the selected method of processing the signal.

In high-performance meters it is necessary to consider increase of fluctuating and systematic errors due to so-called instrument (equipment) errors, appearing due to oscillations of feed voltages, mechanical vibrations, fluctuations of temperature, gaps, dry friction, errors of quantization in digital variants of smoothing circuits, and so forth.

Henceforth, for simplicity we assume that there are no errors due to spurious components of the measured variable nor instrument errors where the forms of errors must be accounted for separately.

During the study of fluctuating and dynamic errors it turns out that an essential role in their calculation is played by the relationship between the rate of change of the measured parameter on the one hand, and on the other, the rate of change of random variables in the mixture of the signal with interferences (we call them for brevity fluctuations), which in principle there is no need to measure. In the most general case there certainly must exist rapid fluctuations due to internal noise of the receiver. Furthermore, there may exist random variables, the rate of change of which is commensurate with that for the measured parameter. Inasmuch as the approach during analysis and the method of calculation of error of measurement somewhat differ for various combinations of these variables, below we distinguish three separate cases:

a) case of rapid fluctuations alone;

b) case of a set of rapid and very slow fluctuations;

c) general case.

6.2.2. Characteristics of a Discriminator in the Case of Rapid Fluctuations

Here, it is useful to introduce a single statistical ensemble of fluctuations, in general determined by fluctuations of varied physical nature (see Chapter 1). Width of the spectrum of fluctuation in $z(t;\ \varepsilon(t))$ is assumed larger than the

effective transmission band of the whole closed system of the meter, and all processes in the discriminator are considerably faster than in the meter as a whole.* Therefore, function $z(t, \varepsilon)$ can be considered to consist of two parts — a mean value and certain noise. For a continuous input signal the mean value is equal to

$$a(\varepsilon, t) = \overline{z(t, \varepsilon)}, \tag{6.2.1}$$

and noise has correlation function

$$R(\tau; \varepsilon, t) = \overline{[z(t, \varepsilon) - \overline{z(t, \varepsilon)}][z(t+\tau, \varepsilon) - \overline{z(t+\tau, \varepsilon)}]}. \tag{6.2.2}$$

Due to inertia of subsequent circuits this noise can be considered white. Its spectral density one should consider equal to the spectral density of output fluctuations of the discriminator at low frequencies, introducing here the characteristic

$$S(\varepsilon, t) \equiv S(\omega; \varepsilon, t)|_{\omega=0} = \int_{-\infty}^{+\infty} R(\tau; \varepsilon, t)\, d\tau. \tag{6.2.3}$$

Averaging in (6.2.1) and (6.2.2), designated by the vinculum, is produced over the complete ensemble of fluctuations at the input, and magnitude ε here is considered fixed, which corresponds physically to slowness of its change. Possible dependence of $a(\varepsilon, t)$, $S(\varepsilon, t)$ on time is explained by the fact that statistical properties of input fluctuations may depend on time, but slowly and by known law.

For a pulse periodic signal it is possible to characterize $z(t, \varepsilon)$ by the same functions $a(\varepsilon, t)$, $S(\varepsilon, t)$, if we include in (6.2.1) and (6.2.2) additional averaging in time (for period of modulation T_r):

$$a(\varepsilon, t) = \frac{1}{T_r} \int_{t-T_r}^{t} \overline{z(t_1, \varepsilon)}\, dt_1, \tag{6.2.4}$$

$$S(\varepsilon, t) = \frac{1}{T_r} \int_{t-T_r}^{t} dt_1 \int_{-\infty}^{\infty} d\tau R(\tau; \varepsilon, t_1). \tag{6.2.5}$$

If we now compose from characteristics $a(\varepsilon, t)$, $S(\varepsilon, t)$ a certain statistical equivalent of function $z(t, \varepsilon)$, this equivalent, obviously, will have form

$$z_э(t, \varepsilon) = a(\varepsilon, t) + \sqrt{S(\varepsilon, t)}\,\zeta(t), \tag{6.2.6}$$

where $\zeta(t)$ — white noise of unit spectral density.

Actually, $z_э(t, \varepsilon)$ has identical properties at low frequencies with $z(t, \varepsilon)$,

*If smoothing circuits change in time, then the idea of an effective band is not applicable, but it nevertheless is possible to speak about a certain effective constant reaction time.

and this suffices for further analysis.

We introduce now for $a(\varepsilon, t)$ the name discrimination characteristic and for $S(\varepsilon, t)$, fluctuation characteristic of the discriminator. They are found by means of analysis of passage of the signal and different interferences through the discriminator for fixed mismatch ε. This is a separate problem, sometimes very bulky (see Chapters 7-11). Let us indicate that in nonstatistical consideration the discrimination characteristic is simply the dependence of output voltage of the discriminator on mismatch at a nominal level of input signal, and the fluctuation characteristic is not introduced in general.

If the signal has a pulse character, and fluctuating disturbances formed at the discriminator output in different periods are not correlated, it is sometimes more convenient to present the output of the discriminator in the k-th period in a form which is a discrete analog of (6.2.6):

$$z_k(\varepsilon) = a_k(\varepsilon) + \sigma_k(\varepsilon)\zeta_k, \qquad (6.2.7)$$

where ζ_k — a discrete random process with uncorrelated values and unit variance;

$a_k(\varepsilon)$ — mean value of z_k;

$\sigma_k^2(\varepsilon)$ — variance of the fluctuating component in the k-th period (dependence of functions $a_k(\varepsilon)$ and $\sigma_k^2(\varepsilon)$ on k, as also of functions $a(\varepsilon, t)$, $S(\varepsilon, t)$ on time t, again is considered slow and regular).

Let us stress, however, that writing (6.2.7) on the assumption of smallness of the period of repetition T_r as compared to the time of change of the measured quantity not only does not lead to greater rigor of consideration as compared to (6.2.6), but is also valid in the more frequent case of absence of correlation between disturbances in separate periods. As it is easy to check, formal transition from (6.2.7) to (6.2.6) should be carried out by replacement of current subscript k by current time t, of process with uncorrelated values ζ_k by white noise $\zeta(t)$ of unit spectral density and of magnitude $\sigma_k^2(\varepsilon)$ by $S(\varepsilon, t_k) = \sigma_k^2(\varepsilon)T_r$.

Thus, the notation (6.2.6) in the considered conditions is sufficiently general, and it remains to consider in more detailed form the form of characteristics of the discriminator, assuming, for simplicity, that they do not depend on time.

With satisfaction of conditions of symmetry of the circuit and of characteristics of the mixture $y(t; \lambda(t))$ the discrimination characteristic is an odd, and the fluctuation characteristic is an even function of ε. The discrimination characteristic (Fig. 6.2) for small ε has a linear section. We usually seek

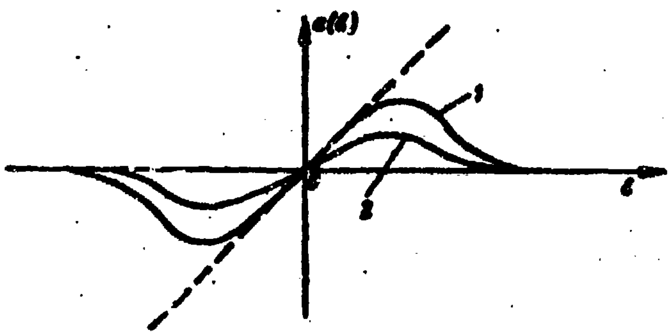

Fig. 6.2. Typical family of discrimination curves: 1) case of small interferences; 2) case of large interferences.

increases of the extent and slope of this section; however, this does not lead in all cases to lowering of errors of measurement. For large ε the discrimination characteristic has dips corresponding to emergence of the parameter beyond the limits of the region in which the discriminator is selective. Analysis of concrete circuits shows that intense interferences lead to decrease of the scale of the discrimination curve along the axis of ordinates and to amplification of dips. This is explained by the normalizing action of certain, generally speaking, nonlinear elements (system of automatic gain control (AGC), clippers, peak detectors, and so forth, see Chapter 2).

The fluctuation characteristic (Fig. 6.3) to a still greater extent depends on the form and level of input interferences. Output fluctuations of a discriminator consist of several components. They appear as a result of interaction among spectral components of interference, beats of interference with the signal, and, in the case of the presence of signal fluctuations, results of interaction of signal components.* Intensity of the last two components depends on ε, increasing where the control signal, i.e., the modulus of the discrimination characteristic is great. Intensity of beats of interference components among themselves also depends on ε, but only

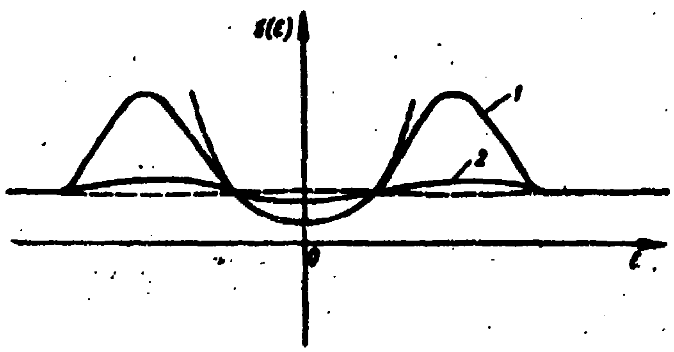

Fig. 6.3. Typical family of fluctuation characteristics: 1) case of small interferences; 2) case of large interferences.

*Strictly speaking, division of output fluctuations into the three shown components can be performed only when using in the discriminator circuit, as elements of nonlinear processing, square-law peak detectors and ideal multipliers. Qualitatively the picture also remains, however, true in other cases, at least for large signals.

due to the normalizing action of the discriminator, overwhelming noises for strong signals and small ε. We have in mind, for instance, decrease of gain of the receiver, and consequently also of the output level of noises due to action of an AGC system with sufficiently accurate tuning on a powerful signal. As a result the fluctuation characteristic frequently has the form of a double-humped curve with a dip near $\varepsilon = 0$, has rises near extremes of the discrimination characteristic and has a certain constant level for very great mismatch ε (Fig. 6.3). Here, $S(\varepsilon)$ as $\varepsilon \to \pm\infty$ is determined only by internal noises of the receiver and interferences. With growth of interferences not carrying information about the measured variable, function $S(\varepsilon)$ gradually is smoothed, until it becomes independent of ε and of the signal-to-interference ratio at the discriminator input.

The described dependences in subsequent chapters will be illustrated by a series of examples. They take place under the above-indicated conditions of symmetry. In other cases the form of the characteristics of the discriminator may vary somewhat. In particular, shift of both characteristics along the axis ε without change of the shape of the curves is possible, so that the new center of symmetry will be point $\Delta \neq 0$. Then we speak of systematic error, introduced by the discriminator. In general, shift is accompanied by distortion of the shape of the curves, so that they no longer have the shown simple form. This occurs, for instance, with the presence along with the useful signal of an interfering signal close in structure.

Presentation of discriminator output in form (6.2.6) should be used under conditions when mismatch ε can take large values, i.e., during the study of questions of lock-in and breakoff (§ 6.3). At a sufficiently low level of interferences random process $\varepsilon(t)$ with high probability takes small values, and approximation of characteristics of the discriminator by very simple functions near point $\varepsilon = 0$ is sufficient:

$$a(\varepsilon)=a_0+K_Д\varepsilon,$$
$$S(\varepsilon)=S_0+S_1\varepsilon+S_2\varepsilon^2, \tag{6.2.8}$$

where

$$a_0=a(0),\ K_Д=\frac{da(0)}{d\varepsilon},\ S_k=\frac{1}{k!}\frac{d^kS(0)}{d\varepsilon^k}\ (k=0,\ 1,\ 2).$$

Ratio $\Delta = a_0/K_Д$ is the systematic error of the discriminator. When conditions of symmetry are satisfied, quantities a_0 and S_1 turn into zero and (6.2.8) takes the

form

$$a(\varepsilon)=K_{д}\varepsilon,\ S(\varepsilon)=S_0+S_2\varepsilon^2. \tag{6.2.9}$$

Coefficient $K_д$ is called the amplification or gain factor of the discriminator. Quantity S_0 characterizes the fluctuating component, not depending on mismatch ε, and S_2 — the component, proportional to ε. For explanation we indicate that (6.2.9) corresponds to the following presentation of discriminator output voltage:

$$z_д(t;\varepsilon)=K_д\eta(t)+K_д(1+\xi(t))\varepsilon. \tag{6.2.10}$$

In this case random process $\sqrt{S(\varepsilon)}\zeta(t)$ is replaced by two uncorrelated processes $K_д\eta(t)$ and $K_д\varepsilon\xi(t)$ with the same total intensity. Here $\eta(t)$ — white noise with dimensionality of the measured parameter and with spectral density proportional to S_0:

$$S_\eta\equiv S_{экв}=S_0/K_д^2, \tag{6.2.11}$$

henceforth called the equivalent spectral density, and $\xi(t)$ — a parametric input in the form of white noise with spectral density, proportional to S_2:

$$S_\xi\equiv S_{пар}=S_2/K_д^2, \tag{6.2.12}$$

henceforth called parametric.

Thus, under the formulated conditions of symmetry output of the discriminator can be characterized by only three quantities: $K_д$, $S_{экв}$ and $S_{пар}$. Their dependence on the character of the input signal (signals) $y(t;\lambda)$ and the method of construction of the discriminator in concrete examples will be clarified in subsequent chapters. Here, we indicate that the gain factor $K_д$ drops with growth of interfences which are not contained in the coded form of the tracked parameter, simultaneously with the above-indicated decrease of the scale of curve $a(\varepsilon)$. Quantity $S_{экв}$ also depends on the level of interferences: for weak interferences it is close to zero, and with growth of interferences it grows without limit. Quantity $S_{пар}$ characterizes the coefficient of parameteric pulsations of control voltage, caused usually by amplitude fadings of the signal. In goniometric and range finding system $S_{пар}$ is considerably lowered when there are present amplitude-demodulators (AGC, clippers, etc). The dependence of $S_{пар}$ on the level of input interferences is determined by the concrete nature of $\lambda(t)$ and the circuit of the discriminator.

It is easy to prove that the introduced magnitudes $S_{экв}$ and $S_{пар}$ are two coefficients of the expansion with respect to ε^2 of function $S_{экв}(\varepsilon)=S(\varepsilon)/K_д^2$, which is sometimes convenient to use instead of $S(\varepsilon)$, called the equivalent fluctuation

characteristic. This term indicates that $S_{экв}(\varepsilon)$ characterizes noises, recalculated in equivalent values of the tracked variable.

Relationship (6.2.10) is conveniently interpreted by the circuit of Fig. 6.4, which can be used during simulation of a complicated system, where a given discriminator is applied. Note that $S_{пар}$ characterizes that component of fluctuations, which in the given equivalent circuit parametrically governs amplification of the radio channel.

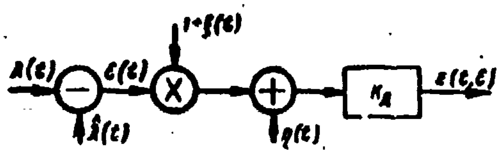

Fig. 6.4. Equivalent circuit of a discriminator: $K_Д$ — inertialess amplifier with gain factor $K_Д$.

Thus, even in the case of linear smoothing circuits with constant parameters the radar tracking meter turns out to possess variable gain in the loop, i.e., one variable parameter. In this connection the fluctuating component, proportional to mismatching, we call parametric. One should not, of course, confuse this component with random changes of the tracked parameter $\lambda(t)$.

It is necessary, however, to stipulate that in a number of combinations of parameters of the circuit and the input signal-to-noise ratio it turns out that $S_2 < 0$, i.e., the level of noises upon the appearance of mismatch even decreases somewhat. Examples of this especially nonlinear effect will be given in subsequent chapters. When $S_2 < 0$ presentation (6.2.10) is already impermissible, and the linear model of Fig. 6.4 loses meaning. The problem can be studied only by a mathematical apparatus, adapted to nonlinear problems.

6.2.3. Accuracy of Measurement in the Absence of Parametric Fluctuations

On the basis of the introduced discriminator characteristics we shall calculate the error of measurement. Let us assume that parametric fluctuations are negligible. We shall study that important and sufficiently general case, when smoothing circuits are linear, but not necessarily with constant parameters, so that their input and output are connected by relationship

$$\hat{\lambda}(t)=\int_{t_0}^{t} h(t,\tau)\,z(\tau)\,d\tau+\lambda_B(t), \tag{6.2.13}$$

where $h(t, \tau)$ — pulse response of smoothing circuits;

$\lambda_B(t)$ — quantity introduced into the meter (Fig. 6.1), not necessarily accurately coinciding with the a priori mean value of parameter $\lambda(t)$, which is considered a random process.

Substituting in (6.2.13) according to (6.2.10) quantity

$$\varepsilon_{э}(t) = K_{д}\varepsilon(t) + K_{д}\eta(t)$$

and considering that $\hat{\lambda} = \lambda - \varepsilon$, we have equation

$$\varepsilon(t) + K_{д}\int_{t_0}^{t} h(t, \tau)\,\varepsilon(\tau)\,d\tau = \lambda(t) - \lambda_{в}(t) - K_{д}\int_{t_0}^{t} h(t, \tau)\,\eta(\tau)\,d\tau. \qquad (6.2.14)$$

We introduce the pulse response $g(t, \tau)$ of the closed system, considering $\hat{\lambda}(t)$ the output quantity. Function $g(t, \tau)$ is determined by integral equation

$$g(t, \tau) + K_{д}\int_{\tau}^{t} h(t, s)\,g(s, \tau)\,ds = K_{д}h(t, \tau). \qquad (6.2.15)$$

Additionally we introduce pulse response $v(t, \tau)$, considering mismatch $\varepsilon(t)$ the output quantity. Function $v(t, \tau)$ satisfies equation

$$v(t, \tau) + K_{д}\int_{\tau}^{t} h(t, s)\,v(s, \tau)\,ds = \delta(t - \tau). \qquad (6.2.16)$$

By direct substitution of expression (6.2.13) in (6.2.14), using (6.2.15) and (6.2.16), it is simple to prove that the solution of equation (6.2.14) has the form

$$\begin{aligned}\varepsilon(t) = &-\int_{t_0}^{t} g(t, s)\,\eta(s)\,ds + \int_{t_0}^{t} v(t, s)\,[\lambda(s) - \overline{\lambda(s)}]\,ds + \\ &+\int_{t_0}^{t} v(t, s)\,[\overline{\lambda(s)} - \lambda_{в}(s)]\,ds = \varepsilon_{фл}(t) + \varepsilon_{дин\,1}(t) + \varepsilon_{дин\,2}(t),\end{aligned} \qquad (6.2.17)$$

where function $\overline{\lambda(t)}$ is added and subtracted.

According to (6.2.17) current error is determined by the joint action of interference $\eta(t)$, passed through the filter with pulse response $g(t, s)$, of random changes of parameter $\lambda(s) - \overline{\lambda(s)}$, and of uncorrected regular changes of parameter $\overline{\lambda(s)} - \lambda_{в}(s)$, passed through the filter with pulse response $v(t, s)$. The first component

$$\varepsilon_{фл}(t) = \int_{t_0}^{t} g(t, s)\,\eta(s)\,ds$$

is naturally called fluctuation error; to terms $\varepsilon_{дин\,1}(t)$, $\varepsilon_{дин\,2}(t)$ we give the designation of dynamic errors. Let us note, however, that in the statistical approach to measured quantities introduction of the last term is somewhat conditional, inasmuch as $\varepsilon_{дин\,1}(t)$ has a random character to the same extent as $\varepsilon_{фл}(t)$.

Considering that the measured parameter is a random process, and that fluctuations are rapid, by (6.2.17) it is possible to easily calculate the mean square

overall measuring error:

$$\sigma^2_{\text{вых}}(t)=\overline{\varepsilon^2(t)}=S_{\text{экв}}\int_{t_0}^{t} g^2(t,s)\,ds+$$
$$+\int_{t_0}^{t}\int_{t_0}^{t} v(t,s_1)\,v(t,s_2)\,R_\lambda(s_1,s_2)\,ds_1 ds_2+$$
$$+\Big[\int_{t_0}^{t} v(t,s)\,\Delta(s)\,ds\Big]^2=\sigma^2_{\text{фл}}(t)+\sigma^2_{\text{дин1}}(t)+\sigma^2_{\text{дин2}}(t). \qquad (6.2.18)$$

Here we use the property of δ-correlation of interference, and there is introduced correlation function $R_\lambda(t_1, t_2)$ of the random part of the measured parameter and we denote $\Delta(t) = \overline{\lambda(t)} - \lambda_{\text{B}}(t)$.

According to (6.2.18) for an arbitrary pulse response of smoothing circuits the variance of fluctuation error $\sigma^2_{\text{фл}}(t)$ is proportional to the equivalent spectral density (6.2.11).

Further simplifications of expressions for separate components of expression (6.2.18) are possible upon concretization of the form of smoothing circuits and of the character of change of $\lambda(t)$.

a) Assume stationariness of the random part of parameter $R_\lambda(t_1, t_2) = R_\lambda(t_1 - t_2)$, full correction of its regular part ($\Delta(t) = 0$) and constancy of parameters of smoothing circuits. Fourier transforms from functions $g(t, \tau) = g(t - \tau)$ and $v(t, \tau) = v(t - \tau)$ with respect to (6.2.15) and (6.2.16) in this case are easily expressed in terms of the amplification factor of the discriminator and the frequency response of the smoothing circuits

$$H(i\omega)=\int_0^{\infty} h(\tau)\,e^{-i\omega\tau}d\tau$$

in the form

$$G(i\omega)=\frac{K_{\text{д}}H(i\omega)}{1+K_{\text{д}}H(i\omega)},$$
$$V(i\omega)=\frac{1}{1+K_{\text{д}}H(i\omega)}. \qquad (6.2.19)$$

Finally instead of (6.2.18) we can obtain

$$\sigma^2_{\text{вых}}=\sigma^2_{\text{фл}}+\sigma^2_{\text{дин}}=2S_{\text{экв}}\Delta f_{\text{эф}}+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{S_\lambda(\omega)\,d\omega}{|1+K_{\text{д}}H(i\omega)|^2}, \qquad (6.2.20)$$

where $S_\lambda(\omega)=\int_{-\infty}^{+\infty}R_\lambda(\tau)e^{-i\omega\tau}d\tau$ – spectral density of the random part of parameter $\lambda(t)$;

$\Delta f_{\text{эф}}$ – effective transmission band of a closed-cycle meter:

$$\Delta f_{\text{эф}}=\frac{1}{2\pi}\int_0^{+\infty}|G(i\omega)|^2\,d\omega=\frac{1}{2}\int_0^{\infty}g^2(t)\,dt. \qquad (6.2.21)$$

It would be more systematic to determine the effective band by normalized relationship

$$\Delta f_{эф} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{|G(i\omega)|^2}{|G(0)|^2}\, d\omega. \qquad (6.2.22)$$

However, inasmuch as due to the large gain in the loop, usually being $G(0) \approx 1$, there is no difference between (6.2.21) and (6.2.22).

As follows from (6.2.20), variances of fluctuation and dynamic errors in the considered case do not depend on time, where the first is expressed by the simple and widely known formula

$$\sigma^2_{\phi л} = 2S_{экв}\Delta f_{эф}. \qquad (6.2.23)$$

b) Let us consider an example, when smoothing circuits are constant, and the random part of the parameter is expressed by a polynomial in time t with random factors μ_k:

$$\lambda(t) = \sum_{k=0}^{m} \mu_k t^k \ (\overline{\mu_k} = 0,\ \overline{\mu_i \mu_k} = M_{ik},\ t > 0) \qquad (6.2.24)$$

Fluctuation error is expressed as before by formula (6.2.23), and the mean square dynamic error according to (6.2.18) has the form:

$$\sigma^2_{дин}(t) = \sum_{i,k=0}^{m} M_{ik}\varepsilon_i(t)\varepsilon_k(t), \qquad (6.2.25)$$

where

$$\varepsilon_k(t) = \int_0^t v(t-\tau)\tau^k d\tau = \frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty} V(p)\frac{k!}{p^{k+1}}e^{pt}dp, \qquad (6.2.26)$$

and $V(p)$ is given by formula (6.2.19).

With increase of the time of measurement, (6.2.26) passes to

$$\lim_{t\to\infty} \varepsilon_k(t) = \lim_{p\to 0} V(p)\frac{k!}{p^k}. \qquad (6.2.27)$$

It follows from this that a regime with a strictly finite magnitude of variance of stationary dynamic error exists only in the presence in the numerator of $V(p)$ of a factor p to a power, not smaller than the power of polynomial (6.2.24), depicting behavior of the parameter. The numerator of $V(p)$, according to (6.2.19), is determined in turn by the denominator of the transfer function $H(p)$ of the smoothing circuits. The latter, in the presence of factor p^n, are called astatic functions of the n-th order. Astaticism is attained by circuits, close in properties to

ideal integrators. Thus, a smoothing circuit in the form of a single integrator ensures astaticism of the first order:

$$H(p)=\frac{K_{\text{и}}}{p},\ V(p)=\frac{p}{p+K_{\text{и}}K_{\text{д}}}, \tag{6.2.28}$$

and with a double integrator with correction we have astaticism of the second order:

$$H(p)=\frac{K_{\text{и}}(1+p\tau)}{p^2},\ V(p)=\frac{p^2}{p^2+K_{\text{и}}K_{\text{д}}(1+p\tau)}. \tag{6.2.29}$$

In general form the mean square dynamic error in steady-state regime finally has the form

$$\sigma^2_{\text{дин}}=\sum_{i,k=0}^{m} M_{ik}\lim_{p\to 0}\frac{i!\,k!\,V^2(p)}{p^{i+k}}. \tag{6.2.30}$$

Let us consider in more detail a series of simple examples.

1. The smoothing circuit has the form of an ideal integrator (6.2.28). The effective transmission band of a closed-loop meter here is equal to

$$\Delta f_{\text{эф}}=K_{\text{и}}K_{\text{д}}/4. \tag{6.2.31}$$

For the case of a stationary parameter with spectral density $S_\lambda(\omega) = 2\sigma_\lambda^2 T/[1 + (\omega T)^2]$, where σ_λ^2 – variance, we have mean square dynamic error

$$\sigma^2_{\text{дин}}=\frac{\sigma^2_\lambda}{1+1/TK_{\text{и}}K_{\text{д}}}. \tag{6.2.32}$$

For a parameter in the form of polynomial (6.2.24) establishment of error is observed only at m = 0 and 1. When m = 0 error $\sigma^2_{\text{дин}} = 0$, and when m = 1

$$\sigma^2_{\text{дин}}=\frac{M_{11}}{(K_{\text{и}}K_{\text{д}})^2}. \tag{6.2.33}$$

Error in position (i.e., due to the constant term of the random variable) is equal to zero, and error in acceleration or with respect to a higher derivative (in the case of the presence in the law of change of parameter of corresponding terms) grows without limit.

2. The smoothing circuit has the form of an RC-circuit:

$$H(p)=K_0/(1+p\tau).$$

The effective band and dynamic error for a stationary parameter with the above-mentioned spectral density have the form:

$$\Delta f_{\text{эф}}=\frac{K_0K_{\text{д}}}{4\tau},$$
$$\sigma^2_{\text{дин}}=\sigma^2_\lambda\left(\frac{K_0}{1+K_0}\right)^2\frac{1}{T+\tau/(1+K_0)}. \tag{6.2.34}$$

If we turn to the case of a parameter in the form of a polynomial, then it is easy to prove that establishment of error occurs only for constant parameter, and its variance is equal to $M_0/(1 + K_0K_Д)^2$. However, in the case of a large gain factor and large time constant of the smoothing circuit, for $M_{11} \neq 0$ we have in a sufficiently extended interval of time

$$\sigma^2_{уст} \approx M_{11}\left(\frac{T}{K_0K_Д}\right)^2, \tag{6.2.35}$$

i.e., the closed-loop system has approximately the same properties as in the case of an ideal integrator with gain factor K_0/T.

All that has been presented shows that in the absence of parametric fluctuations analysis of accuracy of meters is sufficiently simple and leads to results, widely known from classical automatic control theory.

6.2.4. Accuracy of Measurement in the Presence of Parametric Fluctuations

In the presence of parametric fluctuations it is possible to replace expression (6.2.14) by equation

$$\varepsilon(t) + K_Д \int_{t_0}^{t} h(t, s)[1 + \xi(s)]\varepsilon(s)\,ds =$$
$$= \lambda(t) - \lambda_п(t) - K_Д \int_{t_0}^{t} h(t, s)\eta(s)\,ds. \tag{6.2.36}$$

Again introducing pulse responses $g(t, \tau)$ and $v(t, \tau)$, according to (6.2.15) and (6.2.16), we can obtain

$$\varepsilon(t) = -\int_{t_0}^{t} g(t, s)\eta(s)\,ds + \int_{t_0}^{t} v(t, s)[\lambda(s) - \lambda_п(s)]\,ds -$$
$$-\int_{t_0}^{t} g(t, s)\xi(s)\varepsilon(s)\,ds. \tag{6.2.37}$$

Formula (6.2.37) still is not the solution of equation (6.2.36), inasmuch as quantity $\varepsilon(t)$ enters into both parts of (6.2.37). Results convenient for use can be derived from (6.2.37), using smallness of variance of $\xi(t)$, usually observed in practice. The measure of this smallness will be shown below. According to the method of successive approximations we shall seek a solution of (6.2.37) in the form of the series

$$\varepsilon(t) = \varepsilon_0(t) + \varepsilon_1(t) + \varepsilon_2(t) + \dots,$$

where $\varepsilon_0(t)$ has the zero, $\varepsilon_1(t)$ the first, $\varepsilon_2(t)$ the second, etc., order of smallness.

Then, for the zero approximation $\varepsilon_0(t)$ we again have solution (6.2.17). The first correction is equal to

$$\varepsilon_1(t)=\int_{t_0}^{t}\int_{t_0}^{s} g(t, s)\xi(s)\, g(s, \tau)\, \eta(\tau)\, ds\, d\tau - \\ -\int_{t_0}^{t} g(t, s)\xi(s)\,[\varepsilon_{\text{дин}_1}(s)+\varepsilon_{\text{дин}_2}(s)]\, ds. \tag{6.2.38}$$

Analogously the second correction will be expressed in the form

$$\varepsilon_2(t)=-\int_{t_0}^{t} ds \int_{t_0}^{s} d\vartheta \int_{t_0}^{\vartheta} d\tau\, g(t, s)\xi(s)\, g(s, \vartheta)\xi(\vartheta)\, g(\vartheta, \tau)\, \eta(\tau)+ \\ +\int_{t_0}^{t} ds\int_{t_0}^{s} d\tau\, g(t, s)\xi(s)\, g(s, \tau)\xi(\tau)\,[\varepsilon_{\text{дин}_1}(\tau)+\varepsilon_{\text{дин}_2}(\tau)]. \tag{6.2.39}$$

The mean square of $\varepsilon(t)$ during series expansion is equal to

$$\sigma^2_{\text{вых}}(t)=\overline{\varepsilon_0^2(t)}+2\overline{\varepsilon_0(t)\,\varepsilon_1(t)}+[2\overline{\varepsilon_0(t)\,\varepsilon_2(t)}+\overline{\varepsilon_1^2(t)}]+\ldots,$$

where averaging is conducted both for the ensemble of signals, and also for the ensemble of parameters. Averaging with the help of (6.2.17), (6.2.38) and (6.2.39), we obtain, in order:

1.

$$\overline{\varepsilon_0^2(t)}=S_{\text{экв}}\int_{t_0}^{t} g^2(t, s)\, ds+ \\ +\int_{t_0}^{t}\int_{t_0}^{t} \upsilon(t, s_1)\,\upsilon(t, s_2)\, R_\lambda(s_1, s_2)\, ds_1 ds_2+\Big[\int_{t_0}^{t}\upsilon(t, s)\,\Delta(s)\, ds\Big]^2= \\ =\sigma^2_{\text{фл}\,0}(t)+\sigma^2_{\text{дин}_1}(t)+\sigma^2_{\text{дин}_2}(t). \tag{6.2.40}$$

2. $\overline{\varepsilon_0(t)\varepsilon_1(t)} = 0$ due to noncorrelation and zero mean values of functions $\xi(t)$, $\eta(t)$ and $\lambda(t) - \overline{\lambda(t)}$.

3.

$$\overline{\varepsilon_0(t)}\,\overline{\varepsilon_2(t)}=S_{\text{пар}}\iint g(t, \tau_1)\, g(\tau_2, \tau_2)\,[\overline{\varepsilon_{\text{дин}_1}(t)\,\varepsilon_{\text{дин}_1}}(\tau_2)+ \\ +\varepsilon_{\text{дин}_2}(t)\,\varepsilon_{\text{дин}_2}(\tau_2)]\, d\tau_1 d\tau_2+S_{\text{пар}}S_{\text{экв}}\iint g(t, \tau_1)\, g(\tau_2, \tau_1)\times \\ \times g(t, \tau_2)\, g(\tau_2, \tau_2)\, d\tau_1 d\tau_2. \tag{6.2.41}$$

The value in (6.2.41) of pulse response of the closed-loop system with coinciding arguments $g(t, t)$ one should consider equal to zero, inasmuch as even in approximation of pulse response of smoothing circuits by functions breaking at these points (inertial link of the 1st order, link of the 2nd order with correction, and

so forth) it is impossible not to allow for the fact that the discriminator has inertia (delay), not smaller than the interval of correlation of that fluctuating voltage at its output, which we consider white noise. Consequently, there do not exist fluctuating components instantly returning through the feedback circuit and emphasizing fluctuations in the discriminator.

Considering this circumstance, we have $\overline{\varepsilon_0(t)\varepsilon_2(t)} = 0$.

4.

$$\overline{\varepsilon_1^2(t)} = S_{пар}S_{экв}\iint g^2(t,\tau_1)\,g^2(\tau_1,\tau_2)\,d\tau_1\,d\tau_2 + S_{пар}\int g^2(t,\tau)\left[\sigma^2_{дин_1}(\tau)+\sigma^2_{дин_0}(\tau)\right]d\tau. \tag{6.2.42}$$

Considering (6.2.40) and (6.2.42) with an accuracy of terms of the second order of smallness we have total error of measurement:

$$\sigma^2_{вых}(t) = \sigma^2_{вых_0}(t) + S_{пар}\int g^2(t,\tau)\left[\sigma^2_{дин_1}(\tau)+\sigma^2_{дин_0}(\tau)\right]d\tau + S_{пар}S_{экв}\int g^2(t,\tau_1)\,g^2(\tau_1,\tau_2)\,d\tau_1 d\tau_2, \tag{6.2.43}$$

where $\sigma^2_{ВЫХ_0}(t)$ – variance of output error for a meter when we disregard parametric fluctuations, expressed by formula (6.2.40).

Simpler relationships can be obtained in the case of a stationary parameter and constant smoothing circuits for $\Delta = 0$:

$$\sigma^2_{вых} = \sigma^2_{вых_0}(1+\sigma^2_{пар}). \tag{6.2.44}$$

Here, there is introduced $\sigma^2_{пар} = 2S_{пар}\Delta f_{эф}$ – variance of parametric fluctuations, smoothed in the loop.

After analogous calculations to terms of the fourth order of smallness we have

$$\sigma^2_{вых} = \sigma^2_{вых_0}(1+\sigma^2_{пар}+\sigma^4_{пар}). \tag{6.2.45}$$

According to (6.2.45) total output error due to parametric fluctuations is increased by a factor of $1 + \sigma^2_{пар} + \sigma^4_{пар}$, i.e., the additional component of error is greater, the greater the error of measurement in the absence of parametric fluctuations. This circumstance corresponds to description of the circuit of Fig. 6.4, where parametric fluctuations characterize changes of gain of the discriminator, having no influence only when $\varepsilon = 0$.

Quantity $\sigma^2_{пар}$ in the stationary case can be expressed in terms of variance σ^2_ξ and width of the spectrum Δf_ξ of function $\xi(t)$, assumed above to be white noise, in the form

$$\sigma^2_{пар} = 2S_{пар}\Delta f_{эф} = \sigma_\xi^2\,\frac{\Delta f_{эф}}{\Delta f_\xi}. \tag{6.2.46}$$

From (6.2.46) it follows that parametric fluctuations can be ignored when the effective band of a tracking meter is considerably narrower than the width of the spectrum of these fluctuations, and their variance is much less than unity. Only in this case, instead of (6.2.43) and (6.2.45), it is possible to use simpler formulas (6.2.18) and (6.2.20), and properties of the discriminator are determined only by quantities $K_{д}$ and $S_{экв}$.

We shall indicate here that the obtained formulas, containing coefficient $S_{пар}$, are not applicable if $S_{пар} < 0$, and also upon reaching $\sigma^2_{пар}$ of large values, when approximation, based on use of only the first terms of the series expansion and leading to formulas (6.2.44), (6.2.45), does not give a correct answer. In these cases the tracking meter should be considered a nonlinear system. Such a consideration will be given in § 6.3.

6.2.5. Case of a Set of Rapid and Very Slow Fluctuations

Besides rapid fluctuations in the input signal there may exist disturbances, varying considerably more slowly than the measured parameter. We consider, for instance, very slow fadings of the signal, in which its amplitude and phase for very considerable time intervals can be considered constant, or stable interfering reflection. In these cases it is useful to write the input signal in the form $y(t, \lambda(t), \nu(t))$, where $\lambda(t)$ — tracked variable, and $\nu(t)$ — random, slowly varying parameter (or parameters) of the signal, which, in principle, there is no necessity to measure. We naturally introduce here the same characteristics of a discriminator as for the case of rapid fluctuations, but we consider them to additionally depend on $\nu(t)$. The equivalent of output voltage of the discriminator is presented in the form

$$z_{\bullet}(t)=\dot{a}(\varepsilon, \nu)+\sqrt{S(\varepsilon, \nu)}\zeta(t), \qquad (6.2.47)$$

where $a(\varepsilon, \nu)$, $S(\varepsilon, \nu)$, as before, are the discrimination and fluctuation characteristics.

Relative white noise $\zeta(t)$ here is formed only by rapid fluctuations, as distinguished from (6.2.6). For a low level of noises instead of (6.2.47) it is sufficient to use relationship

$$z_{\bullet}(t)=K_{д}(\nu)\varepsilon+\sqrt{S_{\bullet}(\nu)}\zeta(t). \qquad (6.2.48)$$

The basic performance characteristic of tracking — mean square error — is found just as in Paragraph 6.2.3. In particular, in the absence of rapid

parametric fluctuations in the stationary case we have a formula, analogous to (6.2.20):

$$\sigma^2_{\text{вых}}(\nu) = 2S_{\text{экв}}(\nu)\,\Delta f_{\text{эф}}(\nu) + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{S_\lambda(\omega)\,d\omega}{|1+K_\text{д}(\nu)\,H(i\omega)|^2}, \tag{6.2.49}$$

where

$$S_{\text{экв}}(\nu) = S_0(\nu)/K_\text{д}^2(\nu);$$

$$\Delta f_{\text{эф}}(\nu) = \frac{1}{2\pi}\int_0^{\infty}\left|\frac{K_\text{д}(\nu)\,H(i\omega)}{1+K_\text{д}(\nu)\,H(i\omega)}\right|^2 d\omega.$$

According to (6.2.49), on ν there depend both dynamic error (through the gain factor in the loop), and also fluctuation error (through equivalent spectral density $S_{\text{ЭКВ}}(\nu)$ and effective band width $\Delta f_{\text{эф}}(\nu)$).

In formula (6.2.49) we average the ensembles of rapid fluctuations and of parameter $\lambda(t)$. It is possible to use this formula when we are interested in the process of measurement at time intervals, in which random parameter $\nu(t)$ does not noticeably change. In another case it is useful to average $\sigma^2_{\text{ВЫХ}}$ with respect to ν:

$$\overline{\sigma^2_{\text{вых}}} = \overline{\sigma^2_{\text{дин}}(\nu)} + 2\overline{S_{\text{экв}}(\nu)\,\Delta f_{\text{эф}}(\nu)}. \tag{6.2.50}$$

As a simple example we shall consider the case when slow fluctuations are expressed only in stationary amplitude modulation of the amplification factor of the discriminator, and S_0 does not change in time. In other words, there occur slow parametric fluctuations.

Introducing amplification factor $\overline{K}_\text{Д}$ averaged with respect to ν,

$$K_\text{д}(\nu) = \overline{K}_\text{д}(1+\xi(t)),$$

where $\xi(t)$ – slowly varying stationary random function, we have fluctuation error

$$\sigma^2_{\text{фл}} = \overline{\frac{S_0}{\overline{K}_\text{д}^2}\frac{1}{(1+\xi)^2}\int_{-\infty}^{+\infty}\left|\frac{\overline{K}_\text{д}(1+\xi)\,H(i\omega)}{1+\overline{K}_\text{д}(1+\xi)\,H(i\omega)}\right|^2 d\omega} \approx \tag{6.2.51}$$

$$\approx \overline{S}_{\text{экв}}\int_{-\infty}^{+\infty}\overline{|\overline{G}(i\omega) - \overline{G}^2(i\omega)\xi + \overline{G}^3(i\omega)\xi^2|^2}\,d\omega =$$

$$= \overline{S}_{\text{экв}}\,\overline{\Delta f_{\text{эф}}}\,(1+k\sigma^2_{\text{пар}}).$$

Here there are introduced the averaged equivalent spectral density $\overline{S}_{\text{ЭКВ}} = S_0/\overline{K}^2_\text{Д}$, the frequency response of the closed loop

$$\overline{G}(i\omega) = \frac{\overline{K}_\text{д}\,H(i\omega)}{1+\overline{K}_\text{д}H(i\omega)}$$

and effective bandwidth $\overline{\Delta f}_{\text{эф}} = \frac{1}{2\pi}\int_0^{\infty} |\overline{G}(i\omega)|^2 d\omega$.

In relationship (6.2.51) we perform expansion with respect to ξ, and there is given the designation $\sigma^2_{\text{пар}} = \overline{\xi^2} \ll 1$ for variance of the parametric input, and by k we denote the numerical coefficient

$$k = \frac{\int_{-\infty}^{+\infty} |\overline{G}(i\omega)|^2 [\,|\overline{G}(i\omega)|^2 - 2\overline{G}^2(i\omega)|\, d\omega}{\int_{-\infty}^{+\infty} |\overline{G}(i\omega)|^2 d\omega}. \qquad (6.2.52)$$

As we proved, the presence of very slow fluctuations does not lead to noticeable change of our approach to analysis of meters; it is only necessary to consider that $S_{\text{экв}}$ differs for rapid and slow fadings of the signal.

6.2.6. General Case

The most general case one should consider that in which, besides rapid interferences in the signal, there exist random variables with a rate of change, comparible with Paragraph 6.2.5, so that

$$z_0(t) = a(\varepsilon, \nu(t)) + \sqrt{S(\varepsilon, \nu(t))}\,\zeta(t). \qquad (6.2.53)$$

However, $\nu(t)$ varies in characteristics of relationship (6.2.53) too fast, so that it is possible not to take this into account during determination of pulse response of the closed loop and of measurement error. Instead of (6.2.14), in this case we have equation

$$\varepsilon(t) + \overline{K}_{\text{д}} \int_{t_0}^{t} h(t,\tau)(1+\xi(\tau))\,\varepsilon(\tau)\,d\tau = \lambda(t) - \lambda_{\text{н}}(t) - \\ - \sqrt{\overline{S}_0}\int_{t_0}^{t} h(t,\tau)\sqrt{s(\tau)}\,\zeta(\tau)\,d\tau,$$

where $\overline{K}_{\text{д}}$, $\overline{S}_0$ — the amplification factor and the spectral density averaged for ν:

$$\xi(t) = \frac{K_{\text{д}}(\nu(t))}{\overline{K}_{\text{д}}} - 1,$$

$s(t) = S(\nu(t))/\overline{S}_0$ – normalized random functions, characterizing change of amplification factor and spectral density in accordance with change of $\nu(t)$; and $\zeta(t)$, as before, is white noise of unit spectral density.

If we look for fluctuation error in the form

$$\varepsilon_{\text{фл}}(t) = \sqrt{\overline{S}_{\text{экв}}}\int_{t_0}^{t} g(t,\tau)\zeta(\tau)\,d\tau \quad (\overline{S}_{\text{экв}} = \overline{S}_0/\overline{K}^2_{\text{д}}),$$

assuming for simplicity that $\Delta(t) = 0$, for pulse response of a closed loop, $g(t, \tau)$, randomly depending on the input signal through $\nu(t)$, we can obtain equation

$$g(t, \tau)+\bar{K}_{\text{д}} \int_{\tau}^{t} h(t, s)[1+\xi(s)] g(s, \tau) ds=\bar{K}_{\text{д}} h(t, \tau) \sqrt{s(\tau)}, \tag{6.2.54}$$

similar to (6.2.15), where $h(t, \tau)$ – pulse response of the smoothing circuits.

If we solve this equation for $g(t, \tau)$, fluctuation error will be expressed superficially by the known formula

$$\sigma_{\text{фл}}^{2}(\nu(t))=\bar{S}_{\text{экв}} \int_{t_0}^{t} g^{2}(t, \tau) d\tau, \tag{6.2.55}$$

and for a steady-state regime for stationary $s(t)$ and $\xi(t)$, after averaging with respect to ν, we have

$$\overline{\sigma_{\text{фл}}^{2}(\nu)}=\bar{S}_{\text{экв}} \int_{0}^{\infty} \overline{g^{2}(t, t-\xi)}\, d\xi. \tag{6.2.56}$$

However, in common form (6.2.54) can not be solved, and it is necessary to limit oneself to consideration of the case of a small coefficient of modulation of the gain factor of the discriminator ($\overline{\xi^2(t)} = \sigma_{\text{пар}}^2 \ll 1$) and constant smoothing circuits. Seeking our solution in the form

$$\begin{aligned} g(t, \tau)=g_{0}(t-\tau) \sqrt{s(\tau)}+g_{1}(t-\tau, \tau) \sqrt{s(\tau)}+ \\ +g_{2}(t-\tau, \tau) \sqrt{s(\tau)}+\ldots, \end{aligned} \tag{6.2.57}$$

where g_0 has the zero, g_1 the first, and g_2 the second order of smallness, for the zero approximation we obtain equation

$$g_{0}(t)+\bar{K}_{\text{д}} \int_{0}^{t} h(t-x) g_{0}(x) dx=\bar{K}_{\text{д}} h(t),$$

whence

$$g_{0}(t)=\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\bar{K}_{\text{д}} H(i\omega)}{1+\bar{K}_{\text{д}} H(i\omega)} e^{i\omega t} d\omega.$$

Further, for the first approximation we have equation

$$\begin{aligned} g_{1}(t, \tau)+\bar{K}_{\text{д}} \int_{0}^{t} h(t-x) g_{1}(x, \tau) dx= \\ =-\bar{K}_{\text{д}} \int_{0}^{t} h(t-x) \xi(x+\tau) g_{0}(x) dx, \end{aligned}$$

which gives as a result

$$g_{1}(t-\tau, \tau)=-\int_{0}^{t-\tau} g_{0}(t-\tau-x) g_{0}(x) \xi(x+\tau) dx.$$

Analogously for the second approximation we obtain

$$\begin{aligned} g_{2}(t-\tau, \tau)= \\ -\int_{0}^{t-\tau} dx \int_{0}^{x} dy\, g_{0}(t-\tau-x) g_{0}(x-y) g_{0}(y) \xi(x+y) \xi(y+\tau). \end{aligned}$$

Finally, the solution of (6.2.54) has the form

$$g(t,\tau)=$$
$$=\sqrt{s(\tau)}\left\{g_0(t-\tau)-\int_0^{t-\tau}g_0(t-\tau-x)g_0(x)\xi(x+\tau)\,dx+\right.$$
$$\left.+\int_0^{t-\tau}dx\int_0^{x}dy\,g_0(t-\tau-x)g_0(x-y)g_0(y)\xi(x+\tau)\xi(y+\tau)\right\}. \tag{6.2.58}$$

From this, by (6.2.56), limiting ourselves to terms of the second order of smallness and considering that $\overline{\xi(t)} = 0$, for variance of the mean fluctuation error we have

$$\overline{\sigma^2_{\phi\text{л}}}=\overline{S}_{\text{экв}}\int_0^{\infty}\left[g_0^2(s)+\right.$$
$$+\int_0^{s}\int_0^{s}g_0(s-x)g_0(x)g_0(s-y)g_0(y)M(x,y)\,dxdy+$$
$$\left.+2\int_0^{s}dx\int_0^{x}dy\,g_0(s)g_0(s-x)g_0(x-y)g_0(y)M(x,y)\right]ds, \tag{6.2.59}$$

where we considered that $\overline{s(t)} = 1$, and

$$M(x,y)=\overline{s(t)\xi(t+x)\xi(t+y)}. \tag{6.2.60}$$

As can be seen from (6.2.59), only the first, the basic component of variance of mean fluctuation error can be rewritten in familiar form $2\overline{S}_{\text{экв}}\overline{\Delta f}_{\text{эф}}$, where in the definition of $\overline{\Delta f}_{\text{эф}}$ there enters average amplification factor. Other components have more complicated form and are expressed through the pulse response of the zero approximation $g_0(t)$ and the estimator $M(x, y)$ of random functions $s(t)$ and $\xi(t)$. In the particular case when the modulation of spectral density and of the gain factor of the discriminator due to slow interferences do not depend on one another additional components in $\sigma^2_{\phi\text{л}}$ are expressed through spectral density $S_{\text{пар}}(\omega)$ of parametric influence $\xi(t)$, inasmuch as then $M(x, y) = \overline{\xi(x)\xi(y)}$. This leads to relationship

$$\sigma^2_{\phi\text{л}}=\overline{S}_{\text{экв}}\left\{2\overline{\Delta f_{\text{эф}}}+\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}S_{\text{пар}}(\omega_1)\times\right.$$
$$\left.\times|\overline{G}(i\omega_2)|^2\left[|\overline{G}(\omega_2-\omega_1)|^2+2\overline{G}(\omega_2)\overline{G}(\omega_2-\omega_1)\right]d\omega_1d\omega_2\right\}. \tag{6.2.61}$$

In the two extreme cases of very rapid and very slow parametric fluctuations from (6.2.61) we have

$$\lim_{S_{\text{пар}}(\omega)\to\text{const}}\overline{\sigma^2_{\phi\text{л}}}=2\overline{S}_{\text{экв}}\overline{\Delta f_{\text{эф}}}\left[1+2S_{\text{пар}}\overline{\Delta f_{\text{эф}}}\right], \tag{6.2.62}$$

$$\lim_{S_{\text{пар}}(\omega)\to\sigma^2_{\text{пар}}2\pi\delta(\omega)} \overline{\sigma^2_{\text{фл}}} = 2\bar{S}_{\text{экв}}\overline{\Delta f_{\text{эф}}}\,[1 + k\sigma^2_{\text{пар}}], \tag{6.2.63}$$

where k — numerical coefficient, given in (6.2.52). Formula (6.2.62) is already known from Paragraph 6.2.4, which is obvious upon comparison of it with (6.2.44). In its derivation we again used the fact that g(0) = 0. Formula (6.2.63) coincides with formula (6.2.51), derived earlier in Paragraph 6.2.5.

This investigation shows in what sense one should understand in every case the equivalent spectral density, how characteristics of the discriminator are introduced, and how by them we calculate errors of measurement. The given relationships are valid for tracking meters of any parameter of modulation of a radar signal and can serve, thereby, as a basis for analysis of accuracy of tracking radar meters.

§ 6.3. Questions of Breakoff of Tracking

We analyzed tracking meters for the influence on them of small noises and interferences, when errors of measurement are small, and the linear approximation introduced in the preceding section is valid for characterization of the discriminator. However, for large noises and interferences, error of measurement becomes comparable with the width of the linear section of the discrimination characteristic or even exceeds it. This, first, leads to insufficiency of linear approximation during calculation of accuracy both due to nonlinearity of the discrimination, so also due to the complicated form of the fluctuation characteristics. Secondly, here probability of breakoff of tracking sharply increases, where mismatch exceeds the width of the discrimination curve, and the useful signal, in general, ceases to act on the discriminator.

Investigation of nonlinear phenomena in tracking meters, occurring with intense interferences, is the subject of the present paragraph. It turns out that a suitable mathematical device for analysis in this case are diffusion and related differential equations, obtained during the study of Markovian random processes. Therefore, to facilitate understanding of the conducted analysis we give in Paragraph 6.3.1 certain information from the theory of Markovian processes and differential equations, describing their statistical properties.

6.3.1. Markovian Random Processes and Related Differential Equations

The device of Markovian processes has a wide range of applications and recently has been widely used in theoretical radio engineering [20], being the best means

of investigation of properties of nonl... ..lements of radio channels, fluctuations in vacuum-tube oscillators, and so forth. ... call Markovian a random process y(t), for which conditional probability density $P(y_n|y_{n-1}, \ldots, y_1)$ satisfies relationship

$$P(y_n|y_{n-1}, \ldots, y_1) \equiv P(y_n|y_{n-1}) \ (y_i = y(t_i)), \tag{6.3.1}$$

i.e., depends functionally only on the value of the process at one preceding moment of time. Conditional probability density $P(y_n|y_{n-1}) \equiv W(y_n|y_{n-1})$ we call probability density of transition from state y_{n-1} at time t_{n-1} to state y_n at time t_n. It is of basic value in the theory of Markovian processes, since by it we express n-dimensional probability densities:

$$P_n(y_0, \ldots, y_{n-1}) = P_1(y_0) \prod_{i=1}^{n-1} W(y_i|y_{i-1}), \tag{6.3.2}$$

where $P_1(x)$ — one-dimensional distribution.

In the general theory it is proved [20] that for sufficiently broad conditions function $W(y|y_0)$ obeys two partial differential equations (Fokker-Planck-Kolmogorov equations). If differentiation is performed with respect to finite values of y, t, the equation is called direct and is recorded in the form

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial y}\{A(y, t) W\} + \frac{1}{2}\frac{\partial^2}{\partial y^2}\{B(y, t) W\}, \tag{6.3.3}$$

if however, W is differentiated with respect to initial values of y_0, t_0, the equation is called reverse and has the form

$$\frac{\partial W}{\partial t_0} = A(y_0, t_0)\frac{\partial W}{\partial y} + \frac{B(y_0, t_0)}{2}\frac{\partial^2 W}{\partial y^2}. \tag{6.3.4}$$

Functions A(y, t) and B(y, t) we call coefficients of drift and diffusion, respectively.

The method of finding these coefficients follows from another theoretical proposition of [20]. Let us assume that function y(t) obeys ordinary differential equation

$$\frac{dy}{dt} = f(y, t) + g(y, t)\xi(t), \tag{6.3.5}$$

where f(y, t), g(y, t) — functions of a broad class;

$\xi(t)$ — fluctuating influence with a wide spectrum, which it is possible to approximate by white noise with unit spectral density [the correlation function is equal to $\overline{\xi(t)\xi(t+\tau)} = \delta(\tau)$].

Then y(t) is a Markovian process, the probability density of transition of which obeys equations (6.3.3) and (6.3.4) with coefficients

$$A(y, t)=f(y, t), \quad B(y, t)=g^2(y, t). \tag{6.3.6}$$

Deeper mathematical consideration shows the presence in the coefficient of drift of an additional component $\frac{1}{2} g(y, t) \frac{\partial g(y,t)}{\partial y}$ [20]. However, proceeding from the physical meaning of the processes occurring in meters, which we will be interested in below, this component should not be considered. The explanation of this was the circumstance, already noted in § 6.2, that the discriminator has inertia or delay, not smaller than the interval of correlation of that fluctuating voltage at its output, which is approximated by white noise. As a result there does not occur an instantaneous return in the feedback circuit of fluctuating disturbances, which could correlate with fluctuations in the discriminator and thereby change the coefficient of drift.

It remains to explain how to pass to a differential equation of type (6.3.5) from the equation obtained during the analysis of certain meters, of form

$$\frac{dy}{dt}=\zeta(y, t), \tag{6.3.7}$$

where the time of correlation of the random component in $\zeta(y, t)$ is small as compared to the effective time constant of the system. As shown [20], for this one should take as f(y, t) and g(y, t) functions

$$\begin{gathered} f(y, t)=\overline{\zeta(y, t)}, \\ g(y, t)= \\ =\left\{\int_{-\infty}^{+\infty} \overline{[\zeta(y, t)-\overline{\zeta(y, t)}][\zeta(y, t+\tau)-\overline{\zeta(y, t+\tau)}]}\, d\tau\right\}^{1/2}. \end{gathered} \tag{6.3.8}$$

Equation (6.3.5) corresponds to transmission of white noise through an inertial link of the first order, nonlinear and with variable gain. Subsequently we will prove that namely (6.3.5) and related equations can be used for investigation of nonlinear phenomena in tracking meters of very simple form.

If the dynamic system, under the influence of fluctuations, has a more complicated form, it is not possible to describe it by the Markovian process of the 1st order, which was considered above. Here it is convenient to use the idea of a Markovian process of the n-th order. Let us introduce vector random process $\{y_1(t), \ldots, y_n(t)\} = \mathbf{y}(t)$, components of which are scalar random processes $y_i(t)$. Let us assume that $\mathbf{y}(t)$ obeys system of equations

$$\frac{d\mathbf{y}(t)}{dt}=\boldsymbol{\zeta}(\mathbf{y},\ t),$$

where $\boldsymbol{\zeta}(\mathbf{y}, t)$ – vector function.

Then, if times of correlation of random components in $\boldsymbol{\zeta}(\mathbf{y}, t)$ are small as compared to the time scale of the system as a whole, the probability density of transition $W(\mathbf{y}, t|\mathbf{y}_0, t_0)$ of process $\mathbf{y}(t)$ from value $\mathbf{y}_0$ at time t_0 to $\mathbf{y}$ at time t obeys equation

$$\frac{\partial W}{\partial t}=-\sum_{i=1}^{m}\frac{\partial}{\partial y_i}\{A_i(\mathbf{y},\ t)W\}+ +\frac{1}{2}\sum_{i,\ j=1}^{m}\frac{\partial^2}{\partial y_i\partial y_j}\{B_{ij}(\mathbf{y},\ t)W\}, \tag{6.3.9}$$

where

$$A_i(\mathbf{y},\ t)=\overline{\zeta_i(\mathbf{y},\ t)};$$
$$B_{ij}(\mathbf{y},\ t)= =\int_{-\infty}^{+\infty}\overline{[\zeta_i(\mathbf{y},\ t)-\overline{\zeta_i(\mathbf{y},\ t)}][\zeta_j(\mathbf{y},\ t+\tau)-\overline{\zeta_j(\mathbf{y},\ t+\tau)}]}\,d\tau.$$

As components $y_i(t)$ we can with equal success take different interconnected random processes, a random process and its derivatives to the (n-1)-th order inclusively, or the value of one and the same random process y(t) at different moments of time

$$y_1(t)=y(t+\tau_1),\quad y_2(t)=y(t+\tau_2),\ldots,y_n(t)=y(t+\tau_n),$$

where $\tau_1, \ldots, \tau_n$ – constants.

If we determine $y_i(t)$ by the last method, the probability density of transition

$$W(\mathbf{y},\ t\,|\,\mathbf{y}_0,\ t_0)= =W(y_1,\ t+\tau_1,\ldots,y_n,\ t+\tau_n\,|\,y_{01},\ t_0+\tau_1,\ldots,y_{0n},\ t_0+\tau_n)$$

can be found by solution of equation (6.3.9). This means that the conditional probability density of values of the process at a certain n moments of time (and, consequently, of any one value of y(t) at one moment t) depends functionally on values of the process in n preceding moments of time. Such process is called an n-th order Markovian. Nonlinear phenomena in tracking meters, described by n-th order equations, lead to the necessity of investigating these processes and, correspondingly, to solution of equations of form (6.3.9). Considering, however, that such equations for $n > 1$ usually cannot be solved, let us turn to a more

detailed study of one-dimensional equations.

Solution of (6.3.3) reveals the evolution in time of the probability density of transition, which for an initial condition in the form of a δ-function simply repeats one-dimensional probability density.

In certain conditions [20] (primarily for independence of A(y) and B(y) from time) there is finally established stationary distribution, which it is possible to express through coefficients A(y) and B(y), equating ∂W/∂t in (6.3.3) to zero:

$$W_{\text{стац}}(y) = \frac{C}{B(y)} \exp\left\{2 \int \frac{A(y)\,dy}{B(y)}\right\}, \tag{6.3.10}$$

where C — normalizing constant, and integral is taken as indefinite.

No less frequent, however, are cases when a stationary distribution does not exist or we are interested in the actual process of becoming stationary. Then, it is necessary to solve the complete equation (6.3.3), which is generally a very complicated mathematical problem. Here, it is usually necessary to be given so-called boundary conditions.

If, after the first time y(t) exceeds a certain magnitude y_1 further realization of y(t) does not interest us, by analogy with the process of diffusion, we talk about the presence at point y_1 of an "absorbing screen." With the help of mathematical condition

$$W(y_1, t) = 0 \tag{6.3.11}$$

realizations of y(t), even once touching the screen before moment t, are automatically ejected from consideration after this moment of time.

If, after arriving at point y_1, any realization of y(t) immediately turns back, we talk of the presence of a "reflecting screen." The mathematical condition of reflection is equality

$$\frac{\partial W(y_1, t)}{\partial y} = 0. \tag{6.3.12}$$

Solving equation (6.3.3) under these conditions, we can find how W(y, t) changes in time. However, we are not always interested in the complete structure of probability density. Sometimes it is convenient to limit ourselves to cruder characteristics, as which we can consider in the first place variance of fluctuation error, during calculation of which, in distinction from the case of § 6.2, we must allow for nonlinear factors. In principle it is not difficult, for instance, to determine the variance of stationary distribution (6.3.10), if it exists.

We may be interested also in probability characteristics of another type, for instance average time of stay of a realization of $y(t)$ in a given region on the condition of a certain concrete initial value $y_0 = y(t_0)$. For average time of stay of $T(y_0)$ as a function of the initial condition from equation (6.3.4) we can obtain differential equation

$$\frac{B(y_0)}{2}\frac{d^2T}{dy_0^2}+A(y_0)\frac{dT}{dy_0}+1=0, \tag{6.3.13}$$

for which on boundaries of the region we have conditions

$$T(y_1)=0, \quad T(y_2)=0. \tag{6.3.14}$$

The physical meaning of these conditions is that a realization located next to the "absorbing" screen due to the purely random character of disturbing factors will immediately intersect it, and this is equivalent to a zero average time of stay.

6.3.2. Formulation of the Problem About Breakoff of Tracking

Above we already indicated that interference of high intensity makes linear approximation for the discrimination characteristic invalid. Mismatch frequently goes into the nonlinear region and can attain such magnitudes, at which the useful signal no longer affects the discriminator. Here, there occurs breakoff of tracking. The shown nonlinearity of the discriminator characteristic is not some imperfection, inherent in certain concrete circuits. Further, we shall show that characteristics of optimum discriminators have similar form. This circumstance increases the importance of investigations of nonlinear regimes of meters and, in particular, questions of breakoff of tracking.

With a nonlinear characteristic of the discriminator the process representing the result of measurement in a number of conditions has variance growing in time. The fact is that the probability of departure from the linear section is always different from zero, so that during prolonged measurement short duration failure certainly sets in. Here, the meter practically becomes an open storage unit of noises, and mismatch $\varepsilon(t)$ experiences random changes, in no way connected with the parameter of the useful signal. The circumstance that in a certain time interval realization of $\varepsilon(t)$ can be similar to realization of the process in a linear system does not change the essence of the matter, since we are interested in the whole set of realizations, including those, in which nonlinear properties of the system have

already appeared. Since with passage of time the probability of failure increases, it is clear that variance of process $\varepsilon(t)$ increases. It is however, not the only possible crude characteristic. It is possible to apply other forms of probability characteristics, where their reasonable selection depends on the type and method of use of the meter.

Before considering such characteristics in reference to the phenomenon of breakoff of tracking, we will make the following remarks. From what has been presented it follows that with observance of certain conditions nonlinear phenomena in meters, including the phenomenon of breakoff of tracking, can be studied by solution of diffusion equations (and equations connected with them). As a result there are found various estimators of failure. Use of diffusion equations is based in this case on the practical inertialess nature of the discriminator as compared to the smoothing circuits, since only upon observance of this condition and during application of smoothing circuits of the first order for mismatch is there obtained an equation of form (6.3.5), where $\xi(t)$ is white noise.

In the case of a low level of noises or interferences the formulated condition is not satisfied sufficiently well in all cases. However, with growth of interference, not depending on the measured parameter (internal noises, interferences, etc.), the scale of the discrimination characteristics $\alpha(\varepsilon)$ greatly decreases, and the transmission factor of the discriminator $K_{\text{Д}} = \frac{d\alpha(0)}{d\varepsilon}$ drops. Here, the inertia of the closed loop grows. Such a dependence on examples will be explained in subsequent chapters. From the point of view of questions of failure most interesting, namely, is a high level of interferences. Thus, conditions of applicability of diffusion equations are obviously satisfied where, in view of nonlinearity of the problem, it is most expedient to use this mathematical device.

Let us turn to estimators, which are best found from solution of diffusion equations for problems of failure. Let us assume, first of all that the meter is the only means of signal selection and tracking, there are no automatic means to shift to search upon disruption of tracking, and the final effect of work of the radar depends on the presence of tracking at every given moment of time. Then, if there are no limitations on output values of the measured variable, naturally as the characteristic of failure we take the probability density of mismatch, which, as a function of time, should be found from the solution of the diffusion equation without boundary conditions.

More frequently, however, in smoothing and control circuits there exist limiters of various types. Here, the phenomenon of breakoff of tracking leads to the following. In the process of tracking at a certain moment of time under the influence of some large fluctuating overshoot mismatch becomes larger than the width of the discrimination curve and tracking is disrupted. Then, for a certain random interval of time mismatch randomly varies under the influence of interference alone. Here, however, it remains finite due to the limitation, and therefore, under the influence of fluctuating overshoots of reverse sign it sooner or later decreases to a magnitude in which tracking again is renewed. This process is repeated again, and upon the expiration of a certain time establishments of it can be considered stationary. Therefore, if moments of measurement of the tracked variable are sufficiently removed from the beginning of tracking (observation time is great), and if we are interested in accuracy of measurement with breakoff of tracking, then as the characteristic we should take variance of measurement as $t \to \to \infty$. It can be found proceeding from stationary distribution (6.3.10) with idealization of the phenomenon of limitation in the form of a "reflecting screen" [boundary conditions are taken in form (6.3.12)]. As shown below, with growth of intensity of interference this stationary variance grows, and there occurs a threshold effect, corresponding to sharp increase of errors due to short duration failures when interference exceeds a certain critical level.

If the time of observation of the output quantity of the meter is small, the described approach to investigation of breakoff phenomena is unacceptable. Here characteristics of the first failure after beginning of tracking take on special interest, especially if we consider that short duration failure of a meter of one parameter of a signal usually also leads to vanishing of the signal in meters of its other parameters. In order to characterize the first short duration failure, we take the following definition. By short duration failure at time t we understand exceeding by the magnitude of mismatch $\varepsilon(t) = \lambda(t) - \hat{\lambda}(t)$ between the true and measured values of the parameter of certain fixed levels $(-\Delta, +\Delta)$ under the condition that at the initial moment mismatch was sufficiently small [within limits $(-\Lambda, +\Delta)$] and up to moment t was always in this interval. With smoothing circuits of the first order this leads to a boundary value problem for $W(t, \varepsilon)$ in the form (6.3.3), where on the boundaries we are given conditions of absorbing screens.

Screens are considered located on boundaries of the discriminator range, immediately behind "slopes" of the discrimination characteristic (Fig. 6.5). These boundaries are determined by the bandwidth of filters in the frequency discriminators, duration of gate pulses in time discriminators, the width of the radiation pattern in angular discriminators, etc. It is necessary to note that in selection of coordinates of screens there exists a certain arbitrariness. No matter how we choose these coordinates, realizations of mismatch are probable, which after intersecting the screen almost immediately intersect it in the opposite direction, and there will be essentially no failure. It is clear that such phenomena are not considered during application of the above definition of failure. Therefore, the described approach to analysis of failure phenomena leads only to approximate solutions, allowing us, however, to find certain laws governing performance.

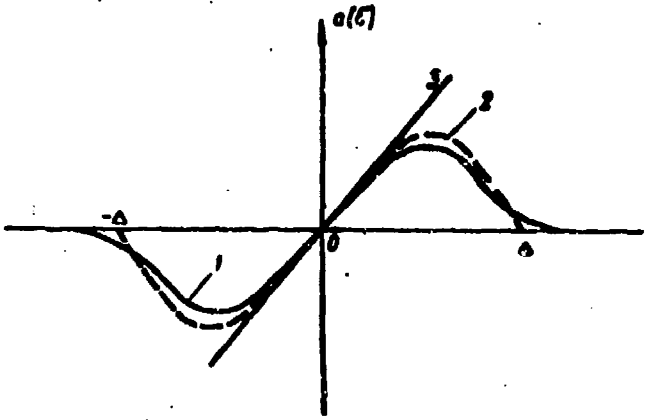

Fig. 6.5. Discrimination characteristic and its approximations: 1) true relationship; 2) sinusoidal approximation; 3) linear approximation.

By direct solution of diffusion equation (6.3.3) with boundary conditions (6.3.11) we can find the probability density of mismatch by which it is easy to calculate the probability of absence of failure up to moment t. It is more expedient, however, to use a simplified characteristic of failure — average time to failure T_{c6}. This parameter is the solution of equation (6.3.13), ensuing from the diffusion equation. Average time to failure decreases with growth of intensity of interference. Just as for variance of stationary distribution, here there exists a threshold effect, expressed in the fact that with decrease of intensity of interference and related variance of error of a linearized system for certain values of these magnitudes there starts a sharp growth of average time to failure. This gives us the possibility of finding the cutoff value of intensities of interferences, at which there occurs short duration failure (in the above-indicated sense).

Direct solutions of diffusion equations for the problems interesting us in conditions of a high level of interferences are given in [21, 22, 23, 24].

Considering, however, the expediency of application for analysis of failure of criteria of stationary variance and average time to failure, we shall give a solution of two other problems, noted above. Here, we shall consider general laws; applications of the obtained solutions will be given in subsequent chapters devoted to investigation of meters of concrete parameters of radar signals.

6.3.3. Stationary Distribution with Limitations

Let us consider a tracking meter of the type in Fig. 6.1. Output voltage of the discriminator in the case of the presence of rapid fluctuations alone, according to § 6.2, can be recorded in the form

$$z(t, \varepsilon) = a(\varepsilon) + \sqrt{S(\varepsilon)}\,\xi(t), \qquad (6.3.15)$$

where $a(\varepsilon)$, $S(\varepsilon)$ — discrimination and fluctuation characteristics, respectively; $\xi(t)$ — white noise of unit spectral density.

Analysis of phenomena interesting us can ultimately be reduced to the case of smoothing circuits in the form of a single integrator, where

$$\frac{d}{dt}\hat{\lambda}(t) = K_{\text{и}} z(t). \qquad (6.3.16)$$

Considering that $\lambda - \hat{\lambda} = \varepsilon$, from relationships (6.3.15) and (6.3.16) we find the differential equation, describing the meter as a whole:

$$\frac{d\varepsilon}{dt} = \dot{\lambda} - K_{\text{и}} a(\varepsilon) - K_{\text{и}}\sqrt{S(\varepsilon)}\,\xi(t), \qquad (6.3.17)$$

where $\dot{\lambda} = d\lambda/dt$ — rate of change of the measured variable.

We assume that $\dot{\lambda}$ = const and introduce equivalent discrimination $a_{\text{экв}}(\varepsilon)$ and fluctuation $S_{\text{экв}}(\varepsilon)$ characteristics according to equalities

$$a_{\text{экв}}(\varepsilon) = \frac{a(\varepsilon)}{K_{\text{д}}}, \quad S_{\text{экв}}(\varepsilon) = \frac{S(\varepsilon)}{K_{\text{д}}^2}. \qquad (6.3.18)$$

Then (6.3.17) is rewritten in the form

$$\frac{d\varepsilon}{dt} = K_{\text{и}}K_{\text{д}}\left[\varepsilon_V - a_{\text{экв}}(\varepsilon)\right] - K_{\text{и}}K_{\text{д}}\sqrt{S_{\text{экв}}(\varepsilon)}\,\xi(t), \qquad (6.3.19)$$

where $\varepsilon_V = \dot{\lambda}/K_{\text{и}}K_{\text{д}}$ — magnitude of dynamic error in a linearized system (see § 6.2).

According to results of Paragraph 6.3.1 the corresponding diffusion equation will be recorded in the form

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial\varepsilon}\left[K_{\text{и}}K_{\text{д}}\left(\varepsilon_V - a_{\text{экв}}(\varepsilon)\right)W\right] + \frac{(K_{\text{и}}K_{\text{д}})^2}{2}\frac{\partial^2}{\partial\varepsilon^2}\left[S_{\text{экв}}(\varepsilon)W\right]. \qquad (6.3.20)$$

Consequently, the coefficient of drift in tracking meters is determined by the discrimination, and the coefficient of diffusion, by the fluctuation characteristics.

The stationary solution, if it exists, is determined by relationship (6.3.1.). In the presence of limitations at points l_1, l_2 it should be normalized in this range, so that

$$\left.\begin{aligned}
W_{\text{стац}}(\varepsilon) &= \frac{C_1}{S_{\text{экв}}(\varepsilon)} \exp\left[-\frac{2}{K_{\text{п}}K_{\text{д}}}\int \frac{a_{\text{экв}}(\varepsilon)-\varepsilon_V}{S_{\text{экв}}(\varepsilon)}\,d\varepsilon\right] \\
&\qquad (l_1<\varepsilon<l_2), \\
\overline{\varepsilon^2} &= C_1\int_{l_1}^{l_2} \frac{\varepsilon^2}{S_{\text{экв}}(\varepsilon)} \exp\left[-\frac{2}{K_{\text{п}}K_{\text{д}}}\int \frac{a_{\text{экв}}(\varepsilon)-\varepsilon_V}{S_{\text{экв}}(\varepsilon)}\,d\varepsilon\right] d\varepsilon, \\
C_1^{-1} &= \int_{l_1}^{l_2} \frac{1}{S_{\text{экв}}(\varepsilon)} \exp\left[-\frac{2}{K_{\text{п}}K_{\text{д}}}\int \frac{a_{\text{экв}}(\varepsilon)-\varepsilon_V}{S_{\text{экв}}(\varepsilon)}\,d\varepsilon\right] d\varepsilon.
\end{aligned}\right\} \quad (6.3.21)$$

Being limited to the case of a constant fluctuation characteristic ($S_{\text{экв}}(\varepsilon) = S_{\text{экв}}$), which is a good approximation during large noises, instead of (6.3.21), we have

$$\left.\begin{aligned}
W_{\text{стац}}(\varepsilon) &= C_2 \exp\left\{-\frac{1}{\sigma_{\text{д}}^2}\int [a_{\text{экв}}(\varepsilon)-\varepsilon_V]\,d\varepsilon\right\}, \\
\overline{\varepsilon^2} &= C_2\int_{l_1}^{l_2} \varepsilon^2 \exp\left\{-\frac{1}{\sigma_{\text{д}}^2}\int [a_{\text{экв}}(\varepsilon)-\varepsilon_V]\,d\varepsilon\right\} d\varepsilon, \\
C_2^{-1} &= \int_{l_1}^{l_2} \exp\left\{-\frac{1}{\sigma_{\text{д}}^2}\int [a_{\text{экв}}(\varepsilon)-\varepsilon_V]\,d\varepsilon\right\} d\varepsilon,
\end{aligned}\right\} \quad (6.3.22)$$

where

$$\sigma_{\text{д}}^2 = 2S_{\text{экв}}\Delta f_{\text{эф}} = S_{\text{экв}}\frac{K_{\text{п}}K_{\text{д}}}{2}$$

– variance of fluctuating error of measurement in a linearized system.

Further concretization of the solution requires assignment of an approximation for $a_{\text{экв}}(\varepsilon)$. Let us assume that

$$a_{\text{экв}}(\varepsilon) = \frac{\Delta}{\pi}\sin\frac{\pi\varepsilon}{\Delta}, \quad l_1=-l, \quad l_2=l, \quad \varepsilon_V=0,$$

where as before Δ – half-width of the selected range.

Then*

*These results were obtained by I. P. Zel'nikov.

$$W_{\text{стац}}(\varepsilon)=\begin{cases}\dfrac{1}{2\Delta}\dfrac{\exp\left[u\left(1+\cos\pi\frac{\varepsilon}{\Delta}\right)\right]}{n-1+e^{u}I_0(u)} & \text{when } |\varepsilon|<\Delta;\\ \dfrac{1}{2\Delta}\dfrac{1}{n-1+e^{u}I_0(u)} & \text{when } \Delta<|\varepsilon|<l,\\ 0 & \text{when } |\varepsilon|>l,\end{cases} \tag{6.3.23}$$

where $n=\frac{l}{\Delta}$; $u=\frac{1}{\pi^2}\left(\frac{\Delta}{\sigma_{л}}\right)^2\approx 0{,}1\left(\frac{\Delta}{\sigma_{л}}\right)^2$ – dimensionless quantity, proportional to the square of the ratio of the width of the selected range to mean square error in a linearized system.

For variance $\sigma^2_{НЛ}$ of error, considering the phenomenon of breakoff and restoration, we obtain from (6.3.23) the general expression

$$\sigma^2_{нл}=\frac{\Delta^2}{n-1+e^{u}I_0(u)}\left\{\frac{1}{3}(n^3-1)+\right.$$
$$\left.+e^{u}\left[\frac{1}{3}I_0(u)+\frac{4}{\pi^2}\sum_{k=1}^{\infty}(-1)^k\frac{I_k(u)}{k^2}\right]\right\}. \tag{6.3.24}$$

In particular, for very large noises, when $u < 1$ and the probability of breakoff is great,

$$\left(\frac{\sigma_{нл}}{\Delta}\right)^2\approx\frac{1}{3}\,\frac{n^3-1+e^{u}I_0(u)\left[1-\frac{12}{\pi^2}\frac{I_1(u)}{I_0(u)}\right]}{n-1+e^{u}I_0(u)}. \tag{6.3.25}$$

According to (6.3.25) when $u \ll 1$, $\sigma^2_{нл}\approx\frac{l^2}{3}$, i.e., variance corresponds to uniform distribution of ε in segment $(-l, l)$.

In the reverse case, when $u > 1$ (probability of breakoff is small),

$$\left(\frac{\sigma_{нл}}{\Delta}\right)^2\approx\frac{\frac{1}{3}(n^3-1)\sqrt{2\pi u}\,e^{-2u}+\frac{1}{u}\frac{1}{\pi^2}}{1+\sqrt{2\pi u}\,(n-1)\,e^{-2u}}. \tag{6.3.26}$$

When $u \gg 1$, from (6.3.26) it follows that

$$\sigma^2_{нл}\approx\sigma^2_{л},$$

i.e., breakoff practically does not occur, and the linear approximation in analysis is fully sufficient. Ratio $\sigma_{Л}/\Delta$, at which there occurs increase of the probability of breakoff, can be easily determined by the graph of Fig. 6.6, where there is plotted ratio $\sigma_{НЛ}/\Delta$ as a function of $\sigma_{Л}/\Delta$. The phenomenon of breakoff occurs approximately at $\sigma_{Л}/\Delta \approx 0.08$ to 0.12. Here $\sigma_{НЛ}/\Delta$ starts to increase sharply and is rapidly stabilized around level, determined by the limitation. Inasmuch as

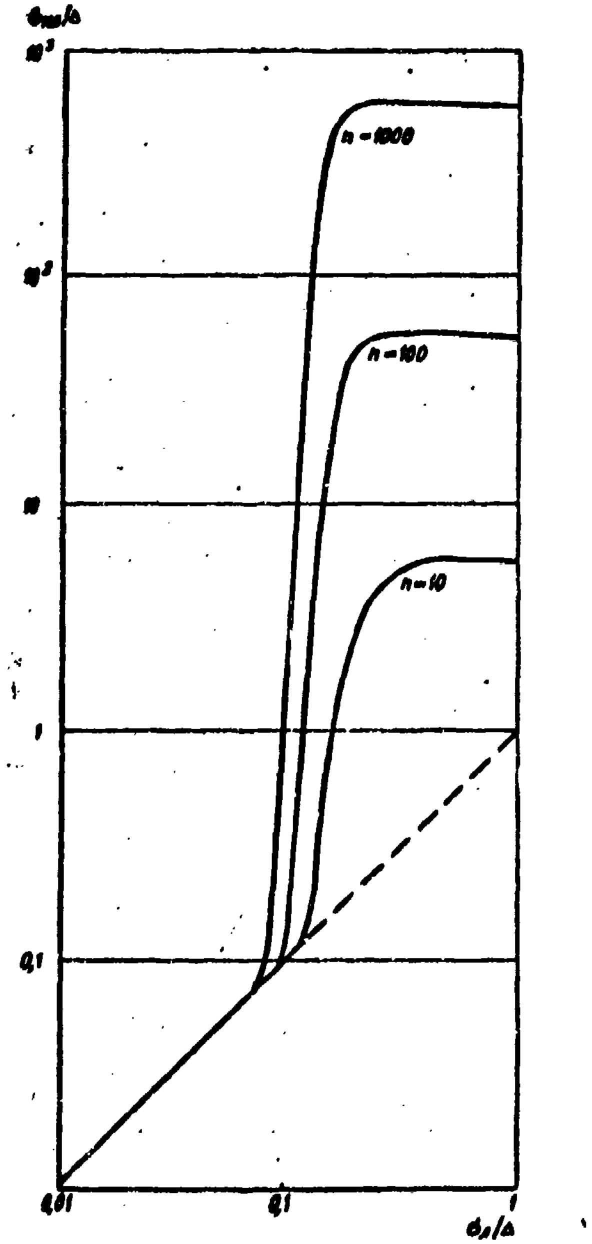

Fig. 6.6. Dependence of variance of stationary distribution on variance in linear approximation.

ratio $\sigma_Л/\Delta$, at which the probability of breakoff sharply increases, depends little on n, it is possible to state that breakoff is a threshold phenomenon, starting when $\sigma_Л/\Delta$ exceeds a certain cutoff value, when fluctuation error comprises about 0.1 the half-width of the selected range.

Since with such a stationary distribution and ratios $\sigma_Л/\Delta$, somewhat below cutoff, there is implied a very large observation time, estimation of this cutoff value by the above method will be somewhat understated, which we shall prove in the following paragraph. A factor, acting in the opposite direction, is the presence of dynamic error, which we ignored in the considered example. Increased probability of breakoff here is explained by reduction of the distance to the edge of the discrimination characteristic, where restoring action of the discriminator ceases.

6.3.4. Average Time to Breakoff

Let us consider now the nonstationary phenomenon of breakoff, defined as emergence of the magnitude of mismatch beyond the limits of a certain segment $(-\Delta, \Delta)$, where Δ – half-width of the range of selection (see Fig. 6.5). According to Paragraphs 6.3.1 and 6.3.3 average time to breakoff is determined by equation

$$\frac{(K_n K_Д)^2}{2} S_{экв}(\varepsilon)\frac{d^2T_{ср}}{d\varepsilon^2} - K_n K_Д\left[a_{экв}(\varepsilon) - \varepsilon_v\right]\frac{dT_{ср}}{d\varepsilon} + 1 = 0,$$

solving which, again on the assumption of zero dynamic error and constant fluctuation characteristic, for $\varepsilon = 0$ we have

$$T_{c6}(0)=\frac{1}{4\Delta f_{э\phi}\sigma^2_{Л}}\int\limits_0^{\Delta} dy \int\limits_0^{y} \exp\left[-\frac{1}{\sigma^2_{Л}}\int\limits_{y}^{z} a_{вых}(\xi)\, d\xi\right] dz. \qquad (6.3.27)$$

To obtain a solution for an arbitrary magnitude $\sigma^2_{Л}$ is difficult here. Therefore, we primarily investigate the range of intense noises, when it is possible to expand the exponential function under the integral sign in powers of the exponent, being limited to a small number of terms. Again given a sinusoidal approximation of the discrimination characteristic, we have asymptotic series

$$T_{c6}(0)=\frac{1}{8\Delta f_{э\phi}}\left(\frac{\Delta}{\sigma_{Л}}\right)^2\left[1+0{,}08\left(\frac{\Delta}{\sigma_{Л}}\right)^2+16\cdot 10^{-4}\left(\frac{\Delta}{\sigma_{Л}}\right)^4+ +o\left(\left(\frac{\Delta}{\sigma_{Л}}\right)^6\right)\right], \qquad (6.3.28)$$

convenient, however, only for comparatively large values of $\sigma_{Л}/\Delta$.

Graphic presentation of $\Delta f_{э\phi} T_{c6}$ as a function of $\sigma_{Л}/\Delta$ in logarithmic and linear scales is given in Fig. 6.7. The linear scale helps us to understand the threshold effect of failure, in this case expressed in the fact that average time to failure sharply decreases (from a quantity of the order of 1000 to a quantity of order of units), when $\sigma_{Л}/\Delta > 0.1$. The logarithmic scale permits us to definitize the concrete value of T_{c6}. However, for small $\sigma_{Л}/\Delta$ formula (6.3.28) gives a somewhat lowered value of T_{c6}. More exact calculation, conducted by B. L. Karelov, showed that the threshold value of $\sigma_{Л}/\Delta$ is approximately 0.2. In other words, threshold values, determined in terms of average time to breakoff and in terms of stationary variance, differ (approximately by a factor of 2). This is understandable, since with stationary variance there is implied a very large observation time, during which breakoff certainly occurs. Average time to breakoff is determined by realizations of mismatch, for which breakoff occurs for the first time.

Due to the approximate nature of analysis such a difference between results of two calculations should be considered immaterial, and, therefore, it is possible to use either of them during practical analysis. If smoothing circuits have a more complicated form, then, in general, critical ratio $\sigma_{Л}/\Delta$ changes, but the obtained results can serve for approximate calculations even in these cases.

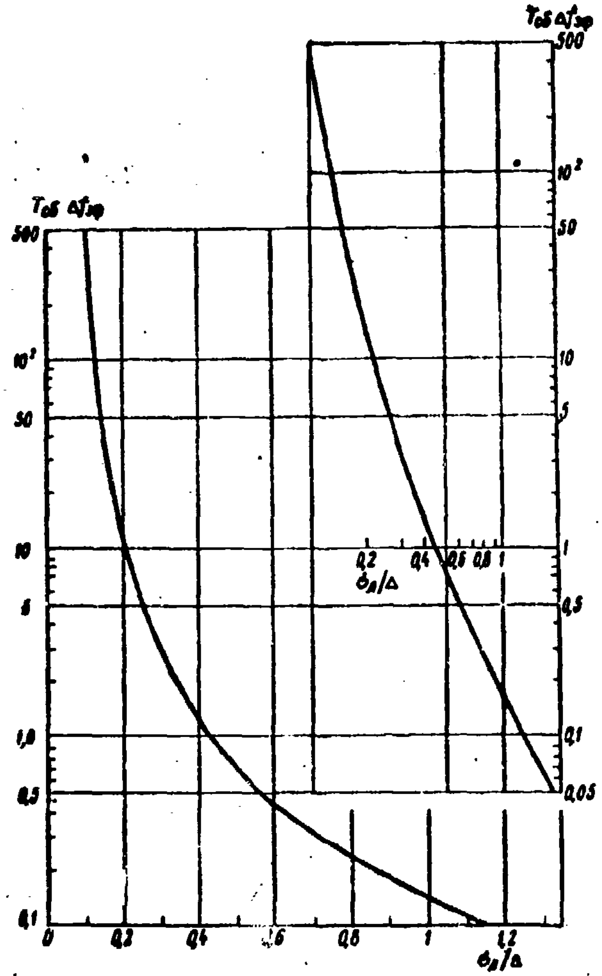

Fig. 6.7. Dependence of average time to failure on variance in linear approximation.

From analysis of tracking meters of concrete forms on the assumption of their linearity we can find $\sigma_Л$ for different forms of interferences. The value of the critical magnitude of ratio $\sigma_Л/\Delta$ gives, thereby, a possibility of finding the intensity of interferences, leading to failure of these meters.

§ 6.4. General Analysis of Nontracking Meters

Besides the meters considered above, built on the principle of a servo system, in practice they also use nontracking measuring circuits. An example is a range finder with frequency modulation, in which the received signal is mixed directly with the radiated signal, and the measure of range is the number of zeroes of the resulting voltage [1]. Then there are Doppler speed meters built on the same principle of counting zeroes [2], angular data meters using two (or several) motionless radiation patterns of antennas for direct reading of the angle, multifilter systems and systems of gated amplifiers for measurement of range and speed, and so forth. For all such circuits it is possible to establish certain general properties of the method of measurement and to derive rules for determining accuracy.

6.4.1. Basic Features of Circuit Construction and Components of Measurement Errors

In general the considered meters, as also tracking meters, are divided into two parts (Fig. 6.8) — an inertialess unit for producing a signal, on the average proportional to the measured quantity (conditionally we call it the estimator unit), and open smoothing circuits. In practice it is rarely possible to immediately obtain voltage which is on the average proportional to the measured parameter λ(t). Thus, determination of angle by signals from two antennas with fixed patterns gives a nonlinear, and in a large interval even a monotonic, dependence of the mean value of output voltage of the coordinate; the average of zeroes in an FM range finder with time lags, comparable to the period of frequency modulation, is not proportional to the range, etc. Therefore, processing of signals in the discriminator can often be divided into two stages.

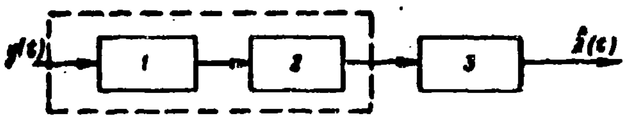

Fig. 6.8. General schematic of a nontracking meter: 1 — the main part of the estimator unit; 2 — inertialess converter; 3 — smoothing circuits.

In the first stage (unit 1 in Fig. 6.8) there will be formed a low-frequency voltage with a mean value, which is related to λ by a monotonic (not necessarily linear) dependence in necessary limits of values of the parameter:

$$z_1(t, \lambda) = a(\lambda, t) + \xi(t, \lambda), \tag{6.4.1}$$

where $a(\lambda, t)$ — mean value, for which $\frac{\partial a(\lambda, t)}{\partial \lambda} > 0$;

$\xi(t)$ — fluctuating disturbance.

The second stage (unit 2 in Fig. 6.8) may consist of a certain inertialess nonlinear operation,

$$z_2(t, \lambda) = B\{z_1(t, \lambda)\}, \qquad (6.4.2)$$

which has as its purpose obtaining of a magnitude, on the average as close as possible to λ. Smoothing circuits 3 in nontracking meters are usually linear and play, in principle, the same role as in tracking meters, but are constructed by an open-loop circuit so that their inertia directly determines the interval of smoothing.

In nontracking meters there are observed in general the same components of errors of measurement as in tracking meters: fluctuating, dynamic, systematic.

In view of the absence of a closed loop systematic errors in the considered circuits, appearing both due to insufficiently correct selection of nonlinear operator B { } in (6.4.2), and also due to incorrect selection of gain in any element of the meter, are not compensated during work of the meter. Appearance of such errors are an essential practical deficiency of the considered circuits. The dependence of these errors on selection of operator B { } will be studied below.

For nontracking meters it would have been possible to investigate the same set of questions appearing during investigation of accuracy as in tracking circuits. We will discuss below only the case when all fluctuating disturbances at the input are rapid as compared to the time of smoothing the meter.

6.4.2. Errors of Measurement During Rapid Fluctuations

In the shown conditions for $z_1(t, \lambda)$ notation, analogous to (6.2.6), is valid:

$$z_1(t, \lambda) = a(\lambda, t) + \sqrt{S(\lambda, t)}\zeta(t), \qquad (6.4.3)$$

where $a(\lambda, t)$ — mean value;

$S(\lambda, t)$ — spectral density at low frequencies;

$\zeta(t)$ — white noise of unit spectral density.

Functions $a(\lambda, t)$ and $S(\lambda, t)$ are obtained by averaging the whole ensemble of input fluctuations for a fixed value of λ. The obvious time-dependence of these functions is determined by regular changes of properties of the ensemble of fluctuations. Subsequently we shall consider $a(\lambda, t)$ and $S(\lambda, t)$ not time-dependent. Then we can write relationship (6.4.2), taking into account (6.4.3), in the form

$$z_2(t, \lambda) = B\{B^{-1}(k\lambda) + [a(\lambda) - B^{-1}(k\lambda)] + \sqrt{S(\lambda)}\,\zeta(t)\}, \tag{6.4.4}$$

where $B^{-1}(\ )$ — nonlinear inertialess operator, the inverse of $B\ (\ )$;

k — proportionality factor.

Such coefficients are frequently encountered in the theory and practice of meters, inasmuch as measured quantities have the physical nature of distance, speed, angles, and the carriers of these quantities in circuits are voltages. After expansion of function $B\ \{\ \}$ in (6.4.4) at point $B^{-1}(k\lambda)$ with preservation of only two members of the expansion, we have

$$\begin{aligned} z_2(t, \lambda) = B\{B^{-1}(k\lambda)\} + B'\{B^{-1}(k\lambda)\}(a(\lambda) - B^{-1}(k\lambda)) + \\ + B'\{B^{-1}(k\lambda)\}\sqrt{S(\lambda)}\,\zeta(t) = \\ = k\lambda + \frac{a(\lambda) - B^{-1}(k\lambda)}{\frac{\partial B^{-1}(k\lambda)}{\partial(k\lambda)}} + \frac{\sqrt{S(\lambda)}}{\frac{\partial B^{-1}(k\lambda)}{\partial(k\lambda)}}\zeta(t), \end{aligned} \tag{6.4.5}$$

where we use the formula for the derivative of an inverse function. If we translate $z_2(t, \lambda)$ into equivalent values of the measured variable, we obtain

$$\frac{z_2(t, \lambda)}{k} = \lambda + \frac{a(\lambda) - B^{-1}(k\lambda)}{k\frac{\partial B^{-1}(k\lambda)}{\partial(k\lambda)}} + \frac{\sqrt{S(\lambda)}}{k\frac{\partial B^{-1}(k\lambda)}{\partial(k\lambda)}}\zeta(t). \tag{6.4.5'}$$

The first term in (6.4.5') is the true value of the measured variable, the second is systematic error of a nontracking discriminator, and the third is fluctuating disturbance with intensity which depends on the current value of λ. We note that zero systematic error can be obtained only for $a(\lambda) = B^{-1}(k\lambda)$, i.e., when operator $B^{-1}(\ )$ with an accuracy of a constant factor will convert $a(\lambda)$ into the true value of the parameter. Then, instead of (6.4.5') we have

$$\frac{z_2(t, \lambda)}{k} = \lambda + \frac{\sqrt{S(\lambda)}}{\frac{\partial a(\lambda)}{\partial\lambda}}\zeta(t). \tag{6.4.6}$$

However, such a relationship is difficult to achieve, inasmuch as the function $a(\lambda)$ in general is nonlinear and depends on the signal-to-noise ratio, and compensation of one nonlinearity by another is technically difficult to perform. Furthermore, as already indicated, error in the gain factor is in no way compensated in the circuit (if we do not take special measures, for instance, introduction of a periodically

fed control signal). Characteristics of the estimator unit are considered by us.

In view of linearity and openness of smoothing circuits further calculation of separate components of errors can be conducted independently. If, for instance, we are to determine fluctuation error in the circuit where there exists relationship (6.4.6), then for smoothing circuits with constant parameters, the frequency response of which* is equal to G(iω) [pulse response g(t)], it is easy to obtain

$$\sigma^2_{\phi\pi}(\lambda) = 2G^2(0)\, S_{\text{экв}}(\lambda)\, \Delta f_{\text{эф}}, \qquad (6.4.7)$$

where $\Delta f_{\text{эф}}$ – effective band, which is determined by relationship

$$\Delta f_{\text{эф}} = \frac{1}{2\pi}\int_0^\infty \frac{|G(i\omega)|^2}{|G(0)|^2}\, d\omega = \frac{1}{2}\,\frac{\int_0^\infty g^2(t)\, dt}{\left[\int_0^\infty g(t)\, dt\right]^2}, \qquad (6.4.8)$$

and

$$S_{\text{экв}}(\lambda) = S(\lambda) \Big/ \left|\frac{\partial a(\lambda)}{\partial \lambda}\right|^2$$

– equivalent spectral density of noises, depending on λ.

From (6.4.8) it follows that $S_{\text{экв}}(\lambda)$ sharply increases in flat sections of the "discrimination characteristic" a(λ), where $\left|\frac{\partial a}{\partial \lambda}\right| \to 0$. This phenomenon is explained by the fact that a small fluctuating disturbance is received here as a considerable deviation of the measured variable.

If the method of processing in the estimator unit in the first stage is already selected and dependence a(λ) has been found, then no circuit manipulations can subsequently increase accuracy in flat sections of a(λ), nor can we obtain uniqueness of measurement for those values of a(λ) where monotonic dependence of this function on λ is disturbed. Inasmuch as in tracking circuits accuracy usually does not depend on the absolute value of the measured variable, this circumstance can be considered one more deficiency of nontracking circuits.

In spite of the superficial simplicity of relationship (6.4.7), we indicate that $\sigma^2_{\phi\pi}(\lambda)$ is conditional variance, calculated, strictly speaking, under the condition that the true value of the parameter is constant and equal to λ. In the case of a variable λ, comparable in rate of change to the inertia of smoothing circuits,

*We have in mind a circuit whose input is a fictitious point, where $U_{BX}(t) = z_2(t, \lambda)/k$.

relationship (6.4.7) should be reexamined.

Other components of measurement of error — dynamic and systematic (the first may also be caused by incorrect input of the mean value $\overline{\lambda(t)}$), which in distinction from errors considered in § 6.2 were omitted here, are calculated analogously to how this was done earlier in § 5.2.

§ 6.5. Selection of the Method of Synthesis of Radar Meters. Theory of Solutions in Problems of Measurement

Before passing directly to synthesis of radar meters, it is necessary in brief to discuss different possible approaches to this problem. Below we discuss a series of nonstatistical and statistical methods for the purpose of finding such method of synthesis which would not simply correspond in meaning to the problem, but would be the most theoretically consistent. The desire to be freed to the greatest extent from preconceived ideas of construction of meters and to obtain their complete circuit analytically is natural. Such a preconceived idea, for instance, is the principle of feedback. If it indeed permits us to obtain highest performance in a number of conditions, then this should follow from theory, and not be postulated a priori.

6.5.1. Different Nonstatistical and Statistical Criteria for Synthesis

One of the most elementary nonstatistical criteria is simplicity of the circuit solution of one or another radio device, its cost, power consumption, and so forth. A related statistical criterion is reliability, determined by the probability of malfunction of the meter in a certain assigned time. It is natural that such criteria always have to be considered in the creation of meters; however, by themselves, at the present stage of development of theory they cannot come near the problem of initial selection of the circuit of a device. Moreover, the tendency of contemporary technical development consists in constant increase of complexity of devices for execution of ever more complicated problems, sometimes sacrificing realiability, costs, power consumption, and so forth. Thus, these very simple criteria evidently are not satisfactory, in any case in the first stage of synthesis.

It is absolutely clear that the criterion of synthesis should be intimately connected with the basic quality of the meter — accuracy of measurement. At first glance the matter is facilitated by the fact that there exists a large adjacent branch of science — the theory of control. Here there has been obtained a mass of results using different, mostly nonstatistical criteria (stability, invariance with respect to different disturbances, minimization of time of flow of the transient, and so forth) and various methods of mathematical solution (classical calculus of

variations, method of the Pontryagin principle of a maximum, linear, nonlinear and dynamic programming, the theory of games, and so forth; see, for instance, [3, 64, 65]).

Insufficiency of the mentioned criteria and methods in reference to synthesis of radar meters becomes evident, if we recall (see § 6.1) the circumstances that work of radar devices always occurs in the presence of random interferences, and selection of the class of possible laws of change of the measured variable a non-statistical variables statistically. Moreover, in general, it is impossible to prove that a meter, possessing very high qualities with respect, for instance, to stability invariance and speed of flow of transients, simultaneously in good during measurement of the randomly varying parameter of mixture of the signal with noise.

Deficiencies of the above mentioned approaches, consisting in their nonstatistical nature, force us to turn to statistical criteria and methods.

First we shall discuss a mixed criterion widely used in the literature — the criterion of a minimum of fluctuating and zero dynamic error for a limited set-up time. According to this criterion we seek a minimum mean square random (fluctuating) error and zero (in steady-state regime) error from a definite change of measured variable. In application to meters here we consider the circuit of discriminator given, and we consider the object of synthesis the smoothing circuit. A deficiency of the criterion is the inconsistency in its approach to the measured variables, of which we already spoke above. Its result is that the meter, optimum according to this criterion, sometimes does not ensure minimum total mean square error.

A strictly statistical criterion can come from information theory. Thus, in [4, 5] it is shown that systems, ensuring minimum mean risk, simultaneously ensure a minimum of uncertainty of the reproduced signal. In [6] for servo systems of certain assigned structure there are introduced information theory concepts, in which accuracy is compared with a number of possible states of the system, and high speed operation is compared with the speed of transmission of information and carrying capacity. However, final relationships between accuracy of measurement and informational properties of complex input signals and methods of their processing, which would be useful for the problem of measurement, have yet to be fixed, so that to apply information theory criteria is difficult.

Very commonly used in radio practice and the practice of servo systems are

considerations about the signal-to-interference ratio at different points of electronic devices. The larger this ratio, the better, it is considered, the device works. This opinion, in general, is incorrect, and in meters this criterion is also difficult to use because it is unclear at what point of the circuit one should measure the signal-to-noise ratio. If, for instance, we consider the discriminator output, then, as is shown in § 6.2, it can be completely characterized by the equivalent spectral density of noises. Although the latter depends monotonically on the signal-to interference ratio at the circuit input, introduced specially for the given forms of signal and interference, it cannot be expressed in terms of this ratio alone.

In a systematic statistical approach to random inputs and measured quantities most natural is the criterion of a minimum of some averaged monotonic function of total error of measurement. Averaging is conducted for the ensemble of noises and parameters. Such an approach in one or another form long ago was applied for finding a certain part of the circuit of a meter.

In the first place here one should note that the extreme case, when the whole circuit of the meter (discriminator, smoothing circuits and the method of their closing) are considered given, with the exception of a small number of circuit parameters (gain factors, time constants, and so forth), which can be selected in the process of development to increase accuracy. This very simple method, however, gives a small degree of confidence that we have obtained a system, close to optimum, inasmuch as during selection of the general idea of the construction it is necessary to lean only on preceding experience of construction and invitation. It is possible only to construct guesses whether a small change of the circuit will yield significanlty better results.

Further it is possible to indicate an approach, in which the general construction of the meter is considered the same as that described in Paragraph 6.2.1, the discriminator circuit is considered given, and only smoothing circuits are optimized. Additionally (sometimes implicitly) it is assumed that the discriminator has linear properties relative to the tracked quantity. During synthesis of smoothing circuits by the criterion of minimum total error there can be applied various statistical criteria. The greatest achievement of the statistical trend from works published until recently is the theory of filtrati [7], put forth in the works of Kolmogorov and Wiener. In this theory it was assumed that from statistical evidence we know only correlation function of the tracked quantity and interferences, and smoothing circuits were sought

in the class of linear filters. These methods are very customary for specialists in the theory of automatic control; however, they, in general, pass over the problem of construction of a radio discriminator. Meanwhile, it is possible to assume that, with respect to total error, optimization of the discriminator is no less important than optimization of smoothing filters.

Everything suggests that the best device of statistical synthesis is that which minimizes total error of measurement with the least number of initial assumptions about the structure of the synthesized device, desirably, without singling out a priori the discriminator and smoothing circuits. As the object of optimum analysis here one should consider the existing mixture of the signal, depending on the measured quantity, with interferences. We know a priori statistical evidence on the mixture and measured quantity, and from considerations connected with the use of results of measurement we select the performance criterion. With such a combination of conditions we should be interested in the branch of mathematics which studies the method of obtaining statistical inferences from an available sampling (realization) of a random process and a priori information. Such a branch is the theory of statistical solutions — a branch of mathematical statistics. One should now turn to the theory of solutions.

6.5.2. Application of the Theory of Solutions to Problems of Measurements

In Chapter 3 we gave a summary of the general propositions of the theory of solutions. Here we will repeat certain propositions, having a direct relationship to problems of measurement. Let us remember that the object of optimum processing (finding a solution) is the mixture of signals with noises $y(t, \lambda)$ at the input, statistical evidence about which is given by the likelihood function

$$P(\mathbf{y}|\boldsymbol{\lambda}),$$

where $\mathbf{y}$ — the whole set of observed values of the mixture.

$\boldsymbol{\lambda}$ — the set of measured variables

When there is observed one mixture $y(t)$ at moments $t_1, t_2, \ldots, t_n$, by $\mathbf{y}$ one should understand the column vector

$$\{y_1, \ldots, y_n\} = \mathbf{y}.$$

If here, for instance, $y(t)$ obeys normal distribution, the likelihood function is recorded in the form of relationship (1.3.2), which is generalized for the case of many input mixtures:

$$P(\mathbf{y}\mid\boldsymbol{\lambda})=(2\pi)^{-\frac{mn}{2}}[\det \mathbf{R}]^{-1/2}\exp\left\{-\frac{1}{2}(\mathbf{y}-\mathbf{a})^{+}\mathbf{W}(\mathbf{y}-\mathbf{a})\right\}=$$

$$=(2\pi)^{-\frac{mn}{2}}[\det\|{}^{(\alpha\beta)}R_{ij}\|]^{-1/2}\times$$

$$\times\exp\left\{-\frac{1}{2}\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{m}{}^{(\alpha\beta)}W_{ij}({}^{(\alpha)}y_i-{}^{(\alpha)}a_i)({}^{(\beta)}y_j-{}^{(\beta)}a_j)\right\}. \quad (6.5.1)$$

Here there is introduced matrix notation; sign $+$ signifies transposition;

$$\mathbf{y}=\{{}^{(1)}y_1^{(2)},\ y_1,\ldots,{}^{(m)}y_1,\ldots,{}^{(1)}y_n^{(2)},y_n,\ldots,{}^{(m)}y_n\}$$

is a block column vector of sampled values with n elements in the form of simple columns $\{{}^{(1)}y_i,\ \ldots\ {}^{(m)}y_i\}=\mathbf{y}_i$, composed of values of all m mixtures at the i-th moment of time; $\mathbf{a}$ – analogous column vector of mean values; $\mathbf{R}=\|\mathbf{R}_{ij}\|$ – complex correlation matrix with matrix-elements $\mathbf{R}_{ij}=\|{}^{\alpha\beta}R_{ij}\|$, consisting of values of the function of cross-correlation of the α and β mixtures at moments t_i and t_j, respectively; $\mathbf{W}=\mathbf{R}^{-1}$ – matrix, the reciprocal of $\mathbf{R}$.

In addition to all this matrices $\mathbf{a}$, $\mathbf{R}$ and $\mathbf{W}$ may depend on parameters $\boldsymbol{\lambda}$ of input mixtures.

For continuous realizations of y(t), as noted above, it is convenient to pass to functionals of probability density $P(y(t)|\lambda)$. Sometimes it is more convenient to pass to a functional limit already at the end of analysis, during interpretation of the obtained operations.

On the basis of the observed realization a radar set selects solution $\hat{\boldsymbol{\lambda}}(\mathbf{y})$ from the possible set of solutions. In problems of measurement the space of solutions ("estimates") in structure is like the space of possible values of measured variables, which will become evident somewhat latter. Inasmuch as $\hat{\boldsymbol{\lambda}}(\mathbf{y})$ does not depend functionally on $\boldsymbol{\lambda}$ [although statistically it does depend on $\boldsymbol{\lambda}$ through $\mathbf{y}(t,\boldsymbol{\lambda})$], the statistical nature of the observed mixture (mixtures) leads to the possibility of random errors.

For estimating the significance ("value") of allowed errors of solution there is introduced loss function $I(\hat{\boldsymbol{\lambda}},\boldsymbol{\lambda})$, depending on parameters of the signal included in the mixture and on the taken estimates. The more the accepted solution corresponds to the truth, the less the value of $I(\hat{\boldsymbol{\lambda}},\boldsymbol{\lambda})$. Performance of a device is determined by the average (for all realizations of $\mathbf{y}$) value of the loss function

$$r(\boldsymbol{\lambda},\hat{\boldsymbol{\lambda}})=\int I(\hat{\boldsymbol{\lambda}},\boldsymbol{\lambda})P(\mathbf{y}\mid\boldsymbol{\lambda})d\mathbf{y}, \quad (6.5.2)$$

called the function of conditional risk, inasmuch as this function is determined on the condition of an assigned value of λ. As follows from §6.1, measured parameters of a signal also can reasonably be considered random variables, and we can assign them a prior distribution $P_0(\lambda)$. Then there is also introduced the function of mean risk

$$R(P_0, \hat{\lambda}) = \int r(\lambda, \hat{\lambda}) P_0(\lambda)\, d\lambda = \iint l(\hat{\lambda}, \lambda) P(y\,|\,\lambda) P_0(\lambda)\, dy\, d\lambda, \qquad (6.5.5)$$

which is the averaging of conditional risk for a priori distribution of parameters. The optimum operation of estimation we find from the condition of a minimum mean or conditional risk for an arbitrarily selected loss function. Radio interpretation of operations of optimum estimation reveals the structures of an optimum radar meter.

If a priori distribution $P_0(\lambda)$ is given, and the optimum meter is found by minimization of mean risk $\min_{\hat{\lambda}} R(P_0, \hat{\lambda})$, the estimate is called a Bayes estimate. In another case it is necessary to seek the so-called minimax estimate. It minimizes (by all rules of estimation) the maximum value of conditional risk for all possible values of the parameter. Cases when there are assigned no limitations on the value of the parameter are very rare, and usually one should seek the minimax estimate with certain limitations, statistical and nonstatistical.

Basically we shall use the Bayes methods of synthesis; however in § 6.9 certain attention will also be paid to minimax methods.

Properties of the theory of statistical solutions are such that in it it is possible to introduce certain a priori assumptions about the structure of the synthesized device, starting synthesis from a certain point of the circuit. However, it is possible also not to introduce any assumptions, considering as the input value the output signal of the antenna unit or even the electromagnetic field in its aperture [52]. Obviously, the last cases are the most interesting and to the largest extent characterize the merit of the theory of solutions. Chronologically, however, earlier there were offered such solutions of the problem of measurement, where a series of elements of the circuit of a meter were assumed assigned, and the theory of solutions was applied for partial synthesis.

Thus, in the 1950's there was developed a method of synthesis of optimum meters on the basis of a branch of the theory of statistical solutions – the theory of estimation [10, 11]. It was assumed that the measured parameter of motion of the target, coded in the input signal, is constant for a certain time of observation.

Proceeding from complete statistical evidence on the mixture of the signal with interferences there was found an optimum circuit for determining the unknown parameter for this time. Such a method is completely applicable, for instance, for direction finders which determine angular position for a small interval of time.

If, however, we turn to the general case of measurement of variable coordinates, the theory of estimation, assuming invariability of parameters in time, strictly speaking, cannot be applied. This does not exclude, however, certain compromise solutions of the problem, consisting in assigning a general idea of the construction of the meter in that form which is known in practice, considering smoothing circuits linear. Here, it is postulated that the discriminator should give an estimate of current mismatch between the true and measured values of the tracked parameter for time intervals during which these mismatches almost do not change. In such an approach it was possible to use the ideas and methods of the theory of estimation for synthesis of discriminators in meters of the class interesting us.

With respect to partial synthesis of smoothing circuits on the assumption of linearity of the discriminator with respect to the measured quantity there is applied the theory of optimum filtration, but with a new conceptual basis. It seems that with an additive mixture of the measured quantity and interference, with Gaussian distributions of each of them, Wiener filters are absolutely optimum, i.e., there does not exist a class of operators executing this task more successfully. Unfortunately, the formulated conditions are so limiting that in practice in pure form they are almost never encountered. Therefore, it is clear that the theory of filtration should somehow be adapted to nonlinear problems.

Consequently, all presented methods of partial synthesis, even that conducted on the basis of the theory of solutions, left doubt about the correctness of selection of the general idea of construction of meters. We desire complete synthesis, emanating only from statistical evidence about signals and parameters. On the assumptions of Markovian and Gaussian parameters this problem was solved in [16, 18, 25, 60]. To these results we shall pass somewhat later.

Here we shall make a number of remarks, simultaneously showing the promising nature of the devices of the theory of solutions and its limitations, which, surely, will be surmounted with further development of the theory.

An essential limitation of the theory of solutions is the necessity for the presence of a sizeable quantity of statistical evidence about the signal and the parameter. This circumstance led certain authors to the pessimistic conclusion that

the theory of solutions simply shifts uncertainty during synthesis from circuit structure to statistical properties of input signals. However, in defense of the theory of solutions it is possible to repeat all the arguments of § [illegible] concerning the statistical nature of measurements. In practice we always have some statistical evidence or physical considerations about them. For further firming up of the propositions of the theory of solutions it is urgent to investigate from all sides the criticality of synthesized circuits with respect to statistics established in them and to find classes of optimum operations ensuring for a given class of statistic qualities which are lower than optimum by not more than a given quantity. Interesting, also, are questions of optimization for limited statistical evidence.

With the question of apriorism in statistical evidence there are also connected classes of interferences on the meter input. In a number of cases, especially where organized interferences are possible, it is impossible to decide beforehand what will be the exact statistical characteristics of input signals. Therefore, it is reasonable to consider construction of an optimum meter, primarily of its discriminator, from two directions.

1) Simpler is synthesis of an optimum meter for those interferences which are present always (for instance, internal noises) or with high probability (for instance, chaotic reflections), and whose influence can be decreased by rational selection of the form of the signal and construction of the circuit. Here, it is possible to consider the statistical evidence on the mixture of the signal with interferences completely assigned, considering, possibly, several parameters of interference unknown.

2) Somewhat more complicated is protection from interferences which may appear only with low probability or with a probability which it is impossible to estimate. Among interferences of this kind we include those, protection from which for assigned characteristics of radar system cannot even be ensured by optimum methods of construction of the receiver, but, in principle, such protection nevertheless can be found in the course of changing other parameters of the system of processing of the signals (for instance, genreal construction of devices of secondary processing in the presence of active interference). It is futile to synthesize an optimum receiver (meter) on the assumption of the presence of all such interferences. This would remove us from the optimum for the case of mixture of the signal with interferences of the first group, not to talk of extraordinary increase of receiver complexity. Therefore, most expedient is synthesis of

automatic devices for indication of interferences of the second group for the purpose of sharp variation of parameters of the system of processing of signals on an instruction from the automatic device.

A modified variant should also be synthesized, based as far as possible on statistical evidence about the given form of interferences. In this direction there have not been obtained many results, if we are considering complicated forms of interferences.

Variation of parameters of a receiver in the course of work, in principle, is already self-tuning of the meter circuit in accordance with the character of the received signal. At present, problems of analysis and synthesis of automatic systems with self-tuning (true, of somewhat different type) are the subject of the widest attention [3]. However, in our applications, inasmuch as we are interested in the problem of measurement with full assignment of statistical evidence, optimum circuits are obtained without a priori reliance on ideas of self-tuning.

Sometimes circuits of optimum meters contain self-tuning elements, and in other cases they are absent; however, the method of synthesis always guarantees absolute optimality of a circuit if we keep initial data constant. If in the distribution of probabilities of a mixture we start to consider unknown some additional parameter, and assign it its own distribution of probabilities, the principle of synthesis and quality of work of the circuit will not be changed whether we treat measurement of the additional parameter as self-tuning or not. Thus, a priori introduction of ideas of self-tuning will most likely be useful when we have incomplete knowledge of when there is a change of statistical properties of the mixture of the signal, interferences and the parameter.

There is also possible a game-theory approach to meters. At present the theory of solutions sometimes is included in the theory of statistical games, considering that particular case when the "game" is conducted against an "unintelligent antagonist," whose behavior in a statistical sense is known beforehand. If, however, the enemy is intelligent and tries to change the form of interference to bring on maximum error of measurement, and there is the possibility of changing characteristics of the meter for constant preservation of its optimality, there is observed a non-degenerate game-theory situation, which is very interesting for practical applications. Unfortunately, only the first steps have been made in this direction.

Everything said speaks to the fact that the devices of the theory of solutions are completely timely and at present are a powerful means of solution of the problem

of synthesis of radar meters.

6.5.3. Theory of Statistical Estimates

We shall now discuss in greater detail the theory of statistical estimates of parameters of distributions [8]. This is what we call the branch of mathematical statistics in which the probability density of a sampling (functional of the density of distribution of realizations) depends on one or several constant, but unknown parameters. These parameters must be estimated from sampled values of the input variable.

The most well-developed and convenient methods of finding estimates are the method of maximum a posteriori probability and the maximum likelihood method. The first method consists in differentiation with respect to parameter λ (we shall now consider it unique) of the product of the a priori distribution of parameter $P_0(\lambda)$ and likelihood function $P(\mathbf{y}|\lambda)$. With an accuracy of a factor, not depending on λ, this product is the a posteriori, i.e., the knowledge-from-experience probability density of the parameter, which follows from the known formula of Bayes

$$P(\lambda|\mathbf{y}) \equiv \tilde{P}(\lambda) = \frac{P_0(\lambda)\, P(\mathbf{y}|\lambda)}{\int P_0(\lambda)\, P(\mathbf{y}|\lambda)\, d\lambda} = \frac{1}{k} P_0(\lambda)\, P(\mathbf{y}|\lambda), \tag{6.5.4}$$

where k is a factor, not depending on λ.

The result of differentiation is equated to zero, and we seek the root of the obtained equation which explicitly depends on $\mathbf{y}$:

$$\frac{\partial}{\partial \lambda}[P_0(\lambda) P(\mathbf{y}|\lambda)] = 0, \tag{6.5.5}$$

which is taken as the very best estimate.

According to the maximum likelihood method we differentiate the probability function, which gives the so-called likelihood equation

$$\frac{\partial}{\partial \lambda} P(\mathbf{y}|\lambda) = 0. \tag{6.5.6}$$

Its root is called the maximum likelihood estimate. Although the theory of estimation was developed before the general theory of solutions, with the appearance of the latter it was possible to interpret the theory of estimation from more general propositions. Here, we consider that the space of solutions is a segment of a straight line, on which there are plotted possible "estimates" of parameter "$\hat{\lambda}$."
As the loss functions we usually consider:

the simple loss function

$$l(\lambda, \hat{\lambda}) = C - \delta(\lambda - \hat{\lambda}) \quad (C = \text{const}), \tag{6.5.7}$$

the quadratic loss function

$$l(\lambda, \hat{\lambda}) = (\lambda - \hat{\lambda})^2, \tag{6.5.8}$$

a loss function with saturation

$$l(\lambda, \hat{\lambda}) = 1 - \exp\{-a(\lambda - \hat{\lambda})^2\}$$

or function

$$l(\lambda, \hat{\lambda}) = |\lambda - \hat{\lambda}|.$$

If we select simple loss function (6.5.7), mean risk (6.5.3) is equal to

$$R(P_0, \hat{\lambda}) = C - \int P(\mathbf{y} | \hat{\lambda}) P_0(\hat{\lambda})\, d\mathbf{y}. \tag{6.5.9}$$

For its minimization, equivalent to maximization of the integral in the right part of (6.5.9), it is necessary that for any fixed **y** there be found a solution $\hat{\lambda}$, corresponding to a maximum of the integrand. In other words, the optimum estimate corresponds to a maximum of the a posteriori probability, i.e., is a root of equation (6.5.5). Hence, it is clear in what sense the estimate of maximum a posteriori probability is optimum. It is easy to prove that such an estimate additionally maximizes the probability of a correct solution. We also note that with a wide a priori distribution, the estimate of maximum a posteriori probability passes to the estimate of maximum likelihood.

We now take quadratic loss function (6.5.8). Here, mean risk (6.5.3) will take the form

$$R(P_0, \hat{\lambda}) = \iint (\lambda - \hat{\lambda})^2 P(\mathbf{y} | \lambda) P_0(\lambda)\, d\lambda\, d\mathbf{y}. \tag{6.5.10}$$

It has a physical meaning of error of measurement of λ, averaged over **y** and λ. In order to minimize $R(P_0, \hat{\lambda})$, it is necessary and sufficient for any **y** to minimize by selection of $\hat{\lambda}$ the integral over λ in (6.5.10). Differentiating this integral with respect to $\hat{\lambda}$ and equating the result to zero, we have an equation for finding the estimate

$$\int (\lambda - \hat{\lambda}) P(\mathbf{y} | \lambda) P_0(\lambda)\, d\lambda = 0. \tag{6.5.11}$$

Formal solution of (6.5.11) is the conditional mathematical expectation of λ for the given **y**, i.e., the mean value of the a posteriori distribution:

$$\hat{\lambda}(\mathbf{y}) = \overline{\lambda(\mathbf{y})} \equiv \frac{\int \lambda P(\mathbf{y} | \lambda) P_0(\lambda)\, d\lambda}{\int P(\mathbf{y} | \lambda) P_0(\lambda)\, d\lambda} \equiv \int \lambda \tilde{P}(\lambda)\, d\lambda. \tag{6.5.12}$$

In distinction from the operator of estimation of maximum a posteriori probability the structure of the operator of the optimum estimate for a quadratic loss

function depends on the form of the whole function of the a posteriori probability.

If one assigns a loss function of any form differing from the considered one in general there will be obtained other operators. And although the criterion of minimum mean square error, ensuing from application of a quadratic loss function, is the most customary, the variety of optimum solutions causes certain dissatisfaction. We would like to obtain a solution which would be optimum for a sufficiently broad class of cases. It turns out that for certain limiting assumptions the theory of Bayes estimates permits such a solution.

To explain this circumstance we first indicate that estimates for simple and quadratic loss functions coincide where the a posteriori distribution of probabilities is a function which is symmetric with respect to a certain point. This is clear from the expressions obtained. In the theory of estimation strict mathematical proof of this fact is connected with the property of efficiency of estimates, by which we mean reaching the lower bound of variance of an estimate relative to the true value. It appears [8] that in a certain class of cases estimates of maximum a posteriori probability are efficient. Especially frequently there is noted asymptotic efficiency, i.e., efficiency for a time of observation, considerably greater than the interval of correlation of the slowest random variables of the mixture y(t). Proof of efficiency thereby simultaneously establishes the fact that two classes of estimates are identical.

We shall prove that with symmetry of a posteriori distribution any symmetric loss function leads to the same optimum estimate. Let us assume that the a posteriori probability has the form

$$\widetilde{P}(\lambda)=F(\lambda-\lambda_0),$$

where F(x) is a function, symmetric with respect to zero, and the loss function is also symmetric, so that

$$l(\lambda-\hat{\lambda})=l(\hat{\lambda}-\lambda),\quad \frac{\partial l(\lambda-\hat{\lambda})}{\partial\hat{\lambda}}=-\frac{\partial l(\hat{\lambda}-\lambda)}{\partial\hat{\lambda}}.$$

Inasmuch as at the point of the optimum estimate the derivative of mean risk with respect to the estimate is equal to zero, at this point this equation should be satisfied:

$$J\equiv\int\frac{\partial l(\lambda-\hat{\lambda})}{\partial\hat{\lambda}}\widetilde{P}(\lambda)\,d\lambda=0. \qquad (2.5.12')$$

We express $\widetilde{P}(\lambda)$ in the left part of this relationship through $F(\lambda-\lambda_0)$ and set

$\hat{\lambda} = \lambda_0$. Then, changing the sign of the argument of integration, it is easy to prove that J = -J, from which it follows that J = 0. This proves our statement.*

The value of λ_0, determining the point of symmetry of a posteriori probability, and equal to any estimate using a symmetric loss function, is, obviously, already the familiar conditional mathematical expectation (6.5.12). Thereby, conditional mathematical expectation in a wide range coincides with the maximum value of a posteriori probability and can serve as a universal optimum estimate. Thus, the method of a maximum a posteriori probability, and for a wide a priori distribution, the method of maximum probability, too, under conditions very little limited give a single Bayes estimate of the measured parameter.

Further it is useful to note one mathematical law, proved in the theory of estimation. The fact is that with a large time of observation and with a comparatively low level of input interferences the logarithm of the likelihood function (as a function of the parameter) is close in form to a parabolic curve, located near the true value of the parameter:

$$\ln P(\mathbf{y}|\lambda) \approx \ln P(\mathbf{y}|\lambda_И) + \frac{\partial}{\partial\lambda} \ln P(\mathbf{y}|\lambda_И)(\lambda - \lambda_И) + \\ + \frac{1}{2}\frac{\partial^2}{\partial\lambda^2} \ln P(\mathbf{y}|\lambda_И) \frac{(\lambda - \lambda_И)^2}{2}, \qquad (6.5.13)$$

where $\lambda_И$ – true value of the parameter of the mixture y(t).

From sampling to sampling the vertex of this parabola shifts along both axes, and also changes in width. Naturally this parabola can be expanded at any point, close to points of the true and maximum-probable value, limited in all to three terms of the expansion. In any case we arrive at a curve, coinciding with (6.5.13).

If we now pass from the logarithm to the actual likelihood function, the given dependences will be expressed in the fact that the likelihood function will be approximated by a Gaussian curve, located somewhere near the point of the true value $\lambda_И$, pulsating and shifting from sampling to sampling. Here, as it is easy to show [8], the width of the curve, characterized by the coefficient of sharpness of the parabola (6.5.13) $-\frac{\partial^2}{\partial\lambda^2}\ln P(\mathbf{y}|\lambda)$, on the average remains equal to the mean square scattering of the position of its maximum value from the true $\lambda_И$, and

*In such a proof it is necessary to use the assumption of an infinite domain of determination of the function of the a priori distribution and the loss function, although in practical applications this condition is usually not satisfied. However, in the same applications interesting only are cases when the width of the a posteriori probability distribution is considerably less than that of the a priori. Here, finiteness of boundaries of the domain of determination of the a priori distribution is not important.

both mean values are connected with variance of efficient estimate $\sigma^2_{\text{эф}}$ according to the formula

$$\sigma_{\text{эф}}^{-2} = -\left\langle \frac{\partial^2}{\partial\lambda^2} \ln P(\mathbf{y} \mid \lambda) \right\rangle_y = \left\langle \left[\frac{\partial}{\partial\lambda} \ln P(\mathbf{y} \mid \lambda) \right]^2 \right\rangle_y. \qquad (6.5.14)$$

Such laws occur for $P(\mathbf{y}|\lambda)$ near the true value of the parameter. Besides the main peak of the probability function there are also parasitic peaks, occurring both due to nonideal properties of the signal, and also due to noise disturbances. In the theory of estimation, where the parameter is strictly constant, it is possible to show that the relative magnitude of parasitic peaks with increase of the time of observation (the size of the sampling) gradually decreases. We shall discuss the influence of parasitic peaks in somewhat greater detail in § 6.6.

It is useful to explain properties of joint estimates of several parameters. The likelihood function is determined in this case in the space of all measured quantities $\{\lambda_1, \ldots, \lambda_l\} = \lambda$. Expansion (6.5.13) is replaced by

$$\ln P(\mathbf{y} \mid \lambda) \approx \ln P(\mathbf{y} \mid \lambda_{\text{и}}) + \sum_{i=1}^{l} \frac{\partial}{\partial\lambda_i} \ln P(\mathbf{y} \mid \lambda_{\text{и}}) (\lambda_i - \lambda_{\text{и}i}) +$$
$$+ \frac{1}{2} \sum_{i,j=1}^{l} \frac{\partial^2}{\partial\lambda_i \partial\lambda_j} \ln P(\mathbf{y} \mid \lambda_{\text{и}}) (\lambda_i - \lambda_{\text{и}i}) (\lambda_j - \lambda_{\text{и}j}).$$

The matrix, composed from mean values of second derivatives,

$$\mathbf{A} = \left\| -\left\langle \frac{\partial^2}{\partial\lambda_i \partial\lambda_j} \ln P(\mathbf{y} \mid \lambda_{\text{и}}) \right\rangle \right\|$$

characterizes "sharpness" of the multi-dimensional peak and sometimes is called a Fischer information matrix. Its value is that reciprocal matrix $\mathbf{\Sigma} = \mathbf{A}^{-1}$ consists of mean second moments of minimum errors of measurement of the parameter attained in the case of the existence of jointly efficient estimates. If coding of separate parameters in the signal y(t) are independent, then $\mathbf{\Sigma}$ – diagonal matrix with elements $\sigma^2_{\text{эф} i}$, equal to variances of minimum attainable errors of measurement of separate parameters. In this case the meaning of a minimum does not require further explanations. If, however, forms of coding, and consequently also of error of measurement, are interrelated, it is necessary to introduce into consideration a multi-dimensional ellipsoid, second central moments of which are determined by the matrix of second moments of errors of measurement. Then the introduced ellipsoid of scattering for any method of estimation of parameters will wholly contain in itself the ellipsoid with second moments, determined by matrix $\mathbf{\Sigma}$, which explains the meaning of the minimum of matrix $\mathbf{\Sigma}$ for interrelated coding of separate parameters.

Radar applications of the theory of estimation consist, primarily, of the following. In the course of reflection of the sounding radar signal from a target parameters of the probability density of the mixture of the reflected signal and interferences at the receiver input turn out to depend on the coordinates and velocity of the target. Measuring (estimating) the modulation delay of the signal, its average frequency or direction of arrival, it would be possible, thereby, to determine necessary target position data. If during a certain interval of time these coordinates can be considered constant, there appears the classical problem of the theory of estimation: for a given realization of the mixture of signal and interferences, observed in a given interval of time, determine with minimum errors unknown parameters of the signal in this mixture.

First results in the synthesis of optimum radar meters were obtained, namely, on the basis of the theory of estimation [10, 11]. Intervals of observation were selected in such a way that during them target position data indeed remained constant. Under conditions when most widely used were pulse radars, the structure of the pulse signal, i.e., the division of it into periods of pulse trains (or pulse packs), gave the possibility of conveniently dividing the whole interval of observation into time intervals, satisfying the shown requirement of theory. Not being interested in questions of unification of data of separate periods, they investigated potential properties of measurement in each of them.

It is possible to indicate the following methods of synthesis of devices for processing signals, obtained in the literature on the basis of the theory of estimation. In [10, 11] and in a considerable number of later works they explicitly or implicitly assume that there is the possibility of consecutive or parallel finding of values of the likelihood function in the whole a priori domain of the measured parameter, as a result of which there can be found a point of maximum value which is the estimate. It requires no explanation that, in general, obtaining of the whole "scan" of the likelihood function is technically complicated, with the exception of a parameter in the form of time delay of modulation.

In [12, 13] there is offered another, simpler method of finding the maximum likelihood estimate, based on the assumption that we know the value of parameter λ_1, removed from the true value of the parameter of the input mixture a distance less than the width of the basic peak of the likelihood function. We shall explain the idea of this method. We take the expansion of the logarithm of a likelihood function of type (6.5.13) at point λ_1 and, differentiating it with respect to λ, equate it

to zero. Then, transforming the likelihood equation

$$\frac{\partial}{\partial\lambda}\ln P(y\,|\,\lambda_1)+\frac{\partial^2}{\partial\lambda^2}\ln P(y\,|\,\lambda_1)(\lambda-\lambda_1)=0$$

we give an explicit expression for the estimate of maximum likelihood $\hat{\lambda}$:

$$\hat{\lambda}=\lambda_1-\frac{\dfrac{\partial\ln P(y\,|\,\lambda_1)}{\partial\lambda}}{\dfrac{\partial^2\ln P(y\,|\,\lambda_1)}{\partial\lambda^2}}. \qquad (6.5.14')$$

Thus, to find $\hat{\lambda}$ we need to schematically realize the first and second derivatives of the logarithm of the function at point λ_1, assumed to be close to the true value of λ.

If one were to try to extend methods of synthesis based on the theory of estimation to the case of a variable parameter, then this can be done only by means of easing known rigor. Thus, we assume that the circuit of the meter is already given in the form of a tracking system of the type which was studied in § 6.2. With respect to the discriminator we postulate that it should constantly issue with maximum accuracy an estimate of mismatch between the true value of a parameter and that value λ_p, which as a result of measurement proceeds to the discriminator from the smoothing circuits. As follows from (6.5.14'), the optimum estimate of difference $\lambda - \lambda_p = \varepsilon$ is

$$\hat{\varepsilon}=-\frac{\dfrac{\partial\ln P(y\,|\,\lambda_p)}{\partial\lambda}}{\dfrac{\partial^2\ln P(y\,|\,\lambda_p)}{\partial\lambda^2}}. \qquad (6.5.15)$$

Under conditions when in a subinterval there is realized sufficient accumulation, this approximate relationship is satisfied:

$$\frac{\partial^2}{\partial\lambda^2}\ln P(y\,|\,\lambda)\approx\overline{\frac{\partial^2}{\partial\lambda^2}\ln P(y\,|\,\lambda)},$$

i.e., the second derivative may be replaced by a quantity which does not depend on $y(t)$, and the optimum estimate of mismatch is simply proportional to the first derivative of the logarithm of the likelihood function, taken at point $\lambda = \lambda_p$.

However, such methods of synthesis of tracking meters are inconsistent inasmuch as the new assumptions introduced are beyond the frames of the theory of estimation. At the same time we know that measuring systems are always synthesized from conditions of compromise between fluctuating and dynamic errors, where the appearance of the latter is explained namely by changeability of the parameter in time. It would be desirable to find the optimum measuring system immediately as a whole,

proceeding from the requirement of a minimum of resultant (smoothed) errors. Such a possibility, in principle, is given by filtration theory.

Before presenting principles of filtration theory we shall discuss one more application of the theory of estimation to the problem of optimization of radar or other meters. We have in mind determination of parameters of orbits of objects, flying in ballistic curves [14]. It is known that the path of motion of the center of gravity of an arbitrary body in space with determined external forces is determined by six constants parameters μ, for which, in particular, we can consider coordinates of the object and the components of velocity at a certain moment of time. If to outputs of discriminators of meters we add values of coordinates of their tuning with coefficients, equal to the gain factors of discriminators, then, according to results of § 6.2, disregarding parametric fluctuations, we have

$$x(t)=K_{\text{д}}\lambda(t)+K_{\text{д}}\eta(t), \tag{6.5.16}$$

i.e., additive mixtures of measured coordinates $\lambda(t)$ (angles, distances) with noises. If one were to consider that coordinates $\lambda(t)$ depend in turn on constant parameters μ, then for any time of observation we arrive at a problem of the classical theory of estimation. After determining parameters of the orbit it is possible to calculate arbitrary functions of these parameters, for instance, the measured coordinates themselves. It is implicitly assumed that the best estimates of parameters of the orbit lead to the best measurements of coordinates of the body. This is indeed so if errors of separate measurements are small.

6.5.4. Theory of Optimum Linear Filtration

Filtration is continuous reproduction ("measurement," "finding of an estimate") of a certain variable which is a parameter of an observed random process.

The theory of optimum filtration already has a known history and has been given a foundation by the fundamental works of Kolmogorov and Wiener on filtration [7]. Initially this theory was not connected at all with the theory of solutions and was based on the following postulates:

1) parameter $\lambda(t)$ and interference $n(t)$ are additively mixed stationary random processes having different correlation functions. Realization of mixture $y(t) = \lambda(t) + n(t)$ is assigned on an interval, starting from an infinitely removed moment up to the current moment;

2) the sought optimum system should carry out a linear operation on the mixture;

3) the system should be physically realizable, i.e., its pulse response $h(t)$

should turn into zero for negative t;

4) the criterion of optimality is minimum mean square error of reproduction of the parameter or of any linear functional of the parameter — a derivative, an integral, a certain expected value of the parameter, and so forth.

Upon fulfillment of these requirements one can be certain that pulse response h(t) of an optimum Wiener filter for direct reproduction of the actual parameter satisfies the following integral equation (the Wiener-Hoof equation):

$$\int_0^\infty R_{yy}(t-\tau)\,h(\tau)\,d\tau = R_{y\lambda}(t), \qquad (6.5.17)$$

where $R_{yy}(\tau) = \overline{[\lambda(t) + n(t)][\lambda(t+\tau) + n(t+\tau)]}$ — autocorrelation function of mixture y(t);

$R_{y\lambda}(\tau) = \overline{[\lambda(t) + n(t)]\lambda(t+\tau)}$ — cross-correlation function of the mixture and the parameter.

If we immediately try to apply to (6.5.17) the Fourier transform, we easily find the frequency response of an optimum filter, but it will belong to the class of filters unrealizable in real time. By the latter we mean filters whose pulse response is equal to zero for negative time arguments. If we are talking about filtration with direct delivery of the result, this circumstance is equivalent to the appearance of a response before application of a disturbance and contradicts the causality principle. To find physically realizable filters mathematicians have had to apply much more refined methods of the theory of functions of a complex variable. As a result, the solution of equation (6.5.17) has the form [7]

$$H(i\omega) = \frac{1}{\Psi(i\omega)}\left[\frac{S_{y\lambda}(\omega)}{\Psi(-i\omega)}\right]_+ = \frac{1}{2\pi\Psi(i\omega)}\int_0^\infty e^{-i\omega t}dt\int_{-\infty}^{+\infty}\frac{S_{y\lambda}(u)}{\Psi(-iu)}e^{iut}\,du, \qquad (6.5.18)$$

where

$$H(i\omega) = \int_0^\infty h(t)\,e^{-i\omega t}\,dt; \quad S_{y\lambda}(\omega) = \int_{-\infty}^{+\infty} R_{y\lambda}(\tau)\,e^{-i\omega\tau}d\tau,$$

and

$$\Psi(i\omega)\Psi(-i\omega) = S_{yy}(\omega) = \int_{-\infty}^{+\infty} R_{yy}(\tau)\,e^{-i\omega\tau}d\tau, \qquad (6.5.19)$$

is a special expansion (factoring) of complex variable ω into factors with zeroes and poles, correspondingly, in the upper and lower half-planes. Such an expansion is always feasible, for instance, when $S_{yy}(\omega)$ is a rational-fraction function of ω^2 with real coefficients. (In practice, approximation of spectral densities by such

functions is always possible). The operation $[\ ]_+$ in (6.5.18) signifies taking that component of the expression in the brackets, which has poles in the upper half-plane of complex variable ω. Analytically, this operation can also be expressed in the form of two integrals.

With further development of filtration theory it was possible to weaken postulates 1 and 4. Thus, they considered certain forms of a nonstationary parameter, a finite interval of realization and other criteria of optimality.

In particular, it was found that the pulse response of a Wiener filter in more general cases is determined by integral equation

$$\int_{t_0}^{t} R_{yy}(t, s)\, k(s, \tau)\, ds = R_{y\lambda}(t, \tau), \tag{6.5.20}$$

where $R_{yy}(t, \tau)$, $R_{y\lambda}(t, \tau)$ have their former meaning, but due to nonstationariness of the input mixture they depend on two variables.

It is important to emphasize that the class of operators in which we sought the best remained in all cases given.

In spite of the indicated limitations, filtration theory has great methodological value, and in a whole series of practical cases gives an explicit analytical expression for pulse responses of different filters [7]. For spectral densities in the form of rational-fraction functions of ω these filters belong to systems with constant parameters and are executed in the form of a set of RC- and LRC-circuits and linear inertialess amplifiers.

After application of the theory of solutions it was shown that for an additive mixture of Gaussian measured quantities and interferences with Gaussian distributions Wiener filters are absolutely optimum, i.e., there does not exist a class of operators executing this task more successfully [5].

We note two other important results. From [61] it follows that for an additive mixture of processes with non-Gaussian distributions optimum filter-extrapolators give only a small improvement as compared to the Wiener ones, correctly calculated for the correlation function of the non-Gaussian components. Further, in [62] it is shown that for an additive mixture of Gaussian interference with a signal, for which we know only the correlation function, the Wiener filter is optimum in the minimax sense, i.e., ensures the least mean square error for the worst forms of signals with an assigned correlation function. Thereby, Wiener filters for additive mixtures possess optimum properties of a very general nature. Filters, close in their properties to Wiener ones, will also appear in the problem of nonlinear

filtration of especially nonadditive mixtures, for which in § 6.8 we contine their consideration.

Certain authors also proceed along the same path of finding an operator from a given class in solving problems of nonlinear filtration [15]. However, inasmuch as each has the right to select arbitrary classes of nonlinear operators, this path leaves us unsatisfied: it is always possible to hypothesize the presence of a still better solution.

6.5.5. Filtration of a Markovian Parameter by the Method of a Posteriori Probability

Above we proved that in equations for optimum operators there enters the a posteriori probability of a parameter (parameters) for a given realization of the signal (signals). In other words, forming of the a posteriori probability is a sufficient primary operation of an optimum system, after which it only remains to take the solution, best from the point of view of some criterion. A receiver, forming the a posteriori probability for a constant parameter, is known in the literature as a Woodword-Davis receiver. The Bayes formula may, however, also be recorded for a parameter varying in time. Considering that the observation is conducted at discrete moments $t_1, t_2, \ldots, t_n$, it has formula

$$\tilde{P}_n(\lambda_1, \ldots, \lambda_n) = C_n P_n(\lambda_1, \ldots, \lambda_n) P(y_1, \ldots, y_n | \lambda_1, \ldots, \lambda_n), \qquad (6.5.21)$$

where $\lambda_i = \lambda(t_i)$ $(i = 1, \ldots, n)$;

$P_n, \tilde{P}_n$ – multi-dimensional a priori and a posteriori distributions;

$P(y_1, \ldots, y_n | \lambda_1, \ldots, \lambda_n)$ – multi-dimensional probability function;

C_n – normalizing quantity, not depending on λ_i.

Investigation of the operator of a posteriori probability (6.5.21) without concretization of the character of change of the parameter in time is difficult. A broad class of random processes, embracing the mass of cases interesting for application, is the class of Markovian processes (see § 6.3). The problem of forming the a posteriori probability for Markovian varying parameters, arbitrarily coded in the input signal $y(t)$, was studied by R. L. Stratonovich in [16-18]. Although the method of R. L. Stratonovich essentially does not belong to the theory of solutions, it is important for the theory of measurements. We will present it, without holding to the ground work of the account in [16].

We assume that moments of observation of the input mixture $y(t)$ are separate so that all random variables, on which we perform averaging when finding the

likelihood function, at various moments are not connected, so that $P(y_1, \ldots, y_n | \lambda_1, \ldots, \lambda_n)$ is broken up into the product

$$P(y_1, \ldots, y_n | \lambda_1, \ldots, \lambda_n) = \prod_{i=1}^{n} P(y_i | \lambda_i). \tag{6.5.22}$$

Using (6.3.2) and (6.5.22), we record (6.5.21) for two adjacent moments of observation:

$$\tilde{P}_n(\lambda_1, \ldots, \lambda_n) = C_n P_1(\lambda_1) P(y_1 | \lambda_1) \prod_{i=2}^{n} W(\lambda_i | \lambda_{i-1}) P(y_i | \lambda_i),$$

$$\tilde{P}_{n+1}(\lambda_1, \ldots, \lambda_{n+1}) = \\ = C_{n+1} P_1(\lambda_1) P(y_1 | \lambda_1) \prod_{i=2}^{n+1} W(\lambda_i | \lambda_{i-1}) P(y_i | \lambda_i),$$

whence

$$\tilde{P}_{n+1}(\lambda_1, \ldots, \lambda_{n+1}) = \\ = C^0_{n+1} \tilde{P}_n(\lambda_1, \ldots, \lambda_n) W(\lambda_{n+1} | \lambda_n) P(y_{n+1} | \lambda_{n+1}), \tag{6.5.23}$$

where $C^0_{n+1} = C_{n+1} | C_n$.

Let us now turn to final a posteriori probabilities, characterizing the a posteriori distribution of the parameter in the last step with observations in the preceding ones. This is carried out by integration of $\tilde{P}_k$ over all arguments λ_1, except the last one λ_k. Then from (6.5.23) we have

$$\tilde{P}_{n+1}(\lambda_{n+1}) = C^0_{n+1} P_{n+1}(y_{n+1} | \lambda_{n+1}) \int \tilde{P}_n(\lambda_n) W(\lambda_{n+1} | \lambda_n) d\lambda_n. \tag{6.5.24}$$

The physical interpretation of relationship (6.5.24), subsequently important, is sufficiently clear from Fig. 6.9. For forming the a posteriori distribution in one step it is necessary to transform the a posteriori distribution of the preceding moment by a transition function, which permits us to consider the expected character of change of the parameter in the step, and to multiply the result of the transformation by the value of the likelihood function of the next measurement, characterizing the newly arriving information about the parameter.

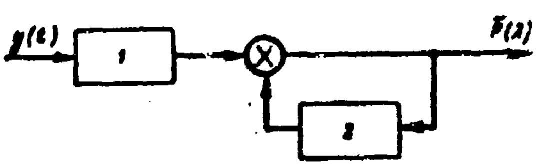

Fig. 6.9. Filter of a posteriori probability: 1 — unit for formation of the likelihood function; 2 — unit for delay and processing of a posteriori probability.

Formula (6.5.24) was derived above on the assumption of discrete observation. However, it has a wider domain of applicability. If the useful component in y(t) has a pulse structure, where all random inputs (noises, interference, fluctuation

of reflecting surface) are independent from pulse to pulse, then after passage to the limit of continuous observation for the likelihood function this relationship is valid:

$$P(y(t)|\lambda_1, \ldots, \lambda_n) = \prod_{i=1}^{n} P(y(t)|\lambda_i), \tag{6.5.25}$$

analogous to (6.5.22). In (6.5.25) as elements there enter likelihood functions for all i-th cycles of observation, depending on the value of parameter λ_i, which remains constant within the limits of the i-th pulse. In other words, if we are not interested in evolution of the a posteriori distribution in intervals between pulses, relationship (6.5.24) remains applicable, where quantization of the parameter is dictated by the character of the input signal y(t), and observation of the realization of the latter is considered continuous.

If, however, the signal has arbitrary structure, and we are interested in the form of the a posteriori distribution continuously, it is necessary to seek some new methods of mathematical description of the a posteriori probability, eliminating the discrete nature of relationship (6.5.24).

With passage to the limit of continuity the analog of relationship (6.5.22) is function

$$P(y(t)|\lambda(t)) = \exp \int_{t_0}^{t} l(y(s), \lambda(s), \tau)\, d\tau, \tag{6.5.26}$$

where (t_0, t) — interval of observation;

$l(y(s), \lambda(s), \tau)$ — certain functional of y(s) and $\lambda(s)$ for $s \in (t_0, \tau)$ (sometimes this is simply function $y(\tau)$, $\lambda(\tau)$, which we shall prove below with examples).

Then relationship (6.5.24) will pass to

$$\tilde{P}(\lambda, t+\Delta) = C(t, \Delta) P_{t,\Delta}(y(t)|\lambda) \int \tilde{P}(\mu, t) W(\lambda, t+\Delta|\mu, t)\, d\mu. \tag{6.5.27}$$

Here

$$P_{t,\Delta}(y(t)|\lambda) = \exp \int_{t}^{t+\Delta} l(y(s), \lambda(s), \tau)\, d\tau$$

— likelihood function in interval $(t, t + \Delta)$, in which we consider the parameter to vary so little that it is practically possible to identify $\lambda(t)$ with its value at the end of the interval.

We shall be interested in parameters in the form of diffusion processes with density of the probability of transition satisfying the Fokker-Planck equation

(see § 6.3):

$$\frac{\partial}{\partial t} W(\lambda, t \mid \lambda_0, t_0) = -\frac{\partial}{\partial \lambda} [A(\lambda) W] + \frac{1}{2} \frac{\partial^2}{\partial \lambda^2} [B(\lambda) W] \equiv Q\{W(\lambda, t \mid \lambda_0, t_0)\}, \qquad (6.5.28)$$

where Q{ } — linear differential operator.

If we consider that $W(\lambda, t|\lambda_0, t_0)$ satisfies the condition of continuity

$$\lim_{t \to t_0} W(x, t \mid x_0, t_0) = \delta(x - x_0),$$

from (6.5.28) it follows that for small Δ

$$\int \tilde{P}(\mu, t) \frac{1}{\Delta} [W(\lambda, t + \Delta \mid \mu, t) - W(\lambda, t \mid \mu, t)]\, d\mu \approx Q\{\tilde{P}(\lambda, t)\}$$

or

$$\int \tilde{P}(\mu, t) W(\lambda, t + \Delta \mid \mu, t)\, d\mu \approx \tilde{P}(\lambda, t) + Q\{\tilde{P}(\lambda, t)\} \Delta. \qquad (6.5.29)$$

We now turn out attention to the fact that for small Δ there is satisfied

$$P_{t,\Delta}(y(t) \mid \lambda) \approx 1 + \Delta \left[\frac{1}{\Delta} \int_t^{t+\Delta} l(y(s), \lambda(s), \tau)\, d\tau \right], \qquad (6.5.30)$$

where the quantity contained in brackets is a finite quantity as $\Delta \to 0$ and from considerations of statistical equivalence can be identified with the integrand. We also consider that $C(t, \Delta)$ for small values of Δ is close to 1, so that it is possible to consider

$$C(t, \Delta) \approx 1 - \nu\Delta, \qquad (6.5.31)$$

where ν is still to be determined. Expanding $\tilde{P}(\lambda, t + \Delta)$ in a series with respect to Δ, substituting in (6.5.27) relationships (6.5.28)-(6.5.31) and being limited everywhere to first powers of Δ, we have equation

$$\frac{\partial}{\partial t} \tilde{P}(\lambda, t) = Q\{\tilde{P}(\lambda, t)\} + [l(\lambda, t) - \nu] \tilde{P}(\lambda, t). \qquad (6.5.32)$$

If we integrate both parts of (6.5.32) over λ and consider decrease of $\tilde{P}$ as $|\lambda| \to \infty$, it turns out that

$$\nu = \int l(\lambda, t) \tilde{P}(\lambda, t)\, d\lambda \equiv \overline{l(\lambda, t)},$$

from which we finally have the differential equation for a posteriori probability,

$$\frac{\partial}{\partial t} \tilde{P}(\lambda, t) = Q\{\tilde{P}(\lambda, t)\} + [l(\lambda, t) - \overline{l(\lambda, t)}] \tilde{P}(\lambda, t). \qquad (6.5.33)$$

Thus, with continuous observation the evolution of a posteriori probability is given by a differential equation of the Fokker-Planck type, the right side of which

contains [illegible] linear operator G, characterizing the transition function, and a component which depends on the input realization. The first term leads to "spreading" of $\widetilde{P}(\cdot, t)$ due to regular and random changes of the parameter; the second, [illegible], leads to "narrowing" of peak of $\widetilde{P}(\lambda, t)$ due to newly arrived information. The measure of information is the proportional difference between the current [illegible] of the logarithm of the likelihood function and the value of the same function, averaged over the a posteriori distribution, which is formed by data arriving earlier.

In §[illegible].3 we already said that to solve an equation of diffusion type is [illegible] even in the absence of randomly varying coefficients in the right part. In the given case the second term contains random components of the type of white noise, but for forming the a posteriori probability it is sufficient to find a technically convenient method of its simulation. Such a problem is solved, for instance, in [[illegible]]. Here we shall avoid simulation of general solutions and their investigation, [illegible] differential equations for certain basic characteristics of the a posteriori distribution.

Let us consider the case of a small a posteriori inaccuracy, when $\widetilde{P}(\lambda, t)$ is a narrow (as compared to the a priori distribution), isolated peak on the axis of values of the parameter. In view of the imposition of a great number of random factors it is reasonable to assume that $\widetilde{P}(\lambda, t)$ with sufficient accuracy is described by a Gaussian curve*:

$$\widetilde{P}(\lambda, t)=\frac{1}{\sqrt{2\pi}\,\sigma(t)}\exp\left\{-\frac{[\lambda-\lambda_0(t)]^2}{2\sigma^2(t)}\right\}, \qquad ([illegible])$$

where $\lambda_0(t)$ — the current vertex of the a posteriori distribution, which we sometimes can take as the optimum estimate of $\lambda(t)$;

$\sigma^2(t)$ — the variance of the a posteriori distribution.

Substituting ([illegible].5.34) in ([illegible].5.33), expanding $A(\lambda)$, $B(\lambda)$, and $l(\cdot, t)$ [illegible] and equating coefficients for identical powers [illegible] $-\lambda_0$, we have equation

$$\left.\begin{aligned}
&\frac{d}{dt}\lambda_0(t)=A(\lambda_0(t))+l'(\lambda_0(t),\, t)\,\sigma^2(t),\\
&\frac{d}{dt}\sigma^2(t)=2\sigma^2(t)\,A'(\lambda_0(t))+B(\lambda_0(t))+\\
&\qquad+\sigma^4(t)\,l''(\lambda_0(t),\, t).
\end{aligned}\right\} \qquad ([illegible])$$

*There are not sufficiently convincing proofs of the applicability of this approximation for a Markovian parameter in [16-18]. A whole series of considerations

In the derivation of (6.5.35) we allowed for a small a posteriori scattering, in which coefficients $A(\lambda)$ and $B(\lambda)$ vary little, so that, e.g., $\sigma^2 B''(\lambda) \ll B(\lambda)$, $\sigma^2 A''(\lambda) \ll A(\lambda)$.

If $\lambda_0(t)$ is considered the optimum estimate, joint solution of equations (6.5.35) is a problem of optimum filtration for a small a posteriori inaccuracy. The solution, naturally, can be produced by a certain dynamic system, simulating these equations. However, detailed study of these equations in [16], where they were first obtained, was not given. The analysis of measurement of varying frequency given in [17, 18] does not exhaust the problem, inasmuch as relationships (6.5.35) in fact are not used there. Therefore, we delay schematic interpretation of these results to § 6.9 and give it after presenting basic results of the theory of optimum meters of Gaussian parameters. In the same place we will study recurrence relationships for curves of an a posteriori distribution, which for a pulse incoherent signal or a discrete observation can be derived directly from (6.5.24).

6.5.6. Limitations of Theory. General Theory of Bayes Filtration

We shall give results flowing from consideration and comparison of different branches of statistical theory in reference to the problem of meter synthesis. The basic deficiency of the theory of estimation is the requirement of constancy of the measured variable during the time of observation. The theory of linear filtration requires additivity of the measured variable and interferences, which, in practice, is usually not satisfied. The method of forming the a posteriori probability of Markovian processes, generally speaking, lacks these deficiencies, but it was developed independently of the theory of solutions, so that for its use in designing a meter one needs certain concretization of the results of this theory. At the same time the theory of solutions in principle permits formulating a general filtration problem, i.e., the problem of finding an optimum resolving filter, under conditions in no way limiting in engineering practice. A general theory of filtration is best constructed on the basis of the Bayes approach.

Again we consider random process $y(t)$ a mixture of useful components and various kinds of interferences, depending on the measured function $\lambda(t)$. As in Paragraph 6.5.5, parameter $\lambda(t)$ is a random function of times. Let $y(t)$ and $\lambda(t)$ take at moments $t_1 < t_2 \dots < t_n$ from interval (t_0, t) values $\{y_1, \dots, y_n\} = \mathbf{y}$, $\{\lambda_1, \dots, \lambda_n\} = \lambda$, respectively. Problem of Bayes filtration consists of construction

[FOOTNOTE CON'T FROM PRECEDING PAGE]
Justifying this approximation for a Gaussian parameter will be given in § 6.6.

of a vector function from samples $\hat{\lambda}(y)$ which is estimate λ, in some measure statistically close to the true realization of the parameter $\lambda(t)$.

The vector analog of the simple loss function is

$$l(\lambda, \hat{\lambda}) = C - \delta(\lambda - \hat{\lambda}) = C - \prod_i \delta(\lambda_i - \hat{\lambda}_i).$$

As it is easy to prove analogously to Paragraph 6.5.4, the optimum estimate here corresponds to a vector, turning into maximum a posteriori distribution (6.5.21). The system of equations of maximum a posteriori probability will be recorded

$$\frac{\partial}{\partial \lambda_j} \ln \tilde{P}(\lambda_1, \ldots, \lambda_n) = 0 \quad (j = 1, \ldots, n). \qquad (6.5.36)$$

In that same case, when the estimate can use only the preceding data, the solution is sought in the form

$$\hat{\lambda}_j = \hat{\lambda}_j(y_1, \ldots, y_j) \quad (j = 1, 2, \ldots).$$

The a posteriori probability density takes the form

$$\begin{aligned} \tilde{P}(\lambda_j) &\equiv P(\lambda_j \mid y_1, \ldots, y_j) = \\ &= \underbrace{\int \cdots \int}_{(j-1)} P(\lambda_1, \ldots, \lambda_j \mid y_1, \ldots, y_j)\, d\lambda_1 \ldots d\lambda_{j-1} = \\ &= k \underbrace{\int \cdots \int}_{(j-1)} P(y_1, \ldots, y_j \mid \lambda_1, \ldots, \lambda_j)\, P_0(\lambda_1, \ldots, \lambda_j)\, d\lambda_1 \ldots d\lambda_{j-1}, \end{aligned}$$

and system of equations (6.5.36) is replaced by

$$\frac{\partial}{\partial \lambda_j} \ln \tilde{P}(\lambda_j) = 0. \qquad (6.5.37)$$

The analog of the scalar quadratic loss function leading to minimum mean square error of measurement for any interval of measurement is in the vector case the quadratic formula

$$l(\lambda, \hat{\lambda}) = (\lambda - \hat{\lambda})^{+} \mathbf{B} (\lambda - \hat{\lambda}) = \sum_{i,j} B_{ij} (\lambda_i - \hat{\lambda}_i)(\lambda_j - \hat{\lambda}_j), \qquad (6.5.38)$$

where $\mathbf{B}$ – certain symmetric non-singular matrix.

Here, mean risk is equal to

$$R(P_0, \hat{\lambda}) = \iint (\lambda - \hat{\lambda})^{+} \mathbf{B} (\lambda - \hat{\lambda})\, P(\mathbf{y} \mid \lambda)\, P_0(\lambda)\, d\lambda\, d\mathbf{y}$$

and is a linear combination of two moments of errors of measurement, allowed at all moments of observation. Thereby, we minimize the ellipsoid of scattering of all joint estimates, which is the most general requirement on it.

In the particular case when matrix B is diagonal, mean risk is simply the weighted sum of variances of errors at separate moments of time. Variation of mean risk on the estimate vector $\hat{\lambda}$ gives a system of equations, analogous to (6.5.11), outside of dependence on the concrete form of matrix B:

$$\int (\lambda - \hat{\lambda}) P(\mathbf{y} \mid \lambda) P_0(\lambda) \, d\lambda = 0. \tag{6.5.39}$$

Again the formal solution of (6.5.39) is the conditional matnematical expectation of vector

$$\hat{\lambda}(\mathbf{y}) = \frac{\int \lambda P(\mathbf{y} \mid \lambda) P_0(\lambda) \, d\lambda}{\int P(\mathbf{y} \mid \lambda) P_0(\lambda) \, d\lambda}. \tag{6.5.40}$$

In expanded form one should understand expressions (6.5.39) and (6.5.40) as

$$\left.\begin{array}{c} \underbrace{\int \cdots \int}_{(n)} (\lambda_j - \hat{\lambda}_j) P(y_1, \ldots, y_n \mid \lambda_1, \ldots, \lambda_n) \times \\ \times P_0(\lambda_1, \ldots, \lambda_n) \, d\lambda_1, \ldots, d\lambda_n = 0, \\ \hat{\lambda}_j = \dfrac{\underbrace{\int \cdots \int}_{(n)} \lambda_j P(y_1, \ldots, y_n \mid \lambda_1, \ldots, \lambda_n) P_0(\lambda_1, \ldots, \lambda_n) \, d\lambda_1, \ldots, d\lambda_n}{\underbrace{\int \cdots \int}_{(n)} P(y_1, \ldots, y_n \mid \lambda_1, \ldots, \lambda_n) P_0(\lambda_1, \ldots, \lambda_n) \, d\lambda_1, \ldots, d\lambda_n} \\ (j = 1, 2, \ldots, n) \end{array}\right\} \tag{6.5.41}$$

and use n relationships jointly.

When the operator of estimation uses only preceding data, there is taken the set of last relationships from (6.5.41):

$$\left.\begin{array}{c} \underbrace{\int \cdots \int}_{(j)} (\lambda_j - \hat{\lambda}_j) P(y_1, \ldots, y_j \mid \lambda_1, \ldots, \lambda_j) \times \\ \times P_0(\lambda_1, \ldots, \lambda_j) \, d\lambda_1, \ldots, d\lambda_j = 0, \\ \hat{\lambda}_j = \dfrac{\underbrace{\int \cdots \int}_{(j)} \lambda_j P(y_1, \ldots, y_j \mid \lambda_1, \ldots, \lambda_j) P_0(\lambda_1, \ldots, \lambda_j) \, d\lambda_1, \ldots, d\lambda_j}{\underbrace{\int \cdots \int}_{(j)} P(y_1, \ldots, y_j \mid \lambda_1, \ldots, \lambda_j) P_0(\lambda_1, \ldots, \lambda_j) \, d\lambda_1, \ldots, d\lambda_j}. \end{array}\right\} \tag{6.5.42}$$

Here the estimate for λ_j is found once, when $t = t_j$; more precise definition of this estimate after finding the new segment of realization of y(t) is not produced. Therefore the solution of (6.5.41) does not coincide with the solution of (6.5.42), with the exception of the last moment of observation, for which corresponding equations from (6.5.41) and (6.5.42) coincide.

For the multidimensional case all those basic tenets which were considered in detail in the theory of estimation (Paragraph 6.5.3) remain basically in force. In particular, there remains in force that proposition that with a posteriori

distribution of form

$$\tilde{P}(\lambda)=F(\lambda-\lambda_0),$$

where F(x) — arbitrary function, symmetric relative to zero, and for a symmetric loss function $(I(\lambda - \hat{\lambda} = I(\hat{\lambda} - \lambda))$ there exists a certain unique solution of the problem of Bayes filtration, coinciding with the operator of conditional mathematical expectation (6.5.40). Proof of this repeats that for the scalar case.

Besides the property that the operator of mathematical expectation (6.5.40) minimizes on the average a large class of loss functions, it also always reduces to zero the value of errors of measurement, averaged in the ensemble of the measured parameter $\lambda(t)$ and signal $y(t)$. This property is expressed by equality

$$\langle\hat{\lambda}\rangle_{\lambda,y}=\langle\lambda\rangle_{\lambda}, \tag{6.5.43}$$

where brackets denote averaging. Relationship (6.5.43) is easy to obtain by multiplying both parts of (6.5.40) by the joint probability density $P(\mathbf{y}|\boldsymbol{\lambda})P_0(\boldsymbol{\lambda})$ of $\mathbf{y}$ and $\boldsymbol{\lambda}$ and integrating over $\mathbf{y}$ and $\boldsymbol{\lambda}$. The property of unbiasedness is especially useful in view of the presence of interfering signals structurally similar to the useful one. Complete absence of bias error in an optimum measuring system in these conditions is far from obvious.

All the given results emphasize the universal character of conditional mathematical expectation. The condition of symmetry of the a posteriori probability, formulated above, is satisfied in practice by a narrow level of signals, and the requirement of symmetry of the loss function is not a substantial limitation at all. In these conditions the theory of optimum filtration is freed from subjective elements connected with arbitrariness in selection of the loss function.

Considering the very attractive properties of the operator of conditional mathematical expectation, one would think it remains only to decipher the mathematical operation in (6.5.40) and translate it into radio engineering language. However, rigorous integration can be performed only in the additive case, when

$$y(t)=n(t)+\lambda(t),$$

where $\lambda(t)$, $n(t)$ — random processes with normal distribution. Here, we arrive at the known Wiener filter. In other cases to rigorously and sufficiently simply decode the operation of conditional mathematical expectation is impossible. Inasmuch as it appears certain that the method of construction of optimum meters should be based on the above-presented theory, the problem of technical creation of an optimum circuit has still greater importance than discovery of general operators.

§ 6.6. Methods of Synthesis of Optimum Meters with Gaussian Statistics of the Parameter. Potential Accuracy of Measurement

6.6.1. Properties of the Likelihood Function and Its Approximation

For the time being we shall assume that the measured parameter λ is constant. Then in the case of Gaussian noises, as was shown in [4, 12], the likelihood function of one parameter monotonically depends on the sum of the autocorrelation function of the useful component of the mixture y(t) and a certain random function, where both functions have as their argument the measured parameter. For illustration we shall consider the likelihood function of time delay of a regular signal u(t), received against a background of white noise with spectral density N_0. The likelihood function here has the form

$$P(y(t)|\tau)=C\exp\left\{-\frac{1}{2N_0}\int_0^T[y(t)-u(t-\tau)]^2\,dt\right\}, \qquad (6.6.1)$$

where y(t) — the observed realization;

τ — time delay;

C — constant, not depending on y and τ.

After substitution in (6.6.1) of the form of realization

$$y(t)=u(t-\tau_0)+n(t),$$

where τ_0 — true (unknown) value of delay, it is possible to separate members, explicitly depending on τ:

$$P(y(t)|\tau)=C'\exp\left\{\frac{1}{N_0}\int_0^T u(t-\tau)u(t-\tau_0)\,dt+\right.$$

$$\left.+\frac{1}{N_0}\int_0^T u(t-\tau)n(t)\,dt\right\}.$$

The first component under the sign of the exponential function is the autocorrelation function with respect to parameter τ, depending on difference $\tau - \tau_0$; the second is the noise disturbance, realizations of which are "signal-like" and stationary with respect to τ [4]. Analogous dependences occur, too, in more complicated cases [12]. With Gaussian input noises the autocorrelation function turns out to be in the

exponent of the exponential function. The exponential function emphasizes the top of the peak of the autocorrelation function and makes the slopes steeper. This leads to an approximately Gaussian curve of the likelihood function in conditions of a smooth top of the autocorrelation function, low level of its side peaks, of which we spoke in Chapter 1, and low level of interferences (Fig. 6.10).

With increase of time of observation the mean value of the exponent of the exponential function increases. As a result peak $P(\mathbf{y}|\lambda)$ narrows, and scattering of its position relative to the true value $\lambda_{\text{И}}$ decreases. This is explained by the fact that $P(\mathbf{y}|\lambda)$ characterizes a result averaged for all observations, in which random disturbances to a certain extent compensate one another.

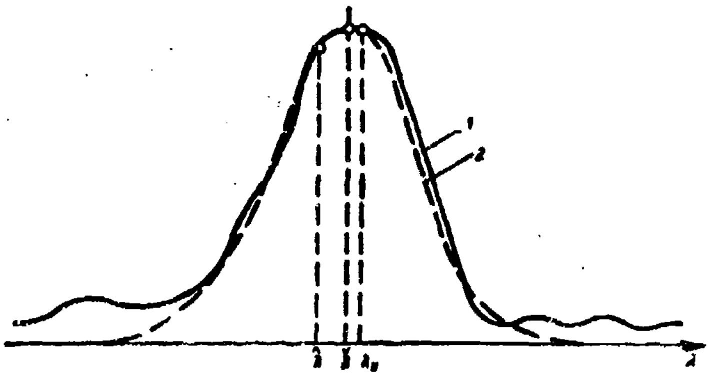

Fig. 6.10. Likelihood function of a constant parameter and its approximation: $\hat{\lambda}$ — value of optimum estimate; $\check{\lambda}$ — maximum likelihood value; $\lambda_{\text{И}}$ — true value of parameter; 1) true dependence; 2) Gaussian approximation.

If we turn now to the case of a parameter varying in time by arbitrary law, in general we can only say a little about the form of the likelihood function.

In general, this function will be multi-dimensional, and with transition to continuous observation it will be a functional, and should be considered in the space of values of a parameter, occurring at separate moments of observation. The produced measurements, forming for a constant parameter a narrowed peak, will for a varying parameter give results depending both on the concretely occurring value of the fluctuating disturbance, and also on the law of change of the parameter.

Sometimes it is possible, indeed, by selection of new parameter which is a function (functional) of the old parameter (for instance, speed, amplitude of the Fourier-harmonic, etc), to arrive at the case of a constant parameter. However, such cases are more the exception and cannot serve as a basis for analysis.

Meanwhile, on certain assumptions, made below, the form of the likelihood function found for a constant measured parameter permits us to qualitatively determine the form of this function for a varying parameter $\lambda(t)$ and on this basis

to find a satisfactory form of its approximate presentation.

In practice usually most interesting are the cases of sufficiently high-quality work of meters, when noises are not too great (but, ultimately, are also not too small, otherwise the problem of optimization loses its importance). Under this condition errors of measurement will be small and the a posteriori distribution will be narrow as compared to the a priori.

Frequently, the rate of change of the measured parameter is very slow as compared to the rate of change of all other random variables in the observed mixture, which are not subject to measurement and for which we realize averaging when finding the explicit form of likelihood function (henceforth we will call them immaterial parameters).

We choose time intervals considerably larger than the interval of statistical coupling of the slowest of the immaterial parameters of the input signal $y(t)$, but considerably smaller than the interval of change of $\lambda(t)$. Upon fulfillment of the shown conditions the logarithm of the likelihood function, as a function of the value of the parameter frozen on these segments, near the main peak can be approximated by its own quadratic expansion for any point, knowingly close to the value of the measured variable. In distinction from the case of a constant parameter the peak of the likelihood function at every subinterval has a finite width, randomly depending on $y(t)$ and the coded function $\lambda(t)$. The form of the likelihood function in the subintervals is illustrated in Fig. 6.11. The multi-dimensional likelihood function in the whole interval of observation here is approximated by the product of likelihood functions obtained in separate subintervals due to the statistical independence of fluctuations of the immaterial parameters. Considering everything we have said and still holding to cases of discrete samplings, we can write the basic multi-dimensional approximation of $P(\mathbf{y}|\boldsymbol{\lambda})$ in the form

$$P(\mathbf{y}\,|\,\boldsymbol{\lambda}) = \exp\Big\{\ln P(\mathbf{y}\,|\,\boldsymbol{\lambda}_0) + \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i} \ln P(\mathbf{y}\,|\,\boldsymbol{\lambda}_0)(\lambda_i - \lambda_{0i}) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial \lambda_i \partial \lambda_j} \ln P(\mathbf{y}\,|\,\boldsymbol{\lambda}_0)(\lambda_i - \lambda_{0i})(\lambda_j - \lambda_{0j})\Big\}, \qquad (6.6.2)$$

where $\boldsymbol{\lambda}_0(t)$ – an arbitrary function, considered at moments of observation t_1, t_2, ..., t_n and assumed close to the true value of the varying parameter.

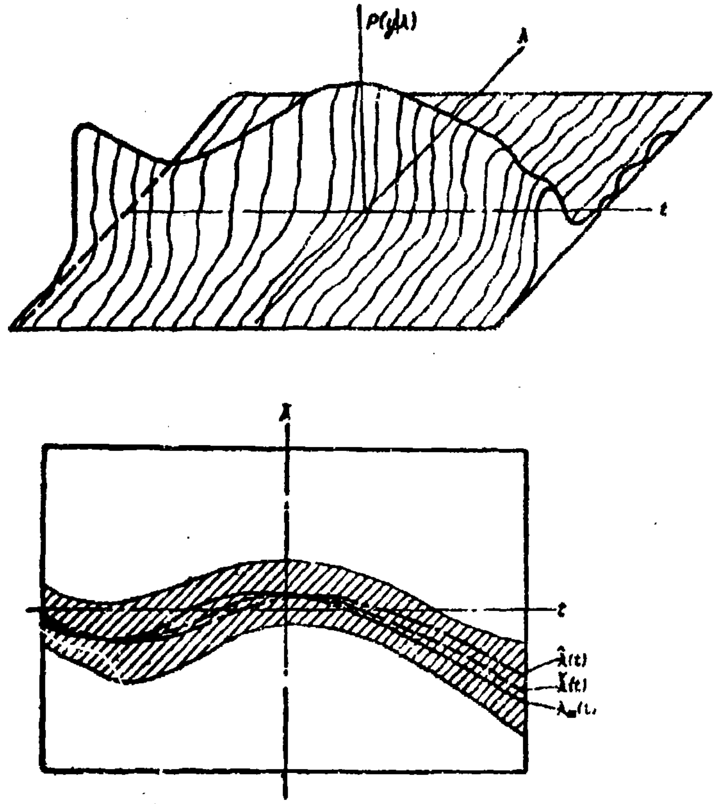

Fig. 6.11. Likelihood function of a variable parameter in the plane of the measured parameter and time.

An approximation of type (6.6.2) is used in the theory of estimation [see formula (6.5.13)] for finding operations of formation of an estimate and proof of asymptotic efficiency of maximum likelihood estimates. Here, there is proven the asymptotic convergence of the quadratic expansion of $\ln P(\mathbf{y}|\lambda)$ to its true value.

Extension of such an expansion to the case of a variable parameter, in general, should be firmly grounded. However, at present a sufficiently satisfactory direct solution of this problem does not exist. Therefore, we will postpone proof of the applied presentation of the likelihood function until we investigate the accuracy of the obtained optimum circuits. In this investigation we shall show that accuracy of meters synthesized by the shown expansion of the likelihood function practically coincides with the maximum possible (potential) accuracy, which proves the validity of this expansion.

Subsequently we will need two concrete expansions of type (6.6.2). The first is conducted at the point of the optimum estimate of parameter $\boldsymbol{\lambda}_c = \hat{\boldsymbol{\lambda}}$ (the meaning of optimality will be explained below). Designating

$$\ln P(\mathbf{y}|\boldsymbol{\lambda}) = L(\boldsymbol{\lambda}), \quad \frac{\partial L(\hat{\boldsymbol{\lambda}})}{\partial \lambda_i} = z_i, \quad -\frac{\partial^2 L(\hat{\boldsymbol{\lambda}})}{\partial \lambda_i \partial \lambda_j} = A_{ij}$$

...d introducing matrix notation with column-vectors $\mathbf{z} = \{z_1, \ldots, z_n\}$, $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_n\}$ and $\hat{\boldsymbol{\lambda}} = \{\hat{\lambda}_1, \ldots, \hat{\lambda}_n\}$ and square matrix $\mathbf{A} = \| A_{ij} \|$, we have instead of (6.6.2)

$$P(\mathbf{y}|\boldsymbol{\lambda}) = P(\mathbf{y}|\hat{\boldsymbol{\lambda}}) \exp\left\{\mathbf{z}^+ (\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}}) - \frac{1}{2} (\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}})^+ \mathbf{A} (\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}})\right\}. \qquad (6.6.3)$$

The second expansion is conducted at the point $\boldsymbol{\lambda}_0 = \check{\boldsymbol{\lambda}}$ of the maximum value of $P(\mathbf{y}|\boldsymbol{\lambda})$ and has the form

$$P(\mathbf{y}|\boldsymbol{\lambda}) = P(\mathbf{y}|\check{\boldsymbol{\lambda}}) \exp\left\{-\frac{1}{2} (\boldsymbol{\lambda} - \check{\boldsymbol{\lambda}})^+ \check{\mathbf{A}} (\boldsymbol{\lambda} - \check{\boldsymbol{\lambda}})\right\}. \qquad (6.6.4)$$

The point of the current position of maximum likelihood $\check{\lambda}(t)$ in general does not coincide with the point of the optimum estimate $\hat{\lambda}(t)$. Actually, according to § 6.5 the latter is the exact maximum of the a posteriori probability, calculated on the basis of the whole interval of observation, so that $\check{\lambda}(t)$ and $\hat{\lambda}(t)$ are correlated as the result of instantaneous measurement and as the smoothed output quantity.

The term with the first derivative in (6.6.4) is rejected as equal to zero, and matrix $\check{\mathbf{A}}$ is determined through derivatives at point $\check{\boldsymbol{\lambda}}$:

$$\check{\mathbf{A}} = \left\| -\frac{\partial^2 L(\check{\boldsymbol{\lambda}})}{\partial \lambda_i \partial \lambda_j} \right\|. \qquad (6.6.5)$$

Relationships (6.6.3) and (6.6.4) will be used below during the synthesis of basic circuits of optimum meters. For clarity in Fig. 6.10 we give a one-dimensional illustration of these expansions.

Let us assume that we nevertheless wanted to more closely account for the structure of the likelihood function. It is expressed with side peaks, determined as side peaks of the autocorrelation function of the signal (see Chapter I), and also purely noise overshoots, which by force of properties of the likelihood function are "signal-similar" in structure. Inasmuch as the exponential function always stresses maxima and suppresses minima of the function which is its argument, for these peaks it is possible to offer a Gaussian approximation. Side peaks of the

autocorrelation function of a signal are rigidly joined in position with the basic peak, and for them it is possible to offer two local approximations, analogous to (6.6.3) and (6.6.4):

$$P(\mathbf{y}|\boldsymbol{\lambda}) \approx P(\mathbf{y}|\hat{\boldsymbol{\lambda}} + \Delta_{(i)}) \exp\left\{z_{(i)}^{+}(\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}} - \Delta_{(i)}) - \right.$$
$$\left. - \frac{1}{2}(\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}} - \Delta_{(i)})^{+} \mathbf{A}_{(i)} (\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}} - \Delta_{(i)})\right\}, \qquad (6.6.6)$$

$$P(\mathbf{y}|\boldsymbol{\lambda}) \approx P(\mathbf{y}|\check{\boldsymbol{\lambda}}_{(i)}) \exp\left\{-\frac{1}{2}(\boldsymbol{\lambda} - \check{\boldsymbol{\lambda}}_{(i)})^{+} \check{\mathbf{A}}_{(i)} (\boldsymbol{\lambda} - \check{\boldsymbol{\lambda}}_{(i)})\right\}. \qquad (6.6.7)$$

Here $\Delta_{(i)}$ — known displacement of the middle of the i-th peak from the center of the basic peak;

$\hat{\boldsymbol{\lambda}}$ — as before, the estimated value of $\boldsymbol{\lambda}$;

$\boldsymbol{\lambda}_{(i)}$ — measured position of i-th maximum;

$$z_{(i)} = \frac{\partial}{\partial \lambda} \ln P(\mathbf{y}|\hat{\boldsymbol{\lambda}} + \Delta_{(i)}),$$

and matrices $\mathbf{A}_{(i)}$, $\check{\mathbf{A}}_{(i)}$ consist of values of second derivatives of $-\ln P(\mathbf{y}|\boldsymbol{\lambda})$, taken for relationships (6.6.6) and (6.6.7), respectively at points $\hat{\boldsymbol{\lambda}} + \Delta_{(i)}$ and $\check{\boldsymbol{\lambda}}_{(i)}$. Properties of quantities $\mathbf{z}_{(i)}$, $\mathbf{A}_{(i)}$, $\check{\boldsymbol{\lambda}}_{(i)}$ can be investigated for every concrete form of $P(\mathbf{y}|\boldsymbol{\lambda})$ absolutely the same as properties of quantities $\mathbf{z}$, $\mathbf{A}$, $\check{\boldsymbol{\lambda}}$ from (6.6.3) and (6.6.4).

If, however, we turn to analysis of side noise peaks, then for them, obviously, only approximation (6.6.7) is suitable, and properties of functions $P(\mathbf{y}|\boldsymbol{\lambda}_{(i)})$ and $\mathbf{A}_{(i)}$ will be different here. Expression (6.6.6) is inapplicable, since the position of noise peaks is not connected with the true value of λ.

Approximations (6.6.6) and (6.6.7) will be used below for deeper study of the structure of optimum meters.

6.6.2. Optimum Tracking Meter for Gaussian Statistics of the Measured Parameter

Now, when there has been prepared a mathematical description of the likelihood function, it is necessary to give the concrete form of the multi-dimensional distribution of probabilities of the measured parameter.

Below right up to § 6.9 we will be interested in Gaussian distribution, which, besides mathematical convenience, is explained by a number of other factors.

First, in a considerable number of important applications the random character of change of $\lambda(t)$ is determined only by the randomness of values of certain initial conditions. The latter do not depend on the will of man and when there is a great

number of disturbing factors they can be considered normally distributed, which leads to a normal (or close to it) distribution of $\lambda(t)$.

Second, change in time of the actual parameter $\lambda(t)$ frequently occurs under the action of a large number of independent random disturbances, which also leads to a normal distribution of probabilities. Finally, the statistics of the parameter may be incompletely known, and may be given only by a correlation function. As we shall prove in § 6.9, with such limited knowledge the assumption of normal distribution ensures the best results as compared to any possible distributions with a given correlation function.

Thus, let us assume that the a priori distribution has the following form:

$$P(\boldsymbol{\lambda})=(2\pi)^{-n/2}[\det \mathbf{R}]^{-1/2}\exp\left\{-\frac{1}{2}(\boldsymbol{\lambda}-\bar{\boldsymbol{\lambda}})^{+}\mathbf{V}(\boldsymbol{\lambda}-\bar{\boldsymbol{\lambda}})\right\}, \tag{6.6.8}$$

where $R = \| R(t_i, t_j) \|$ – correlation matrix of random component $\lambda(t)$;

$\mathbf{V} = \mathbf{R}^{-1}$ – its inverse matrix [in interval (t_0, t_n)];

$\bar{\boldsymbol{\lambda}} = \{\bar{\lambda}_1, \ldots, \bar{\lambda}_n\}$ – column-vector of mean value.

Let us turn to relationship (6.5.42) for a physically realizable operator of filtration.

The first method of finding an optimum meter is direct integration. Substituting (6.6.3) and (6.6.8) in (6.5.42) and designating

$$\boldsymbol{\chi}=\hat{\boldsymbol{\lambda}}-\bar{\boldsymbol{\lambda}}, \quad \boldsymbol{\varepsilon}=\boldsymbol{\lambda}-\hat{\boldsymbol{\lambda}}, \tag{6.6.9}$$

under the sign of the integral performing transformation of variables $(\lambda \to \varepsilon)$ and rejecting factors not depending on ε, we have equation

$$\int\cdots\int \varepsilon_n \exp\left\{\mathbf{z}^{+}\boldsymbol{\varepsilon}-\frac{1}{2}\boldsymbol{\varepsilon}^{+}\mathbf{A}\boldsymbol{\varepsilon}-\right.$$
$$\left.-\frac{1}{2}(\boldsymbol{\varepsilon}+\boldsymbol{\chi})^{+}\mathbf{V}(\boldsymbol{\varepsilon}+\boldsymbol{\chi})\right\}d\varepsilon_1\ldots d\varepsilon_n=0.$$

After grouping terms, using symmetry of matrix $\mathbf{V}$, we obtain from this

$$\int \varepsilon_n \exp\left\{[\mathbf{z}-\mathbf{V}\boldsymbol{\chi}]^{+}\boldsymbol{\varepsilon}-\frac{1}{2}\boldsymbol{\varepsilon}^{+}(\mathbf{A}+\mathbf{V})\boldsymbol{\varepsilon}\right\}d\boldsymbol{\varepsilon}=$$
$$=\frac{\partial}{\partial b_n}\int \exp\left\{\mathbf{b}^{+}\boldsymbol{\varepsilon}-\frac{1}{2}\boldsymbol{\varepsilon}^{+}(\mathbf{A}\qquad)\boldsymbol{\varepsilon}\right\}d\boldsymbol{\varepsilon}=0$$
$$(d\boldsymbol{\varepsilon}=d\varepsilon_1\ldots d\varepsilon_n), \tag{6.6.10}$$

where there is designated

$$\mathbf{b}=\{b_1,\ldots, b_n\}=\mathbf{z}-\mathbf{V}\boldsymbol{\chi}. \tag{6.6.11}$$

Introducing matric $\mathbf{C} = \| C_{ik} \| = [\mathbf{A} + \mathbf{V}]^{-1}$, the inverse of matrix $\mathbf{A} + \mathbf{V}$ in interval (t_0, t_n), instead of (6.6.10), according to [8] we have equation

$$\frac{\partial}{\partial b_n} \exp\left\{-\frac{1}{2}\mathbf{b}^+\mathbf{C}\mathbf{b}\right\} = 0. \qquad (6.6.12)$$

Differentiation in (6.6.12) and substitution of the value of $\mathbf{b}$ from (6.6.11) then gives

$$\sum_{k=1}^{n} C_{nk}\left[z_k + \sum_{i=1}^{n} V_{ki}\chi_i\right] = 0.$$

From this, adding and subtracting from V_{ki} an element of matrix $\mathbf{A}$, considering that $(\mathbf{A} + \mathbf{V})\mathbf{C} = \mathbf{I}$, where $\mathbf{I}$ — unit matrix, and expanding the value of χ, we finally have

$$\hat{\lambda}_n = \sum_{k=1}^{n} C_{nk}\left[z_k + \sum_{i=1}^{n} A_{ki}(\hat{\lambda}_i - \bar{\lambda}_i)\right] + \bar{\lambda}_n. \qquad (6.6.13)$$

Another method of finding of an optimum meter is based on direct study of the expression for the a posteriori probability of $\lambda(t)$. This method in this case is even more convenient, although in general it is not always applicable. After multiplication of the a priori distribution (6.6.8) by approximation of the likelihood function (6.6.3) we bring the logarithm of the formed function to form

$$\ln \tilde{P}(\lambda) = C_0 - \frac{1}{2}(\lambda - \lambda_0)^+ \mathbf{C}^{-1}(\lambda - \lambda_0), \qquad (6.6.14)$$

where scalar C_0, vector $\boldsymbol{\lambda}_0$ and matrix $\mathbf{C}^{-1}$ do not depend on the current argument of $\boldsymbol{\lambda}$. Determining coefficients for different powers of $\boldsymbol{\lambda}$, it is easy to prove that

$$\mathbf{C}^{-1} = (\mathbf{A} + \mathbf{V}), \qquad (6.6.15)$$

$$\boldsymbol{\lambda}_0 = \mathbf{C}[\mathbf{z} + \mathbf{A}(\hat{\boldsymbol{\lambda}} - \bar{\boldsymbol{\lambda}})] + \bar{\boldsymbol{\lambda}}, \qquad (6.6.16)$$

where $\mathbf{C}$ — matrix, the inverse of $\mathbf{C}^{-1}$.

At the same time from the very form of expression (6.6.14) it follows that $\boldsymbol{\lambda}$ has the sense of that point of symmetry (in multi-dimensional space) of the a posteriori distribution which we mentioned in § 6.5. Naturally, it should also correspond to the optimum estimate $\boldsymbol{\lambda}_0 = \hat{\boldsymbol{\lambda}}$. Then from relationship (6.6.16), considered at the last moment of observation t_n, there again ensues expression (6.6.13), determining the optimum meter.

Relationship (6.6.13) relates the value of estimate $\hat{\lambda}_n$ at the last moment of observation with estimates $\hat{\lambda}_k$ in all the preceding moments and in essence is a nonlinear equation for quantities $\hat{\lambda}_1, \ldots, \hat{\lambda}_n$. This nonlinearity is caused by the

fact that values of estimate $\hat{\lambda}$ enter the equation not only directly but also through the coefficient of expansion of the likelihood function $\mathbf{z} = \mathbf{z}(\hat{\lambda})$, $\mathbf{A} = \mathbf{A}(\hat{\lambda})$. If we are drawn away from this dependence, the optimum operator is linear with respect to vector $\mathbf{z} + \mathbf{A}(\hat{\lambda} - \bar{\lambda})$. The matrix of linear transformation $\mathbf{C}$, according to (6.6.15), satisfies equation

$$\mathbf{C}[\mathbf{I} + \mathbf{A}\mathbf{R}] = \mathbf{R} \tag{6.6.17}$$

or in expanded form

$$C_{nk} + \sum_{i,l=1}^{n} C_{ni} A_{il} R_{lk} = R_{nk}, \tag{6.6.18}$$

where, according to the principle of using only the preceding data, it is assumed that $C_{nk} = 0$ when $k > n$.

Relationship (6.6.13) during discrete observation gives the algorithm for construction of a dynamic system, simulating the process of continuous solution of (6.5.42) with use of former results of estimation and newly arriving data. All the prehistory of sampling $\mathbf{y}$ is depicted by estimates of the parameter made in earlier moments: for formation of new estimates there is no need to turn again to former values of the sampling . Such a property of operators of processing are very convenient technically.

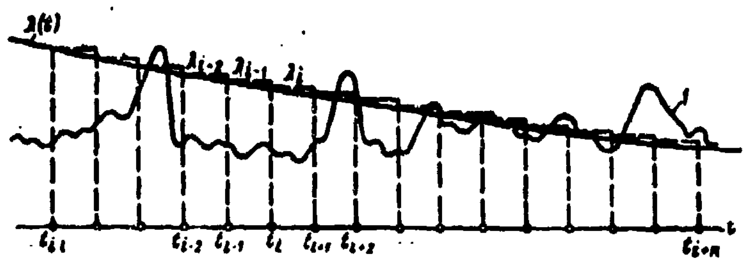

Fig. 6.12. Elementary intervals of observation: 1) envelope of realization of y(t).

Certain simplification of expressions (6.6.13) and (6.6.18) can be produced if we consider that for slow changes of $\lambda(t)$ matrix $\mathbf{A}$ is diagonal, i.e., consists of elements $A_{ik} = K_i \delta_{ik}$. Actually, intervals of statistical coupling of all immaterial parameters of the input mixture $y(t)$ are much smaller than the time of approximate constancy of the correlation interval of parameter $\lambda(t)$. As this is illustrated in Fig 6.12, every moment of observation t_i corresponds to its own frozen value of the parameter $\lambda(t_i)$. First differentiation of the logarithm of the likelihood function with respect to λ_i actually is produced only on the subinterval near t_i. Second differentiation with respect to t_j, which will form matrix $\mathbf{A}$, gives a result which differs from zero only when indices $i = j$. If we allow for this

circumstance, expression (6.6.13) and (6.6.18) will be rewritten in the form

$$\hat{\lambda}_n = \sum_{k=1}^{n} C_{nk}[z_k + K_k(\hat{\lambda}_k - \overline{\lambda}_k)] + \overline{\lambda}_n, \qquad (6.6.19)$$

$$C_{nk} + \sum_{i=1}^{n} C_{ni} K_i R_{ik} = R_{nk}. \qquad (6.6.20)$$

In general, as follows from analysis analogous to that given in [?], the diagonal nature of matrix **A** simultaneously bears witness to the fact that diagonal elements of this matrix with a certain approximation can be replaced by their own mean value ($K_i \approx \overline{K}_i$). However, below we shall show that this somewhat decreases accuracy of measurement, so that such far-reaching simplification is inexpedient.

By direct substitution we prove simply that (6.6.19) can be given a simpler form

$$\hat{\lambda}_n = \sum_{k=1}^{n} G_{nk} z_k + \overline{\lambda}_n, \qquad (6.6.21)$$

where triangular matrix **G** ($G_{nk} = 0$ when $n < k$) is determined by equation

$$C_{nk} + \sum_{i=k}^{n} C_{ni} K_i G_{ik} = G_{nk}. \qquad (6.6.22)$$

We shall establish the sense of quantity z_k. For this we represent this magnitude by a Taylor expansion in the vicinity of point λ_H coinciding with the true value of the parameter of the input mixture. It is obvious that

$$z_k = \frac{\partial L(\hat{\lambda})}{\partial \lambda_k} = \frac{\partial L(\lambda_n)}{\partial \lambda_k} + \sum_{i=1}^{n} \frac{\partial^2 L(\lambda_n)}{\partial \lambda_k \partial \lambda_i} (\hat{\lambda}_i - \lambda_{ni}) + \ldots \approx$$

$$\frac{\partial L(\lambda_n)}{\partial \lambda_k} - \sum_{i=1}^{n} A_{Hki} (\hat{\lambda}_i - \lambda_{Hi}) = \xi_k + K_{Hk}(\lambda_{Hk} - \hat{\lambda}_k), \qquad (6.6.23)$$

where matrix $\mathbf{A}_H = \left\| -\frac{\partial^2 L(\lambda_H)}{\partial \lambda_i \partial \lambda_k} \right\|$ and vector $\mathbf{K}_H$ differ from **A** and **K** only by the fact that values of their elements, as functions of the parameter, are taken, not at point $\hat{\boldsymbol{\lambda}}$, but at a point coinciding with the true value of the parameter λ_H. Inasmuch as $\hat{\boldsymbol{\lambda}}$ and $\boldsymbol{\lambda}$ are assumed close, properties of $\mathbf{A}_H$ and $\mathbf{K}_H$, on the one hand, and of **A** and **K** on the other hand, practically coincide. Quantities ξ_k in formula (6.6.23) do not depend on values of the estimate, and since, according to definition,

$$\xi_k = \frac{\partial L(\lambda_u)}{\partial \lambda_k} = \frac{\partial \ln P(y \mid \lambda_u)}{\partial \lambda_k}, \qquad (6.6.24)$$

it is easy to prove that ξ_k have zero (with respect to the ensemble of input signal y) mean values. In other words, discrete random process $\xi_k (k = 1, \ldots, n)$, taking place even for exact coincidence of the estimate with the true value $\lambda_И$ is a purely fluctuating disturbance.

The second component of z_k, presented in the form of (6.6.23), turns out to be proportional to current mismatch between the true and estimated values of the parameter. Proportionality factor K_k is a random variable with a certain positive mean value. Thus, according to (6.6.23) z_k is the measure of mismatch between the true and estimated values of the parameter. For small $\lambda_{И k} - \hat{\lambda}_k$ this measure on average is linear. In general substantial too are terms with higher powers of $\lambda_{И k} - \hat{\lambda}_k$, so that z_k can be presented by a Taylor expansion in powers of $\lambda_{И k} - \hat{\lambda}_k$ with random coefficients depending only on the realization of the input signal. The described properties of z_k qualitatively coincide with properties of output signals of devices known in practice for detection of the error signal — discriminators (see § 6.2).

Let us turn now to clarification of the sense of quantity K_k, which is the current value of the second derivative of the logarithm of the likelihood function. If parameter λ is to be fixed, the mean value of this derivative, taken at the point of true value $\lambda_И$, will completely characterize accuracy of measurement of the parameter. As already indicated, due to the proximity of $\lambda_И$ and $\hat{\lambda}$ values of the second derivative at two these points practically coincide. Then it may be concluded that the value of K_k characterizes the current accuracy of the relative measurement of the parameter in the k-th moment of observation.

From the theory of estimation, and also the method of least squares we know that for unequal errors of separate measurements results of these measurements should be used for calculation of the final result with weighting factors, inversely proportional to dispersions of separate independent measurements, considered known. Our analysis showed that allowance for unequal accuracy should also be made in the considered case of an optimum meter, where the measure of unequal accuracy, inversely proportional to dispersion, here is the current value of K_i, formed in the actual circuit of the meter.

Obtaining relationships (6.6.19) and (6.6.21) and investigating the ... z_k and K_k, it would be possible to pass to interpretation of ... relationships in the form of block diagrams realizing an optimum filter-meter. However, this is not fully conveniently to do when we have discrete observation, inasmuch as it would be necessary to explain, e.g., how it is possible in a circuit to form a discrete sampling of the input mixture, usually a high-frequency signal. We shall return in the next point to cases when discreteness indeed is justified from physical or technical considerations; but meanwhile we turn to continuous observation, ... direct practical application in radar.

With the help of passage to the limit $t_i - t_{i-1} = \Delta \to 0$, $n \to \infty$ ($i = 1, 2, \ldots, n$; $n\Delta = t - t_0$) we obtain continuous analogs of formulas (6.6.19)-(6.6.24). Designating $t_n = t$, $t_k = \tau$, we note that according to (6.6.20)-(6.6.22) the character of the tendency to a limit of matrix elements of **C** and **G** is the same as for elements of matrix **R**, for which $R_{nk} = R(t_n, t_k) = R(t, \tau)$. Therefore, with passage to the limit one should introduce functions

$$\left.\begin{aligned} c(t, \tau) = c(t_n, t_k) = C_{nk}, \\ g(t, \tau) = g(t_n, t_k) = G_{nk}. \end{aligned}\right\} \qquad (6.6.25)$$

Then from formulas (6.6.19) and (6.6.23) it follows that for the existence of limits it is necessary that z_k seek zero, when $\Delta \to 0$, as Δ, and matrix element A seek zero as Δ^2. This requirement will be satisfied if z_k and A_{ik} can be presented in the form

$$z_k = \int_{t_k-\Delta/2}^{t_k+\Delta/2} z(s)\,ds, \qquad (6.6.26)$$

$$A_{ik} = \int_{t_i-\Delta/2}^{t_i+\Delta/2} ds \int_{t_k-\Delta/2}^{t_k+\Delta/2} d\tau A(s, \tau), \qquad (6.6.27)$$

where $z(t)$ and $A(t, \tau)$ -- certain random functions of time.

The form of these functions will be clarified in examples subsequently; however, it is useful to emphasize that relationship (6.6.26) determines a whole class of statistically equivalent, but generally not equal functions, which give in integration an identical result z_k. Thus, the continuous analog of z_k is determined ambiguously: any statistically equivalent functions will do.

In the same conditions in which matrix **A** is diagonal, i.e., $A_{ik} = K_i\delta_{ik}$, one should consider

$$A(t, \tau) = K(t)\delta(t-\tau), \qquad (6.6.28)$$

where K(t) — limit of quantity K_k/Δ as $\Delta \to 0$, and a δ-function will be formed at the limit from δ_{1k}/Δ. Then expressions (6.6.19)-(6.6.22) reduce to form:

$$\hat{\lambda}(t)=\int_{t_0}^{t} c(t, \tau)[z(\tau)+K(\tau)(\hat{\lambda}(\tau)-\overline{\lambda}(t))]\,d\tau+\overline{\lambda(t)}, \qquad (6.6.29)$$

$$\hat{\lambda}(t)=\int_{t_0}^{t} g(t, \tau)z(\tau)\,d\tau+\overline{\lambda(t)}, \qquad (6.6.30)$$

$$c(t, \tau)+\int_{t_0}^{t} c(t, s)K(s)R(s, \tau)\,ds=R(t, \tau), \qquad (6.6.31)$$

$$c(t, \tau)+\int_{\tau}^{t} c(t, s)K(s)g(s, \tau)\,ds=g(t, \tau). \qquad (6.6.32)$$

Relationship (6.6.29) can be presented by the block-diagram of Fig. 6.13. Input signal y(t) is fed to nonlinear elements 1 and 2. They form, correspondingly, quantities z(t) and K(t). Quantity K(t) directly determines the gain of inertialess amplifier K and, furthermore, as control is fed to filter C. Quantity z(t) is added to output voltage of amplifier K and is fed to inertial filter C, whose pulse response c(t, τ) depends on the magnitude of K(t).

After addition to the a priori mean value of $\overline{\lambda(t)}$ there will be formed an estimate of the parameter, $\hat{\lambda}(t)$. It is used to check characteristics of elements 1, 2. One more loop of intercoupling closes output of filter **C** through amplifier **K**.

The described interpretation circuit is a complicated self-tuning system, containing tunable nonlinear elements 1, 2 and linear filters C and K, pulse responses of which change with change of input signal y(t), i.e., are determined by random factors.

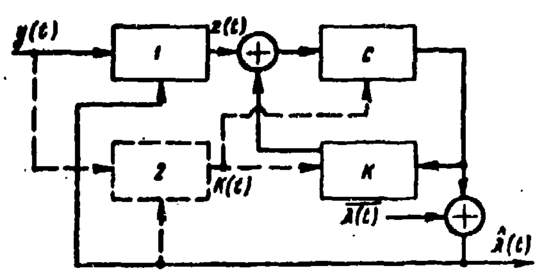

Fig. 6.13. Two-loop variant of the circuit of an optimum tracking meter: 1) discriminator; 2) accuracy unit; C — linear filter with pulse response c(t, τ); K — inertialess amplifier with gain factor K.

Let us give names to nonlinear elements 1 and 2. Making for z(t) an expansion analogous to (6.6.23), we obtain

$$z(t)\approx \xi(t)+K(t)(\lambda_{И}(t)-\hat{\lambda}(t)), \qquad (6.6.33)$$

where the first component ξ(t) — white noise, intensity of which does not depend on the mismatch $\lambda_{И}(t) - \hat{\lambda}(t)$ between the

true and measured values of the parameter, and the second component is proportional to mismatch with a variable proportionality factor K(t). These properties of z(t) qualitatively coincide with properties of the discriminator output signal. Therefore, element 1 is rationally called the optimum discriminator. Common for practically known and optimum discriminators is a similar character of discrimination and fluctuation characteristics. The discrimination characteristic of element 1, as will follow from examples of Chapters VII-XI, has a linear section and at infinity seeks zero.

Quantity K(t) according to (6.6.33) is the variable random gain factor of an optimum discriminator, determined near zero mismatch, and its mean value over ensemble y(t) is the gain factor of the discriminator in the usual meaning. For element 2 to select a name is somewhat more difficult, inasmuch as an analogous element in practical meters is usually absent. Above we indicated that K_K, and consequently, the analogous function K(t) characterizes additionally the current accuracy of measurement. Therefore, we shall call element 2 the accuracy unit.

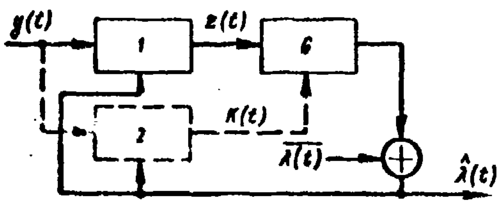

Fig. 6.14. Single-loop variant of the circuit of an optimum tracking meter: 1) discriminator; 2) accuracy unit; G — linear filter with pulse response.

The relationship between the two interpretations of K(t) is obvious, inasmuch as decrease of current accuracy occurs namely due to lowering of the gain factor of the discriminator. In § 6.2 fluctuations of the gain factor of the discriminator were called parametric fluctuations, and it also was shown that they cause increase of error of measurement in practical circuits, in which there are not taken measures to lower their influence. Controlling of smoothing filters in optimum meters is designed to lower the influence of parametric fluctuations.

We pass now to interpreting relationship (6.6.30) in the form of a block diagram. We note that the obtained variant of an optimum meter constitutes a somewhat simpler system with loops of feedback of direct control (Fig. 6.14). Basic elements of this system again are optimum discriminator 1 and a single linear filter G, carrying out smoothing.

Important for work of the system in principle, as also in the variant of Fig. 6.13, is feedback from its output to the discriminator, necessary to change discriminator tuning. In the course of this change at every moment of time the

zero of the discrimination characteristic, having a limited zone of linearity, is tuned approximately to the true value of $\lambda(t)$, coded in the input mixture. Thereby, the system realizes tracking of the realization of process $\lambda(t)$. The structure of linear filter G is determined by equations (6.6.31), (6.6.32), according to which it possesses variable randomly changing parameters. By the correlation function its properties depend on the a priori statistics of change of $\lambda(t)$, and through K(t) on the statistics and realization of input mixture $y(t)$. Forming of K(t) again is carried out by accuracy unit 2, in turn controlled by output data of the system.

Meters close in structure already are used in practice. The circumstance that the optimum solution in any conditions is in principle close to the one practically utilized is very remarkable. Theoretical solution permits us to optimally select the characteristics of separate elements of the meter, in particular to find optimum operations of the discriminator. The need for an accuracy unit and special control of smoothing circuits is a new finding, ensuing from theory.

Let us indicate that the circuits of Figs. 6.13 and 6.14 are absolutely equivalent. Inasmuch as in the first of them there exist two basic coupling loops, closing to the discriminator and the accuracy unit and separately to the adder, standing after the discriminator, and in the second there is a basic loop, we conventionally call the circuit of Fig. 6.13 two-loop, and the circuit of Fig. 6.14 — single-loop variants of an optimum meter.

Further concretization of the solution of the problem of optimum measurement for different statistical properties of input mixture consists of finding operations of formation of quantities z(t) and K(t), and instrumental realization of these operations. Furthermore, it is necessary to establish the algorithm of smoothing in the linear filters for different forms of correlation function of the parameter and different forms of K(t).

For convenience of subsequently finding operations of the discriminator and its characteristics we shall discuss in somewhat greater detail quantities z(t) and K(t). It is obvious that in general likelihood function $P(\mathbf{y}|\boldsymbol{\lambda})$ can be expressed through the product of conditional probabilities in the form

$$P(\mathbf{y}|\boldsymbol{\lambda})=\prod_{k=1}^{n} P_k(y_k|y_{k-1},\ldots,y_1,\lambda_k,\ldots,\lambda_1)=$$
$$=\exp\left\{\sum_{k=1}^{n} l_k(y_k,\ldots,y_1;\lambda_k,\ldots,\lambda_1)\right\}, \qquad (6.6.34)$$

where $P_k(y_k|y_{k-1}, \ldots, y_1, \lambda_k, \ldots, \lambda_1)$ — density of the conditional distribution y_k for assigned $y_{k-1}, \ldots, y_1, \lambda_k, \ldots, \lambda_1$, and t_k = in P_k.

Then, if there exists a likelihood functional, according to (6.6.[illegible]) it can be presented in the form

$$P(y(t)|\lambda(t)) = \exp\left\{\int_{t_0}^{t} l(\tau, \lambda(s), y(s))\, d\tau\right\}, \qquad ([illegible])$$

where $l(\tau, \lambda(s), y(s))$ — certain functional of realization [illegible] and $\lambda(s)$ when $s \in (t_0, \tau)$.

In a particular case $l(\tau, \lambda(s), y(s))$ can be [illegible] a function of $y(\tau), \lambda(\tau)$ at time τ. An example can be the functional of probability density of a [illegible] signal against a background of white noise (6.6.1), which in § [illegible] we shall [illegible] in greater detail.

Let us assume that the integral in formula (6.6.35) can be presented in the [illegible] sum of integrals over intervals of duration Δ. Here, during time the value of parameter $\lambda(t)$ practically does not change, and segments of realization of $y(t)$ in neighboring intervals, if we exclude fringe effects, are not related statistically, so that the integral over an elementary interval actually depends on the value of $\lambda(t)$ only in this interval. Then comparison of relationships ([illegible]) and (6.6.26) shows that output voltage of the discriminator is statistically equivalent to

$$z(t) = \frac{\partial}{\partial\lambda} l(t, \lambda(s), y(s))\Big|_{\lambda(s)=\hat{\lambda}(t)}, \qquad ([illegible])$$

where the functional in $l(t, \lambda(s), y(s))$ is formed for a frozen value of parameter $\lambda(t)$, equal to the value of the estimate at time t, and the realization of $y(s)$ is taken when $s \in (t_0, t)$.

A statistically equivalent function is also obtained when, e.g., we extend functional dependence $l(t, \lambda(s), y(s))$ from realization $y(s)$ to domain $\in (t, +\infty)$, [illegible] as operator l possesses properties filtering with respect to moment t, i.e., acts in an interval considerably smaller than the time of marked change of $\lambda(t)$. The physical realizability of such operators can be achieved by means of insertion of a delay by the magnitude of effective action of the operator.

Analogously to (6.6.36) we determine the operation of the accuracy unit:

$$K(t) = -\frac{\partial^2}{\partial\lambda^2} l(\tau, \lambda(s), y(s))\Big|_{\lambda(s)=\hat{\lambda}(t)}, \qquad ([illegible])$$

by which we easily calculate the average gain factor $\overline{K(t)}$.

We turn now to characteristics of the resultant accuracy of measurement. First we shall study the case of discrete observation. It is obvious that any statistical moment of differences $\lambda - \hat{\lambda}$ is determined by relationship

$$\begin{aligned}&\overline{(\lambda_n - \hat{\lambda}_n)^m (\lambda_k - \hat{\lambda}_k)^l} = \\ &= \iint (\lambda_n - \hat{\lambda}_n)^m (\lambda_k - \hat{\lambda}_k)^l P(\mathbf{y} | \boldsymbol{\lambda}) P_0(\boldsymbol{\lambda})\, d\boldsymbol{\lambda}\, d\mathbf{y} = \\ &= \int \left[\int (\lambda_n - \hat{\lambda}_n)^m (\lambda_k - \hat{\lambda}_k)^l \tilde{P}(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} \right] P(\mathbf{y})\, d\mathbf{y},\end{aligned} \tag{6.6.38}$$

where $\tilde{P}(\boldsymbol{\lambda}) \equiv P(\boldsymbol{\lambda}|\mathbf{y})$ is again the a posteriori distribution of $\boldsymbol{\lambda}$.

Substituting in (6.6.38) the a posteriori distribution

$$\begin{aligned}\tilde{P}(\boldsymbol{\lambda}) = (2\pi)^{-n/2} [\det (\mathbf{V} + \mathbf{A})^{-1/2}] \times \\ \times \exp\left\{ -\frac{1}{2} (\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}})^{+} (\mathbf{V} + \mathbf{A}) (\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}}) \right\},\end{aligned} \tag{6.6.39}$$

which is obtained by multiplication of $P(\mathbf{y}|\boldsymbol{\lambda})$ by $P_0(\boldsymbol{\lambda})$ and introduction of a normalizing factor, we arrive at integrals with multi-dimensional Gaussian functions, calculable in parts or with the help of a multi-dimensional characteristic function.

In particular, as it is easy to prove,

$$\overline{\hat{\lambda}_n} = \overline{\lambda}_n, \quad \overline{\hat{\lambda}_n^{2m+1}} = \overline{\lambda_n^{2m+1}}. \tag{6.6.40}$$

This shows the unbiased nature of the estimate, which is a not unexpected circumstance (see § 6.5). The correlation function of error of measurement is equal to

$$\begin{aligned}R_{nk\,\text{смх}} = \overline{\varepsilon_n \varepsilon_k} = \int_{-\infty}^{+\infty} \Bigg[\int_{-\infty}^{+\infty} \varepsilon_n \varepsilon_k (2\pi)^{-n/2} (\det [\mathbf{V} + \mathbf{A}])^{-1/2} \times \\ \times \exp\left\{ -\frac{1}{2} \boldsymbol{\varepsilon}^{+} (\mathbf{V} + \mathbf{A}) \boldsymbol{\varepsilon} \right\} d\boldsymbol{\varepsilon} \Bigg] P(\mathbf{y})\, d\mathbf{y} = \\ = \int_{-\infty}^{+\infty} [\mathbf{V}_{nk} + \mathbf{A}_{nk}]^{-1} P(\mathbf{y})\, d\mathbf{y} = \\ = \int_{-\infty}^{+\infty} C_{nk}(\mathbf{y}) P(\mathbf{y})\, d\mathbf{y} = \overline{C_{nk}(\mathbf{y})} \quad (n > k).\end{aligned} \tag{6.6.41}$$

Variance of error of measurement is equal to

$$\sigma^2_{\text{смх}\,n} = \overline{\varepsilon_n^2} = \overline{C_{nn}(\mathbf{y})}. \tag{6.6.42}$$

Thus, error is expressed through the value of σ_{nn}, averaged over the ensemble of input signals.

Formulas for the case of continuous observation are obtained by means of passage to the limit from (6.6.41)-(6.6.42):

$$R_{вых}(t, \tau)=\overline{c(t, \tau)}\ (t > \tau), \quad \sigma^2_{вых}(t)=\overline{c(t, t)}=\overline{g(t, t)}. \qquad (6.6.43)$$

6.6.3. The Optimum Tracking Meter in a Number of Specific Discrete Cases

In this paragraph we shall consider specific cases, when from "continuous" formulas of Paragraph 6.6.2 it is expedient or convenient to again pass to their discrete analogs. Let us emphasize that as before we are considering continuous observation of mixture y(t).

First of all we shall study measurement of parameters of an incoherent pulse signal received against a background of arbitrary interferences with a correlation interval smaller than the pulse repetition period. Pulses can have arbitrary shape, duration and form of intrapulse modulation, where in the case of the presence of fluctuations of the signal the values of amplitude and phase of pulses in separate periods will be considered independent. With an incoherent signal for use of the given methods complete independence is not required. It is important only that during the time of action of the pulse the measured parameter does not vary. However, changes of λ(t) during the period may be considerable.

The specific character of this case consists in the fact that even after transition from sampling to continuous realization of y(t) the likelihood functional can be recorded in a form discrete relative to the parameter, somewhat differing from (6.6.39):

$$P(y(t)\,|\,\lambda(t))=\exp\left\{\sum_{k=1}^{n}\int_{(k-1)T_r}^{kT_r} l_k(\tau, \lambda_k, y(s))\,d\tau\right\}, \qquad (6.6.44)$$

where $l_k(\tau, \lambda_k, y(s))$ — function (in general a functional) of y(s), in many respects analogous to function $l(\tau, \lambda(s), y(s))$ from (6.6.44) and practically differing from zero only within limits of pulse duration.

The logarithm of function (6.6.44) can be expanded in powers of $\lambda_k - \hat{\lambda}_k$ analogously to Paragraph 6.6.1. After multiplication by Gaussian a priori distribution (6.6.8) we formally arrive at the same operator of an optimum filter

(6.6.19) as during discrete observation, but by z_k and K_k one should understand (actually within limits of the pulse) the quantities:

$$z_k = \int_{(k-1)T_r}^{kT_r} \frac{\partial}{\partial \lambda} l_k(\tau, \hat{\lambda}_k, y(s))\, d\tau, \tag{6.6.45}$$

$$K_k = -\int_{(k-1)T_r}^{kT_r} \frac{\partial^2}{\partial \lambda^2} l_k(\tau, \hat{\lambda}_k, y(s))\, d\tau. \tag{6.6.46}$$

An optimum meter produces two types of operations — intraperiod (in discriminator and accuracy unit) and interperiod (in the discrete smoothing filter). Transition during smoothing to continuous quantities is formally and technically permissible only when during interval T_r the parameter changes insignificantly. In the general case the smoothing filter remains discrete and its pulse response C_{nk} (or G_{nk}) is determined by matrix equations (6.6.20) or (6.6.22). In distinction from the smoothing filter operations of the discriminator and accuracy unit within each pulse are always continuous. If we allow for the shown peculiarities, then in everything else the case of continuous observation of a pulse signal is formally similar to the case of discrete observation.

We shall discuss one case, when transition to discrete smoothing filters is already permissible from considerations of purely technical convenience. Recently more and more they realize smoothing circuits with the help of digital computers, possibilities of which are extraordinarily broad. Especially evident are advantages of digital computer technology during designing of complicated radar complexes, where the problem of measurement is combined with the problem of further processing and transmission of information. If, in these conditions, outside of dependence on the structure of the carrier-signal of the parameter, we are to quantize in time the data proceeding from discriminators where we select an interval of quantization smaller than the time of noticeable measurement [sic ? change] of the parameter, the smoothing circuits can be synthesized with the help of matrix equations (6.6.20), (6.6.21) and are realized on that digital computer which can execute the set of operations of smoothing with the required speed.

In the particular case when the signal is pulsed, and of the type considered above, and when in a period the measured quantities practically do not change, to decrease the number of operations of smoothing it is permissible to divide smoothing

into two stages. In the first there occurs grouping of data in intervals comparable to the time of change of the parameter, and in the second there occurs discrete smoothing by a filter synthesized according to the indicated matrix equations.

In practice we may encounter other interesting cases of close intertwining of questions of continuous (analog) and discrete (digital) processing of signals, the description of which is beyond the framework of the present section.

6.6.4. Possibility of One Simplification of the Meter Circuit

We shall give certain considerations, allowing us to produce an important simplification of circuits of the found optimum meters. Let us consider the case of continuous observation. With rapid changes of all immaterial parameters y(t) as compared to λ(t) function K(t) is a rapidly changing function as compared to the correlation function of the parameter. Presenting the integral in (6.6.31) in the form

$$\int c(t, s) K(s) R(s, \tau) ds \approx$$
$$\approx \sum_i \int_{t_i}^{t_{i+1}} K(s) ds\, c(t, t_i) R(t_i, \tau),$$

where duration of intervals $\Delta = t_{i+1} - t_i$ is much less than the time of correlation of $R(t, \tau)$, and assuming ergodicity of the random part of function K(t), thanks to which

$$\int_{t_i}^{t_i+\Delta} K(s)\, ds \approx \overline{K(t_i)}\Delta,$$

we obtain

$$\int_{t_0}^{t} c(t, s) K(s) R(s, \tau) ds \approx \int_{t_0}^{t} c(t, s) \overline{K(s)} R(s, \tau)\, ds.$$

Here $\overline{K(t)}$ is the mean value of K(t) in the ensemble of input signals y(t). Here, thanks to smallness of error of measurement $\lambda(t) - \hat{\lambda}(t) = \varepsilon(t)$ the value of $\overline{K(t)}$ can be calculated, not at point $\lambda = \hat{\lambda}$, but at a point equal to the true value of the measured parameter, so that

$$\overline{K(t)} = -\overline{\frac{\partial^2 l(t, \lambda(t), y(s))}{\partial \lambda^2}}. \qquad (6.6.47)$$

Thus, function K(t), expressed through the limit of the matrix of second derivatives of the likelihood functional with respect to values of the measured parameter, in certain cases can be replaced by the a priori known mean value of this function. Equations (6.6.31) and (6.6.32), determining characteristics of optimum filters, here do not change their form, but instead of random function K(t) in them there now appears the known function of time $\overline{K(t)}$. The actual linear filters of block diagrams in Figs. 6.13 and 6.14 become here filters with nonrandom characteristics. The presence of an accuracy unit and circuits for controlling the smoothing filter, becomes unnecessary, and elements of coupling, shown in Figs. 6.13 and 6.14 by the dotted line, disappear. In particular, the circuit of Fig. 6.14 takes the well-known form of a single-loop servo system. It is natural that, inasmuch as $\overline{K(t)}$ does not depend on the current value of y(t) in that same approximation in which it is possible to disregard fluctuations of K(t) during synthesis, variance and the correlation function of measurement error will be expressed in the form

$$\sigma^2_{\text{вых}}(t) = c(t, t) = g(t, t), \tag{6.6.48}$$

$$R_{\text{вых}}(t, \tau) = c(t, \tau). \tag{6.6.49}$$

From material of Paragraph 6.6.2 it is clear that the meaning of the replacement of K(t) and $\overline{K(t)}$ consists in rejection of allowance for unequal accuracy of separate measurements or, which is the same, of allowance for parametric fluctuations. In Paragraph 6.8.6 we shall give an example showing that this refusal leads to certain increase of error of measurement, but, as it was shown in § 6.2 for practical circuits, this increase frequently is small and in a number of cases can be ignored. We note, only, that the introduced simplification deprives the optimum circuit of that flexibility of a system with self-tuning which is manifested upon change of the input signal level. When the average level of the signal changes as compared to design level the previously optimum circuits always become nonoptimal. However, in circuits with adjustment of smoothing circuits there occurs change of the gain factor in the feedback circuit (with respect to the accuracy unit) of the same nature which should occur during change of the design characteristic of the input quantity. At the same time in the absence of adjustment there remain no means for preservation of optimality in the described conditions, and impairment of the quality of measurement may be considerable.

In physical meaning, as follows from (6.6.33) and (6.6.47), $\overline{K(t)}$ is the (averaged) gain factor of the discriminator. On the other hand, considering the discrete case, analogously to [8] one can prove that

$$\overline{K}_k\delta_{ik} = -\overline{\frac{\partial^2 \ln P(y|\lambda)}{\partial\lambda_i\partial\lambda_k}} = \overline{\frac{\partial \ln P(y|\lambda)}{\partial\lambda_i}\frac{\partial \ln P(y|\lambda)}{\partial\lambda_k}} = \overline{\xi_i\xi_k}. \qquad (6.6.50)$$

An analog of relationship (6.6.50) in the continuous case will be

$$\overline{K(t)}\delta(t-\tau) = \overline{\xi(t)\xi(\tau)}, \qquad (6.6.51)$$

where $\xi(t)$ — fluctuating component in the discriminator output, taking place also with zero mismatch.

According to (6.6.51) $\overline{K(t)}$ is also the spectral density of the fluctuating component $\xi(t)$ at the discriminator output. If we translate this component into the equivalent value of the measured quantity by division by $\overline{K(t)}$ similarly to how this was done in § 6.2, the spectral density of the equivalent noise component $\eta(t) = \xi(t)/\overline{K(t)}$ will be quantity $1/\overline{K(t)}$:

$$\overline{\eta(t)\eta(\tau)} = \frac{1}{\overline{K(t)}}\delta(t-\tau). \qquad (6.6.52)$$

Thus, $1/\overline{K(t)}$ is the variable spectral density of equivalent noise at the discriminator input. This interpretation of $\overline{K(t)}$ is exceedingly important, since multiplication of output voltage of an optimum discriminator by an arbitrary constant, immaterial from the point of view of producible operations, changes the gain factor and spectral density of output noise, but leaves constant the equivalent spectral density $1/\overline{K(t)}$. In the frames of these assumptions function $\overline{K(t)}$ (or $\overline{K_k}$) is the unique characteristic of performance of an optimum discriminator. Resultant errors of measurement always depend on this function monotonically: the larger $\overline{K(t)}$ (or $\overline{K_k}$), the less the error.

If statistical characteristics of immaterial parameters y(t) do not depend on time, $\overline{K(t)}$ is a constant [$\overline{K(t)} = K$]. Then from (6.6.31)-(6.6.32) we have the following equations, determining pulse responses of filters $c(t, \tau)$ and $g(t, \tau)$:

$$c(t,\tau) + K\int_{t_0}^{t} c(t,s)R(s,\tau)\,ds = R(t,\tau), \qquad (6.6.53)$$

$$c(t,\tau) + K\int_{\tau}^{t} c(t,s)g(s,\tau)\,ds = g(t,\tau). \qquad (6.6.54)$$

Equation (6.6.53), considered the equation for function $K_C(t, \tau)$, coincides with the equation of an optimum Wiener filter, detecting a signal with correlation function $R(t, \tau)$ from its additive mixture with white noise with spectral density $1/K$. This is understandable, inasmuch as for all ideal assumptions work of an optimum meter in linear conditions is equivalent to work of a linear servo system, presented in any of the variants of Figs. 6.15 and 6.16, to which as input disturbance there proceeds the sum of a "signal" $\lambda(t) - \overline{\lambda(t)}$ and "noise" $\eta(t)$. Due to this all solutions of equation (6.6.53), determining the structure of smoothing circuits, simultaneously are solutions of the corresponding problems of optimum filtration.

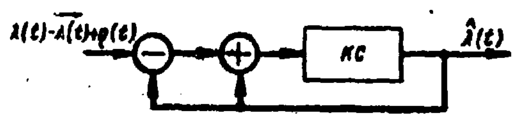

Fig. 6.15. Two-loop variant of an equivalent linear system. KC — linear filter with pulse response $K_C(t, \tau)$.

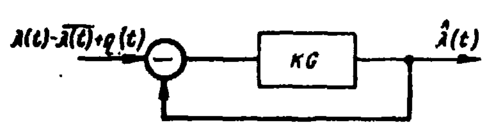

Fig. 6.16. Single-loop variant of an equivalent linear system. KG — linear filter with pulse response $K_G(t, \tau)$.

6.6.5. Optimum Nontracking Meter

The considered type of optimum meters is not the only possible one in the framework of our assumptions. Another circuit of a meter, also realizing potential possibilities of measurement, can be obtained for Gaussian statistics of the parameter on the basis of approximation of the likelihood function by relationship (6.6.4). Cross multiplying (6.6.4) and (6.6.8) and producing transformations of the logarithm of a posteriori probability, analogous to those shown in Paragraph 6.6.2, we arrive during discrete observation at the relationship

$$\hat{\lambda}_n = \sum_{k,j=1}^{n} C_{nk}\check{A}_{kj}(\check{\lambda}_j - \bar{\lambda}_j) + \bar{\lambda}_n, \tag{6.6.55}$$

where $\mathbf{C} = [\check{\mathbf{A}} + \mathbf{V}]^{-1}$ — a matrix, determined in the current interval of observation, and matrix $\check{\mathbf{A}}$ is determined according to (6.6.5) and with a high degree of accuracy coincides with matrices $\mathbf{A}$ and $\mathbf{A}_{::}$.

In view of the diagonalness of matrix $\check{\mathbf{A}}$ relationship (6.6.55) takes the form

$$\hat{\lambda}_n = \sum_{k=1}^{n} C_{nk}K_k(\check{\lambda}_k - \bar{\lambda}_k) + \bar{\lambda}_n. \tag{6.6.56}$$

If we examine the case of continuous observation, (6.6.56) passes into integral relationship:

$$\hat{\lambda}(t)=\int_{t_0}^{t} c(t, s) K(s) [\check{\lambda}(s)-\overline{\lambda(s)}] ds+\overline{\lambda(t)}. \qquad (6.6.57)$$

Relationships (6.6.56) and (6.6.57), giving a final solution to problems of meter synthesis, are basic, and their operations are illustrated by the circuit in Fig. 6.17. Element 1, to which there proceeds realization y(t), is a nonlinear unit, constantly separating the value of the point of maximum likelihood $\hat{\lambda}(t)$ and function K(t), determined through the second derivative of the logarithm of the likelihood function, taken at this point. Then in the circuit there follows a subtractor and linear filter CK with pulse response c(t, s)K(s), to which as the object of filtration there proceeds the difference of the maximum-likelihood and mean value $\check{\lambda}(t)$ - $\overline{\lambda(t)}$, and as an adjustment, function K(t), characterizing the current level of accuracy of measurement. An adder for reverse input of the mean value completes the circuit of the optimum meter.

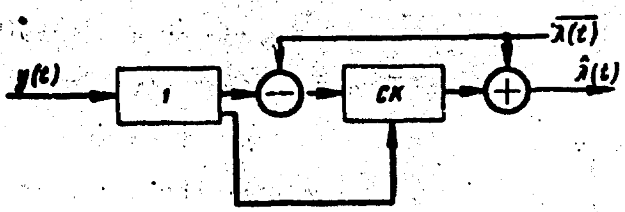

Fig. 6.17. Nontracking variant of an optimum meter: 1) estimator unit; CK — linear filter with pulse response c(t, τ) K(τ).

In distinction from the circuit described in Paragraph 6.6.2, the new circuit of an optimum meter is not closed. The circuit similarity of the two variants consists in the following two factors. First, in both circuits in the first stage there are used low-inertia, nonlinear operations of processing input radio signals. In the tracking variant they are carried out by a discriminator and an accuracy unit; in the nontracking variant these units are replaced by a unit which it is possible to conditionally call an estimator unit.

In both circuits there exist linear smoothing filters; in the tracking variant they close the tracking loop, and here they are constructed on the basis of an open circuit. The internal unity of variants becomes still clearer if we consider that the pulse response of the open-loop smoothing filter is the same as for a linear system equivalent to the closed loop of the tracking variant of a meter, if the latter works with small errors of reproduction of the parameter. This is especially easy to prove when we disregard the variable random part of K(t) (analogously to Paragraph 6.6.3) and have for a constant mean value of K(t):

$$K(t)\approx\overline{K(t)}\equiv K.$$

Then the pulse response of the main part of the low-frequency filter of the circuit in Fig. 6.17 is determined by equation

$$c(t, \tau)+K\int_{t_0}^{t} c(t, s) R(s, \tau) ds=R(t, \tau),$$

coinciding with equation (6.6.53).

By a method, analogous to that given in Paragraph 6.6.2, it is easy to show that the characteristics of accuracy of the new variant of meter are expressed by formulas

$$\overline{\hat{\lambda}(t)}=\overline{\lambda(t)}, \quad \sigma^2_{\text{изм}}(t)=\overline{c(t, t)}=\overline{g(t, t)}.$$

This nontrivial result shows that, in spite of differences in approximation of the likelihood function, in general determining different errors of measurements, both the considered approximations lead to identical errors. They give measuring systems which, for a low level of noises, are equally close to a truly optimum system, which in principle could be obtained by means of direct integration of relationship (6.5.42) without use of any approximations.

However, it must not be concluded that both variants of the circuit of an optimum filter are equally easily realized technically. A very important advantage of the tracking circuit is low criticality to change of a priori information fed into the circuit and relative simplicity of the discriminator as compared to the estimator unit. The discriminator will form the derivative of the logarithm of the likelihood function only at the point of the current result of measurement $\hat{\lambda}(t)$. This operation can be conducted continuously in real time. The estimator unit for issuing the point of maximum likelihood should investigate the whole range of values of λ. This can be carried out by a set of detection channels (without a terminal storage unit), constructed for assigned statistical properties $y(t)$, for all possible values of the measured parameter λ. The circuit should be completed by a device for separation of the channel with the greatest output voltage. In principle such devices, due to their multichannel nature, for technical considerations can not always be realized.

An advantage of the new variant of the meter is basically lessening of requirements for width of the a priori distribution of measurement error in the initial moment of time. Previously this width had to be less than the section of linearity of the discriminator; now it must be somewhat wider than the peak of the likelihood

function. This circumstance is important in the stage of target lock-on. Analogously, the formula of accuracy, previously valid only during linear work of the discriminator, obtains a wider domain of applicability. The section of linearity in the nontracking variant as it were is expanded, which is an advantage attained at the price of complication of the circuit of radio processing of signals.

In the case of a multichannel lock-on system, built from a great number of filters, gated amplifiers, etc., the estimator unit can be realized on the basis of this lock-on system. If channels are placed sufficiently closely, and the probability of appearance in the given region of large noise peaks or extraneous targets is small, such application of a lock-on system does not require additional commentaries.

6.6.6. Potential Accuracy of Measurement

Above we obtained an expression for variance of resultant error of measurement of a parameter changing in time. However, in its derivation from the very beginning we used a specific approximation of the likelihood function. Meanwhile, it is no less interesting to determine the potential minimum variance of measurement, as far as possible not using any approximations. If we could find coincidence of the true limit of accuracy with the previously derived expression, this would be a proof of the immaterial departure from the true optimum when using Gaussian approximations of the likelihood function.

Let us assume that the a priori distribution of probabilities for $\lambda(t)$ is known and can be described by density $P_0(\boldsymbol{\lambda})$. We consider that there exist derivatives $\frac{\partial}{\partial\lambda_k} P(\mathbf{y}|\boldsymbol{\lambda})P_0(\boldsymbol{\lambda})$. We shall prove that for any estimate $\hat{\lambda}(\mathbf{y}) = \{\hat{\lambda}_1(\mathbf{y}), \ldots, \hat{\lambda}_n(\mathbf{y})\}$ of vector $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_n\}$, describing a realization of the variable parameter, we have the inequality

$$\sum_{i,k=1}^{n} \Sigma_{ik} u_i u_k \geqslant \sum_{i,k=1}^{n} C_{ik} u_i u_k, \tag{6.6.58}$$

where $\Sigma_{ik} = \overline{(\lambda_i - \hat{\lambda}_i)(\lambda_k - \hat{\lambda}_k)}$ – elements of a matrix of second moments of errors of $\boldsymbol{\Sigma} = \| \Sigma_{ik} \|$ of the order $(n \times n)$;

$\mathbf{C} = \| C_{ik} \|$ – symmetric matrix of order $(n \times n)$, the inverse of matrix $\mathbf{C}^{-1} = \| C_{ik}^{(-)} \|$ with elements

$$C_{ik}^{(-1)}=\iint \frac{\partial}{\partial \lambda_i} \ln [P(\mathbf{y}|\lambda) P_0(\lambda)] \frac{\partial}{\partial \lambda_k} \ln [P(\mathbf{y}|\lambda) P_0(\lambda)] \times \times P(\mathbf{y}|\lambda) P_0(\lambda)\, dy d\lambda, \quad (6.6.59)$$

$u_i(i = 1, \ldots, n)$ – arbitrary quantities, where the equality in (6.6.58) is achieved when for all $i = 1, \ldots, n$ there are satisfied these relationships:

$$\hat{\lambda}_i - \lambda_i - \sum_{k=1}^{n} C_{ik} \frac{\partial}{\partial \lambda_k} [\ln P(\mathbf{y}|\lambda) P_0(\lambda)]\, d\lambda = 0. \quad (6.6.60)$$

For proof we shall consider functions

$$f_i = (\lambda_i - \hat{\lambda}_i) \sqrt{P(\mathbf{y}|\lambda) P_0(\lambda)}, \qquad \varphi_i = \frac{\partial}{\partial \lambda_i} \ln [P(\mathbf{y}|\lambda) P_0(\lambda)] \sqrt{P(\mathbf{y}|\lambda) P_0(\lambda)} \quad (6.6.61)$$

and form functions

$$\psi_i = \psi_i(\mathbf{y}, \lambda) = f_i + \sum_{k=1}^{n} C_{ik} \varphi_k. \quad (6.6.62)$$

We introduce arbitrary quantities $u_i(i = 1, \ldots, n)$ and compose the sum $\sum_{i=1}^{n} u_i \psi_i(\mathbf{y}, \lambda) = \Psi(\mathbf{y}, \lambda)$. The integral of the square of function $\Psi(\mathbf{y}, \lambda)$, obviously, is larger than or equal to zero. Therefore

$$\begin{aligned} 0 < \iint \Psi^2(\mathbf{y}, \lambda)\, dy d\lambda &= \iint \Big(\sum_{i=1}^{n} u_i \psi_i\Big)^2 dy d\lambda = \\ &= \sum_{i,k=1}^{n} u_i u_k \iint \psi_i \psi_k\, dy d\lambda = \\ &= \sum_{i,k=1}^{n} u_i u_k \iint \Big[f_i f_k + 2 f_i \sum_{j=1}^{n} C_{kj} \varphi_j + \\ &+ \sum_{j,m=1}^{n} C_{ij} C_{km} \varphi_j \varphi_m \Big] dy d\lambda = \\ &= \sum_{i,k=1}^{n} u_i u_k \Big[\Sigma_{ik} + C_{ik} + 2 \sum_{j=1}^{n} C_{ij} \iint \varphi_j f_k\, dy d\lambda \Big], \end{aligned} \quad (6.6.63)$$

since

$$\iint f_i f_k\, dy d\lambda = \iint (\lambda_i - \hat{\lambda}_i)(\lambda_k - \hat{\lambda}_k) P(\mathbf{y}|\lambda) P_0(\lambda)\, d\lambda = \Sigma_{ik}, \quad (6.6.64)$$

$$\iint \varphi_i \varphi_k\, dy d\lambda = \iint \frac{\partial \ln P(\mathbf{y}|\lambda) P_0(\lambda)}{\partial \lambda_i} \times \quad (6.6.65)$$

$$\times \frac{\partial \ln P(\mathbf{y}|\lambda) P_0(\lambda)}{\partial \lambda_k} P(\mathbf{y}|\lambda) P_0(\lambda)\, d\mathbf{y} d\lambda = C_{ik}^{(-1)}, \qquad \text{(6.6.6[illegible] cont'd)}$$

$$\sum_{k,j=1}^{n} C_{ik} C_{lj} C_{kj}^{(-1)} = \sum_{k=1}^{n} \delta_{lk} C_{ik} = C_{il}. \qquad \text{([illegible])}$$

Let us consider the integrals in the right part of inequality (6.6.63):

$$\iint f_i \varphi_k d\mathbf{y} d\lambda = \iint (\lambda_i - \hat{\lambda}_i) \frac{\partial \ln P(\mathbf{y}|\lambda) P_0(\lambda)}{d\lambda_k} P(\mathbf{y}|\lambda) P_0(\lambda) d\lambda d\mathbf{y} =$$

$$= \int (\lambda_i - \hat{\lambda}_i) \frac{\partial P(\mathbf{y}|\lambda) P_0(\lambda)}{\partial \lambda_k} d\mathbf{y} d\lambda. \qquad \text{(6.6.67)}$$

Integrating in parts, we obtain

$$\iint f_i \varphi_k d\mathbf{y} d\lambda = -\delta_{ik}, \qquad (6.6.68)$$

and then (6.6.6[illegible]) reduces to form

$$0 \leqslant \sum_{i,k=1}^{n} u_i u_k \left[\Sigma_{ik} + C_{ik} - 2 \sum_{j=1}^{n} \delta_{ij} C_{jk} \right] =$$

$$= \sum_{i,k=1}^{n} u_i u_k (\Sigma_{ik} - C_{ik}), \qquad \text{(6.6.69)}$$

from which there follows inequality (6.6.58). By virtue of the arbitrariness of u_i the equality in (6.6.63) and, consequently, in (6.6.58) is attained when $\psi_i(\mathbf{y}, \lambda) = 0$, i.e., upon satisfaction of (6.6.60).

Inequality (6.6.58) shows that the ellipsoid of scattering for any estimate $\hat{\lambda}$ wholly includes the ellipsoid corresponding to matrix C. Thanks to arbitrariness of quantities u_i from (6.6.58) it follows that

$$\Sigma_{kk} = \sigma_k^2 \geqslant C_{kk}, \qquad (6.6.70)$$

i.e., the mean square error of measurement of $\lambda(t)$ at any moment of time t_k is limited from below by the magnitude of C_{kk}.

Estimates of variable parameter $\hat{\lambda}(y)$, for which in inequalities (6.6.58), (6.6.69) and (6.6.70) there is achieved the sign of equality, by analogy with the case of a constant parameter, we naturally call efficient. Then matrix $\| C_{ik} \|$ constitutes a matrix of second moments of errors of measurement in the case of efficient estimates and characterizes potential accuracy of measurement. It establishes the lower limit of accuracy of measurement of time-variable parameters of input signal in the case of known a priori statistics. From the degree of difference of C_{ik} and quantities Σ_{ik}, calculated for any concrete rule of formation

of estimate $\hat{\lambda}(y)$, it is possible to judge the nearness of this rule, and of the device carrying out measurement corresponding to it, to the optimum one ensuring potentially possible accuracy of measurement. As the index of efficiency, quantitatively characterizing this nearness it is convenient to select the ratio of variance of errors of measurement in any moment of time interesting us t_k for efficient and efficient estimates, i.e.,

$$\varkappa_k = \frac{\Sigma_{kk}}{C_{kk}}. \tag{6.6.71}$$

Expression (6.6.59) for matrix $\mathbf{C}^{-1}$ can be transformed to a form more convenient for calculations. Introducing again the logarithm of the likelihood function

$$L(\boldsymbol{\lambda}) = \ln P(\mathbf{y} \mid \boldsymbol{\lambda})$$

and considering that

$$\int \frac{\partial L}{\partial \lambda_i} P(\mathbf{y} \mid \boldsymbol{\lambda})\, d\mathbf{y} = \int \frac{\partial P(\mathbf{y} \mid \boldsymbol{\lambda})}{\partial \lambda_i}\, d\mathbf{y} = \frac{\partial}{\partial \lambda_i} \int P(\mathbf{y} \mid \boldsymbol{\lambda})\, d\mathbf{y} = 0, \tag{6.6.72}$$

we obtain

$$C_{ik}^{(-1)} = \int \Big[-\int d\mathbf{y}\, \frac{\partial^2 L(\boldsymbol{\lambda})}{\partial \lambda_i \partial \lambda_k} P(\mathbf{y} \mid \boldsymbol{\lambda}) + \frac{\partial \ln P_0(\boldsymbol{\lambda})}{\partial \lambda_i} \frac{\partial \ln P_0(\boldsymbol{\lambda})}{\partial \lambda_k} \Big] P_0(\boldsymbol{\lambda})\, d\boldsymbol{\lambda}. \tag{6.6.73}$$

Thus, matrix $\mathbf{C}^{-1}$ is presented in the form of the sum of two matrices: matrix

$$\mathbf{A} = \left\| \int \frac{\partial \ln P_0(\boldsymbol{\lambda})}{\partial \lambda_i} \frac{\partial \ln P_0(\boldsymbol{\lambda})}{\partial \lambda_k} P_0(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} \right\|, \tag{6.6.74}$$

depending only on a priori distribution of $\boldsymbol{\lambda}$, and matrix

$$\overline{\mathbf{A}} = \| \int \mathbf{A}_{ik}(\boldsymbol{\lambda}) P_0(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} \|, \tag{6.6.75}$$

where matrix $\| \mathbf{A}_{ik}(\boldsymbol{\lambda}) \| = \mathbf{A}(\boldsymbol{\lambda})$ is determined by expression

$$\mathbf{A}_{ik}(\boldsymbol{\lambda}) = -\int \frac{\partial^2 L(\boldsymbol{\lambda})}{\partial \lambda_i \partial \lambda_k} P(\mathbf{y} \mid \boldsymbol{\lambda}) d\mathbf{y} = \int \frac{\partial L(\boldsymbol{\lambda})}{\partial \lambda_i} \frac{\partial L(\boldsymbol{\lambda})}{\partial \lambda_k} P(\mathbf{y} \mid \boldsymbol{\lambda})\, d\mathbf{y}. \tag{6.6.76}$$

Matrix $\overline{\mathbf{A}}$ depends, in general, both on the method of encoding $\lambda(t)$ in $y(t)$ and properties of input signal $x(t)$, and also on the a priori distribution. As we shall subsequently prove with examples, in a whole series of practically interesting cases matrix $A(\boldsymbol{\lambda})$ does not depend on $\boldsymbol{\lambda}$. Then $\overline{\mathbf{A}} = \mathbf{A}$ and the a priori distribution of probabilities affects potential accuracy of measurement only by means of matrix $\mathbf{A}$.

Matrix **A**, as is clear from the definition, is that matrix which we used in solving the problem of synthesis of an optimum system of measurement. Matrix **A** for Gaussian a priori statistics is equal to matrix $\mathbf{V} = \| V_{ik} \|$. Therefore, in this case,

$$\mathbf{C}^{-1} = \mathbf{A} + \mathbf{V}$$

and

$$\mathbf{C} + \mathbf{CAR} = \mathbf{R},$$

consequently, for Gaussian a priori statistics the matrix of efficient estimates coincides with the same matrix C which appeared in the process of solution of the problem of synthesis and which, as we showed earlier, characterizes accuracy of the synthesized meter in linearized conditions.

It follows from this that the synthesized meter is actually optimum as long as conditions of its linearization are satisfied, inasmuch as it ensures potentially possible accuracy.

6.6.7. Allowance for Side Peaks of the Likelihood Function

Gaussian approximation of a likelihood function of type (6.6.3) or (6.6.4) remains true only near the true value of $\lambda(t)$, if noises are sufficiently low. In the general case during synthesis it is necessary to consider side peaks of the likelihood function, explained both by imperfect form of the autocorrelation function of the useful component y(t), and by purely noise disturbances. Let us show to what sort of circuit changes complication of the approximation of the likelihood function, aimed at more closely reflecting its structure, leads.

In the first place we shall study the case when side peaks of the likelihood function are explained by a specific imperfectness of the useful component of the signal, i.e., corresponding peaks of its autocorrelation function. The position of these peaks is rigidly related to the position of the basic peak, so that the following complete approximation of the likelihood function, based on relationships (6.6.3) and (6.6.6) is convenient:

$$P(\mathbf{y}|\boldsymbol{\lambda}) = \sum_{i=-N}^{N} P(\mathbf{y}|\hat{\boldsymbol{\lambda}} + \Delta_{(i)}) \exp\Big\{\mathbf{z}_{(i)}^{+}(\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}} - \Delta_{(i)}) - \frac{1}{2}(\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}} - \Delta_{(i)})^{+} \mathbf{A}_{(i)}(\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}} - \Delta_{(i)})\Big\}. \qquad (6.6.77)$$

Here $\Delta_{(i)}$ – separation of the basic and i-th side peaks;

$\mathbf{z}_{(i)}$, $\mathbf{A}_{(i)}$ – matrices, expressed as per Paragraph 6.6.1 through the first and second derivatives of ln $P(\mathbf{y}|\lambda)$ with respect to λ at points $\lambda = \hat{\lambda} + \Delta_{(i)}$, where

among the considered peaks 2N + 1 there are included those which exceed the noise background in level.

Multiplying (6.6.77) by Gaussian a priori distribution (6.6.8) and integrating by a method analogous to that presented in Paragraph 6.6.2, taking into account the dia... ...ture of all matrices $\| A_{(i)kj} \| = \| K_{(i)k}\delta_{kj} \|$, we have the following ...sion for the operator of an optimum meter:

$$\hat{\lambda}_n = \sum_{l=N}^{n} \bar{\Lambda}_n^{(l)} \sum_{k=1}^{n} C_{(i)nk} [z_{(i)k} + K_{(i)k} (\hat{\lambda}_k + \Delta_{(i)k} - \bar{\lambda}_k)] + \bar{\lambda}_n, \tag{6.6.78}$$

where

$$C_{(i)} = [A_{(i)} + V]^{-1}; \quad \bar{\Lambda}_n^{(l)} = \frac{\Lambda_n^{(l)}}{\sum\limits_{k=-N}^{N} \Lambda_n^{(k)}},$$

$$\Lambda_n^{(l)} = P(\mathbf{y} \mid \hat{\lambda} + \Delta_{(i)}) \left[\frac{\det (A_{(l)}^{-1} + R)}{\det A_{(l)}^{-1}} \right]^{-1/2} \times$$

$$\times \frac{\exp\left\{-\frac{1}{2} [z_{(i)} + A_{(i)} (\hat{\lambda} - \Delta_{(i)} - \lambda)]^{+} (A_{(l)}^{-1} - C_{(i)}) [z_{(i)} + A_{(i)} (\hat{\lambda} - \Delta_{(i)} - \hat{\lambda})]\right\}}{\exp\left\{-\frac{1}{2} z_{(l)}^{+} A_{(l)}^{-1} z_{(i)}\right\}} =$$

$$= P(\mathbf{y} \mid \hat{\lambda} + \Delta_{(i)}) \frac{P_{em(i)}}{P_{m(i)}}. \tag{6.6.79}$$

Normal transition in (6.6.79) to continuous observation taking into account diagonalness of $A_{(i)}$ gives

$$\hat{\lambda}(t) = \sum_{l=-N}^{N} \bar{\Lambda}^{(l)}(t) \int_{t_0}^{t} c_{(i)}(t, \tau) [z_{(i)}(\tau) + K_{(i)}(\tau) (\hat{\lambda}(\tau) - \Delta_{(i)}(\tau) - \overline{\lambda(\tau)}] \, d\tau + \overline{\lambda(t)}, \tag{6.6.80}$$

$$\left.\begin{aligned} &\bar{\Lambda}^{(l)}(t) = \Lambda^{(l)}(t) \Big/ \sum_{k=-N}^{N} \Lambda^{(k)}(t); \\ &\Lambda^{(l)}(t) = P(y(t) \mid \hat{\lambda}(t) + \Delta_{(i)}) \exp\left\{\frac{1}{2} \int_{t_0}^{t} K_{(i)}(\tau) z_{(l)}^{2}(\tau) \, d\tau - \right. \\ &-\frac{1}{2} \int_{t_0}^{t} K_{(i)}(\tau) [z_{(i)}(\tau) + K_{(i)}(\tau) (\hat{\lambda}(\tau) - \Delta_{(i)} - \overline{\lambda(\tau)})]^{2} \, d\tau + \\ &+\frac{1}{2} \int_{t_0}^{t} \int_{t_0}^{t} c_{(i)}(t, \tau) [z_{(i)}(t) + K_{(i)}(t) (\hat{\lambda}(t) - \Delta_{(i)} - \overline{\lambda(t)})] [z_{(i)}(\tau) + \\ &\left. + K_{(i)}(\tau) (\hat{\lambda}(\tau) - \Delta_{(i)} - \overline{\lambda(t)})] \, dt \, d\tau + \int_{0}^{1} d\mu \int_{t_0}^{t} c_{(i)}(t, t; \mu) \, dt \right\}, \end{aligned}\right\} \tag{6.6.81}$$

$c(t, t; \mu)$ – solution of equation (6.6.31), in which instead of K(s) we take μK(s).

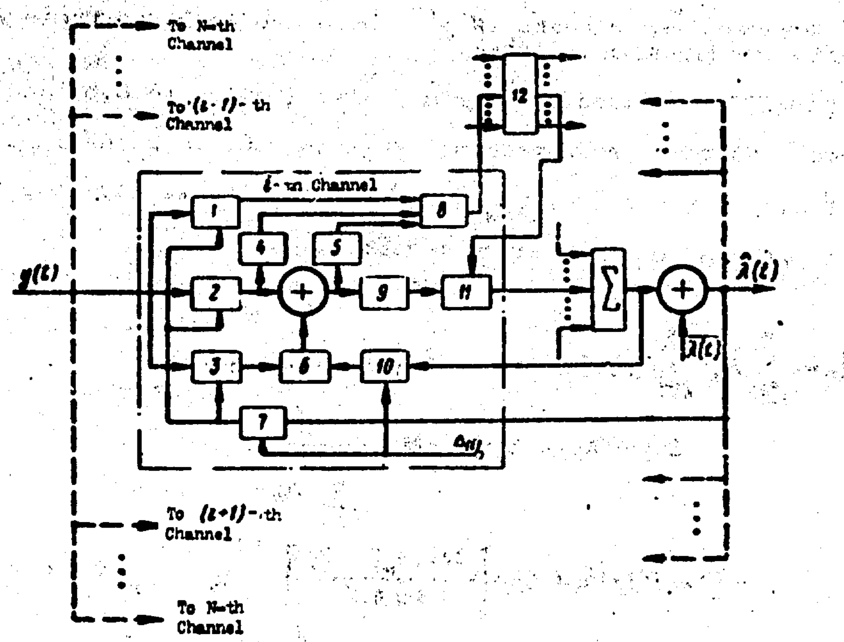

Fig. 6.18. Optimum meter with allowance for side peaks of the autocorrelation function: 1) tuned detection receiver; 2) discriminator; 3) accuracy unit; 4, 5) nonlinear units; 6) inertialess amplifier with gain factor $K_{(i)}$; 7, 10) adders for introducing shift of side peak $\Delta_{(i)}$; 8) unit for formation of unnormalized coefficient $\Lambda^{(i)}(t)$; 9) linear filter with pulse response $c_{(i)}(t, \tau)$; 11) normalizing amplifier; 12) unit for producing normalized coefficients; Σ — adder.

Let us turn to interpretation of operator (6.6.81) in the form of the circuit of (Fig. 6.18). The overall circuit consists of 2N + 1 parallel-working circuits, similar to the optimum meter of Fig. 6.13. Each of them includes a discriminator 2, an accuracy unit 3, adder of the loop of intercommunication and linear filter-amplifiers 9 and 6. As a new element in each i-th circuit there is coupled a device 1, essentially executing the operation of a detection receiver tuned to the i-th peak of the likelihood function. The position of the i-th peak is counted off from the measured value $\hat{\lambda}(t)$ with help of known quantity $\Delta_{(i)}$. Furthermore, in the i-th circuit there are taps from the output of the discriminator and adder of intercommunication to nonlinear units 4 and 5. Output signals of all filters 9 through normalizing amplifiers 11 are fed to the common adder Σ, where they are combined with each other with the help of normalized weights $\tilde{\Lambda}^{(i)}(t)$. The latter are expressed through unnormalized factors $\Lambda^{(i)}(t)$, formed finally in 12 with respect to the input realization of y(t) and data of measurement with the help of detection

receiver 1, nonlinear units 4, and cross-multipliers 8.

At the output of adder Σ there will form the estimated value of the parameter (after subtracting the mean value $\overline{\lambda(t)}$). This quantity is fed to internal feedback circuits through adders for input of shift 10. After addition to $\overline{\lambda(t)}$ there will be formed, finally, the measured value $\hat{\lambda}(t)$, controlled by selection of all units of primary processing, which we consider detection receivers, discriminators and accuracy units. When $\hat{\lambda}(t)$ is fed to the i-th point these devices produce shift of $\hat{\lambda}$ by $\Delta_{(i)}$, similar to that noted above.

Let us discuss the physical meaning of factors $\Lambda_n^{(i)}$, expressed in the discrete case by formula (6.6.79). The first factor $P(\mathbf{y}|\hat{\lambda} + \Delta_{(i)})$, formed in circuit terms by the detection receiver, reflects the magnitude of some peak which characterizes the signal.

The second quantity $P_{\text{сш}(i)}$ can be reduced to form

$$P_{\text{сш}(i)} = [\det(\mathbf{A}_{(i)}^{-1} + \mathbf{R})]^{-1/2} \exp\left\{-\eta_{\text{сш}(i)}^{+}(\mathbf{A}_{(i)}^{-1} + \mathbf{R})^{-1}\eta_{\text{сш}(i)}\right\},$$

where vector quantity $\eta_{\text{сш}(i)} = \mathbf{A}_{(i)}^{-1}\mathbf{z}_{(i)} + \hat{\lambda} + \Delta_{(i)} - \overline{\lambda}$ of the dimensionality of the measured parameter constitutes (after subtracting $\overline{\lambda}$) the true value of the measured coordinates with correlation matrix $\mathbf{R}$, mixed with noise whose correlation matrix is equal to $\mathbf{A}_{(i)}^{-1}$. One can prove this by expanding $z_{(i)}$ analogously to (6.6.23) near the true value of the peak of the side overshoot.

Thus, $P_{\text{сш}(i)}$, formed by nonlinear unit 5, shows the "likelihood" of the given output of the i-th adder of intercommunication as a random variable with correlation matrix $\mathbf{A}_{(i)}^{-1} + \mathbf{R}$.

Analogously we can reduce factor $P_{\text{ш}(i)}$ to the form

$$P_{\text{ш}(i)} = [\det \mathbf{A}_{(i)}^{-1}]^{-1/2} \exp\left\{-\eta_{\text{ш}(i)}^{+}\mathbf{A}_{(i)}\eta_{\text{ш}(i)}\right\},$$

where $\eta_{\text{ш}(i)} = \mathbf{A}_{(i)}^{-1}\mathbf{z}_{(i)}$ with dimensionality of the measured coordinate constitutes equivalent noise at the input of the i-th discriminator (with small mismatches).

Consequently, $P_{\text{ш}(i)}$, formed in 4, shows the likelihood of the given input of the i-th discriminator as a random variable with correlation matrix $\mathbf{A}_{(i)}^{-1}$. As a whole $P_{\text{сш}(i)}/P_{\text{ш}(i)}$ constitutes as it were the likelihood ratio for separation of the parameter from its additive mixture with noise. The less a given peak is suppressed by noise, the bigger factors $P(\mathbf{y}|\hat{\lambda} + \Delta_{(i)})$ and $P_{\text{сш}(i)}/P_{\text{ш}(i)}$. Inasmuch as in the process of work there constantly occurs accumulation at the output of detection receivers and nonlinear units, the true peak obtains weight $\Lambda_n^{(i)}$, more

and more exceeding the weight of all other peaks, until this weight remains the only one differing from zero.

Thus, allowance for side peaks of the considered type in steady-state regime does not lead to circuit deviations from the meter considered earlier. Making the circuit more complex has, on the other hand, the advantage that it makes optional initial lock-on with accuracy equal to the width of the basic peak of the likelihood function; tracking is established even if initial error leads to a hit on a side peak, allowed for in synthesis.

Now let us turn to the case when the peak of the autocorrelation function is the only one, and side peaks of the likelihood function are connected only with the noise background. Here approximation of the likelihood function based on relationships (6.6.4) and (6.6.7) is convenient:

$$P(\mathbf{y}|\boldsymbol{\lambda}) = \sum_{i} P(\mathbf{y}|\check{\boldsymbol{\lambda}}_{(i)}) \exp\left\{-\frac{1}{2}(\boldsymbol{\lambda}-\check{\boldsymbol{\lambda}}_{(i)})\check{\mathbf{A}}_{(i)}(\boldsymbol{\lambda}-\check{\boldsymbol{\lambda}}_{(i)})\right\}. \qquad (6.6.82)$$

It is necessary to stipulate that the number of peaks of the likelihood function can be arbitrarily large with a large domain of definition of λ, so that it is reasonable to approximate by (6.6.63) those peaks which exceed in magnitude a certain threshold level. As we can see, such approximation, being simple in form, is not so in essence. Cross multiplying (6.6.82) with a priori Gaussian distribution (6.6.8) and integrating; for the operator of an optimum meter for the last moment of the time of observation we obtain the following expression

$$\hat{\lambda}_n = \sum_{i} \tilde{\Lambda}_n^{(i)} \sum_{k=1}^{n} C_{(i)nk} K_{(i)k} [\check{\lambda}_{(i)k} - \bar{\lambda}_k] + \bar{\lambda}_n, \qquad (6.6.83)$$

where

$$C_{(i)} = [A_{(i)} + V]^{-1}; \quad A_{(i)} = \| K_{(i)k}{}^2{}_{kl} \|; \quad \tilde{\Lambda}_n^{(i)} = \Lambda_n^{(i)} / \sum_{i} \Lambda_n^{(i)};$$

$$\Lambda_n^{(i)} = P(\mathbf{y}|\check{\boldsymbol{\lambda}}_{(i)}) [\det(A_{(i)}^{-1} + R)]^{-1/2} \times$$
$$\times \exp\left\{-\frac{1}{2}(\check{\boldsymbol{\lambda}}_{(i)} - \bar{\boldsymbol{\lambda}})^{+} (A_{(i)}^{-1} + R)^{-1} (\check{\boldsymbol{\lambda}}_{(i)} - \bar{\boldsymbol{\lambda}})\right\}. \qquad (6.6.84)$$

With transition to continuous observations (6.6.83) reduces to the form

$$\hat{\lambda}(t) = \sum_{i} \tilde{\Lambda}^{(i)}(t) \int_{t_0}^{t} c_{(i)}(t,\tau) K_{(i)}(\tau) [\check{\lambda}_{(i)}(\tau) - \bar{\lambda}(\tau)] d\tau + \bar{\lambda}(t). \qquad (6.6.85)$$

A block diagram interpreting relationship (6.6.85) is presented in Fig. 6.19. It consists of separate parallel circuits, very similar to the nontracking meter of (Fig. 6.17). The estimator unit issues a set of points of maximum value of the

likelihood function and corresponding points of current values of "sharpness" of peaks of $P(y|\lambda)$. After subtracting the mean value from quantities $\hat{\lambda}_{(i)}(t)$ they are processed by linear filters with pulse responses $c_{(i)}(t, \tau)K_{(i)}(\tau)$, and then results are summed with normalized weighting factors $\lambda^{(i)}(t)$. After reverse input of the mean value there will be formed estimate $\hat{\lambda}(t)$.

Partial circuits of Fig. 6.19 have filters, absolutely similar to those considered in Paragraph 6.6.5, and therefore do not require explanation. We need only indicate the physical meaning of factors $\tilde{\lambda}^{(i)}(t)$. In distinction from the past case, besides the measure of the height of peak $P(\mathbf{y}|\check{\lambda}_{(i)})$ these factors have in their composition quantities

$$P_{em(i)} = [\det(\mathbf{A}_{(i)}^{-1} + \mathbf{R})]^{-1/2} \times$$
$$\times \exp\left\{-\frac{1}{2}(\check{\lambda}_{(i)} - \bar{\lambda})^{+} (\mathbf{A}_{(i)}^{-1} + \mathbf{R})^{-1} (\check{\lambda}_{(i)} - \bar{\lambda})\right\}, \qquad (6.6.86)$$

showing the "likelihood" that the position of the i-th peak is a random function with correlation matrix $\mathbf{A}_{(i)}^{-1} + \mathbf{R}$, i.e., consists of a variable parameter with correlation matrix $\mathbf{R}$ and a fluctuating disturbance with correlation matrix $\mathbf{A}_{(i)}^{-1}$. Practically only one of the peaks is true. The others, first, turn out to be of small height, which is necessarily revealed in the process of accumulation, and second, they contain in their position a shift from the true value of λ, which being accumulated in inertial nonlinear unit 4, according to (6.6.86), leads to further decrease of the weight of this peak during formation of the estimate. As a result the circuit again passes into the known circuit of Fig. 6.17, working on one peak.

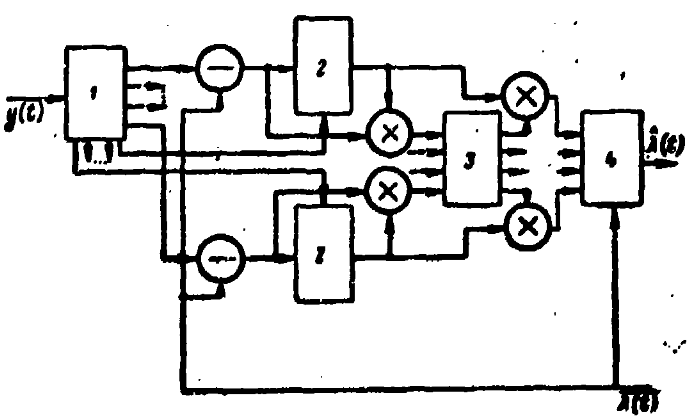

Fig. 6.19. Optimum meter with allowance for noise peaks: 1) estimator unit; 2) linear filter with pulse responses $c_{(i)}(t, \tau)K_{(i)}(\tau)$; 3) unit for producing normalizing factors; 4) general adder.

The conducted investigation shows that for a sufficiently large observation time the meter circuit, synthesized on the basis of a "single-peak" approximation, in structure, and consequently also in performance, will differ little from systems built with finer allowance for the real structure of the likelihood paragraphs. This indicates the sufficient universality of results of the preceding points.

6.6.8. Case of a Partially Assigned Discriminator

Everywhere above we assumed that the input realization does not depend on results of estimation of the parameter, i.e., the method of forming the realization and the method of estimation do not have points of contact. In practice this is not always so. As an example we shall consider a radar goniometer system with a tracking antenna. From the antenna output there proceeds a signal, a certain parameter of modulation of which depends on the mismatch between the direction to the sounded target and the axis of the radiation pattern of the antenna. From physical considerations it is clear that for a maximum level of the reflected signal the axis of the antenna should to the best possible degree coincide with the target direction, but inasmuch as we have at our disposal no more exact judgement about the position of the target than the estimate of this position, the axis of the antenna should coincide with the current estimated position of the target.

Hence it is clear that the input (for the receiver) realization from which the estimate is produced directly depends on the estimate. We are primarily interested in cases when the dependence on the estimate is expressed through the difference between it and the true value of the parameter. It is reasonable to assign these cases to the class of systems with a partially assigned discriminator. Actually, the antenna system, during synthesis considered here to be assigned, to a considerable extent predetermines optimum construction of the subsequent circuits.

The indicated "feedback" between the realization and the estimate deprives us of the possibility of applying without reservations the apparatus developed in the preceding points. Let us start from the fact that conditional risk

$$R(P_{\bullet}, \hat{\lambda}) = \iint l(\lambda, \hat{\lambda}) P(y|\lambda - \hat{\lambda}) P_{\bullet}(\lambda) d\lambda dy \qquad (6.6.87)$$

turns out to depend on $\hat{\lambda}$ both through the loss function and the likelihood function, so that, strictly speaking, its variation should be produced in full by both these functions. This leads to very tiresome calculations and a very complicated meter circuit.

Considering the case of discrete observation, we simplify the problem, assuming that the estimate should be produced for moment t_n from the sampling produced at moments of time $t_1, t_2, \ldots, t_{n-1}$. Then relationship (6.6.87) for a quadratic loss function will be rewritten in the form

$$R(P_0, \hat{\lambda}) = \int \dots \int \sum_{i=1}^{n} (\lambda_i - \hat{\lambda}_i)^2 \times$$
$$\times P(y_1, \dots, y_{n-1} | \lambda_1 - \hat{\lambda}_1, \dots, \lambda_{n-1} - \hat{\lambda}_{n-1}) \times$$
$$\times P_0(\lambda_1, \dots, \lambda_n) d\lambda_1, \dots, d\lambda_n dy_1, \dots, dy_{n-1}. \tag{6.6.88}$$

The solution is produced as if with a delay of one step of quantization. Actually such delay necessarily appears due to the irremediable inertia of feedback. This assumption leads to great simplification of the problem of estimation. Variation must be produced only for estimate $\hat{\lambda}_n$, contained in the loss function. Assuming again that conditions of approximation $P(\mathbf{y}|\boldsymbol{\lambda} - \hat{\lambda})$ are satisfied by the Gaussian function

$$P(\mathbf{y}|\lambda - \hat{\lambda}) \approx P(\mathbf{y}|0) \exp\left\{z^+(\lambda - \hat{\lambda}) - \frac{1}{2}(\lambda - \hat{\lambda})^+ A(\lambda - \hat{\lambda})\right\}, \tag{6.6.89}$$

where

$$z_k = \frac{\partial}{\partial \varepsilon_k} \ln P(\mathbf{y}|\varepsilon)\Big|_{\varepsilon=0}; \quad A_{ik} = -\frac{\partial^2}{\partial \varepsilon_i \partial \varepsilon_k} \ln P(\mathbf{y}|\varepsilon)\Big|_{\varepsilon=0}, \tag{6.6.90}$$

we obtain a solution in two equivalent forms, very similar to (6.6.19) and (6.6.21):

$$\hat{\lambda}_n = \sum_{k=1}^{n-1} C_{nk} [z_k + K_k (\hat{\lambda}_k - \overline{\lambda}_k)] + \overline{\lambda}_n, \tag{6.6.91}$$

$$\hat{\lambda}_n = \sum_{k=1}^{n-1} G_{nk} z_k + \overline{\lambda}_n. \tag{6.6.92}$$

Functions C_{nk}, G_{nk} and K_k introduced here do not require explanations. Relationships (6.6.91) and (6.6.92) differ from (6.6.19) and (6.6.21) only in their upper limit (n - 1 instead of n). It is obvious that for large n and a sufficiently small step of quantization the difference between the two cases is small and at the limit disappears completely, so that for the case of continuous observation

$$\int_{t_0} c(t, \tau) [z(\tau) + K(\tau) (\hat{\lambda}(\tau) - \overline{\lambda(\tau)})] d\tau + \overline{\lambda(t)}, \tag{6.6.93}$$

$$\hat{\lambda}(t) = \int_{t_0}^{t} g(t, \tau) z(\tau) d\tau + \overline{\lambda(t)}. \tag{6.6.94}$$

Relationships (6.6.93) and (6.6.94) completely coincide with (6.6.29) and (6.6.31). However, it is worth recalling that operators for processing a signal in the discriminator and accuracy unit are obtained by differentiation not at the estimated point, but at zero, which is explained by assignment of part of the discriminator, usually carried out in the form of elements of the antenna system.

The shown specific character of solutions of (6.6.93) and (6. 6.94) is also reflected on the form of the block diagrams of optimum meters. In Fig. 6.20 there is a circuit interpreting relationship (6.6.94). In the figure there are separated element 1, forming the realization depending upon the estimate, optimized parts of discriminator 2 and accuracy unit 3, a circuit for input of the mean value, and smoothing filter G. Specially marked is the signal fed to the input $y_{BX}(t; \lambda)$ which usually differs from $y(t; \lambda - \hat{\lambda})$ both in structure, and also in physical nature. Usually this is an electromagnetic wave, proceeding to the antenna aperture. Its parameters depend on true parameters of the target λ.

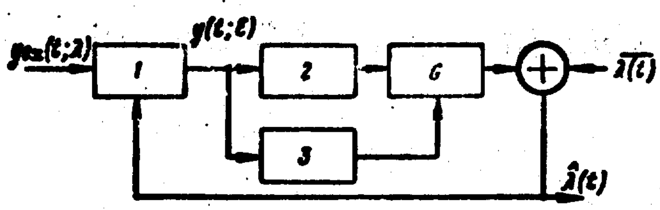

Fig. 6.20. Optimum meter for a partially assigned discriminator: 1) assigned part; 2) optimized part of the discriminator; 3) optimized part of the accuracy unit; G — linear filter with pulse response $g(t, \tau)$.

The result of the present paragraph is the fact that all subsequent results in synthesis of optimum discriminators (§ 6.7) completely pertain also to the case of a partially assigned discriminator, if we consider the specific character of relationships (6.6.90).

6.6.9. Optimum Meter of a Linear Functional of a Parameter with Gaussian Statistics

In a whole series of cases of radar practice there arises the necessity of measurement not only of a parameter coded in the received mixture of signal with noise $y(t)$ but of a certain linear or nonlinear function or functional of the coded quantity. An example is measurement of the speed of an object by readings of a range finder in an incoherent radar or measurement of Cartesian coordinates of the object by a signal in which there are coded the radio (usually spherical) coordinates of this object.

In this section we consider methods of optimum construction of meters of linear functionals $\nu(t)$ of parameters $\lambda(t)$ with Gaussian distribution, directly coded in y(t). The relationship between $\nu(t)$ and $\lambda(t)$ is established by linear integral relationship

$$\nu(t)=\int_{T_1}^{T_2} F(t, s)\lambda(s)ds \qquad (6.6.95)$$

and is wholly determined by the form of function F(t, s). In the particular case when F(t, s) is a k-th order derivative of δ-function

$$F(t, s)=\delta^{(k)}(t-s) \quad (k=0, 1, 2, \ldots),$$

$\nu(t)$ is the k-th derivative of $\lambda(t)$. If $F(t, s) = (t - s)^{k-1}$ $(k = 1, 2, \ldots)$, $T_2 = t$, $\nu(t)$ is the k-multiple integral of $\lambda(t)$. For other examples we can consider a functional which is a certain smoothed or expected value of $\lambda(t)$, etc. If certain functions $\lambda(t)$ and $\nu(t)$ depend on constant random parameters $a_i (i = 1, \ldots, m)$, differing little from mean values $\bar{a}$, so that

$$a_i=\bar{a}_i+\mu_i,$$

it is possible to expand $\lambda(t)$ and $\nu(t)$ in powers of μ_i, limiting ourselves to terms of the first order of smallness with respect to μ_i. Then we obtain relationships

$$\lambda(t)\approx\lambda(t;\ \bar{a}_1, \ldots, \bar{a}_m)+\sum_i \mu_i \frac{\partial}{\partial a_i}\lambda(t;\bar{a}_1, \ldots, \bar{a}_m),$$

$$\nu(t)\approx\nu(t;\ \bar{a}_1, \ldots, \bar{a}_m)+\sum_i \mu_i \frac{\partial}{\partial a_i}\nu(t;\ \bar{a}_1, \ldots, \bar{a}_m),$$

by which it is also possible to approximately establish a linear relationship between $\nu(t)$ and $\lambda(t)$, allowing us to consider $\nu(t)$ a linear functional of $\lambda(t)$.

The method of synthesis of optimum meters of linear functionals is closely connected with the method of synthesis of meters of quantity $\lambda(t)$ directly. As also in paragraph 6.6.2, it is convenient to quantize in time all the considered quantities. An analog to (6.6.95) in this case is relationship

$$\nu_m=\sum_{k=N_1}^{N_2} F_{mk}\lambda_k. \qquad (6.6.\ldots)$$

We shall vary the mean risk for ν_m with quadratic loss function

$$R(P_0, \hat{\nu})=\int\ldots\int(\nu_m-\hat{\nu}_m)^2 P(y_1, \ldots, y_n\,|\,\lambda_1, \ldots, \lambda_n)\times$$
$$\times P_0(\lambda_{N_1}, \ldots, \lambda_{N_2})\,d\lambda_{N_1}, \ldots, d\lambda_{N_2}dy_1\ldots dy_n,$$

where we assume that signal y(t) is accessible to observation at moments $t_1, \ldots, t_n$, in no way connected with moment t_m which is the argument of functional ν_m. As a

result we have equation

$$\int (v_m - \hat{v}_m)^2 P_{(1,n)}(y|\lambda) P_{0(N_1,N_2)}(\lambda) d\lambda = 0, \qquad (6.6.97)$$

where subscripts in the likelihood function and the a priori distribution signify moments of reading. The formal solution of (6.6.97) is

$$\hat{v}_m = \sum_{k=N_1}^{N_2} F_{mk} \frac{\int \lambda_k P_{(1,n)}(y|\lambda) P_{0(N_1,N_2)}(\lambda) d\lambda}{\int P_{(1,n)}(y|\lambda) P_{0(N_1,N_2)}(\lambda) d\lambda} = \sum_{k=N_1}^{N_2} F_{mk} \bar{\lambda}_k(1, n), \qquad (6.6.98)$$

where there is introduced the conditional mathematical expectation $\bar{\lambda}_k(1, n)$ of the parameter at the k-th moment of time, composed on the basis of observation of mixture y(t) at moments $t_1, \ldots, t_n$.

In § 6.5 we indicated that the conditional mathematical expectation in principle is the optimum estimate of the parameter. However, all methods of expanding the operator of conditional mathematical expectation above pertained to a case when the estimate was produced for the k-th moment of time for a sampling, obtained up to the k-th moment. In relationship (6.6.98) conditional mathematical expectation is taken for the sampling, ending with the n-th moment, which can exceed the k-th. As a result we cannot directly use the method of obtaining an optimum estimate developed in Paragraph 6.6.2. In other words, with the requirement of physical realizability of the operator of formation of an estimate of the actual parameter $\lambda(t)$ the linear functional of the estimate, taken according to (6.6.98), will not be the optimum estimate of the linear functional. In the practical sphere, for instance, it follows from this that simple differentiation of readings of the range finder in general will not give the value of speed of the object measured with the least error. Moreover, it is possible to indicate cases when this differentiation leads to great error of measurement, limited only by the passband of the discriminator.

For finding a physically realizable solution of equation (6.6.97) we again assume fulfillment of conditions in which the likelihood function can be approximated by Gaussian curve (6.6.3), and we assume Gaussian statistics of parameter $\lambda(t)$. The expansion of the logarithm of the likelihood function we conduct at the point of the current physically realizable estimate, which can always be required beforehand. Substituting (6.6.3) and (6.6.8) in (6.6.97) and integrating according to the method of Paragraph 6.6.2, taking into account the diagonal nature of the matrix of second derivatives, we have

$$\hat{v}_m = \sum_{k=N_1}^{N_2} F_{mk} \left\{ \bar{\lambda}_k + \sum_{i=1}^{n} C_{ki} [z_i + K_i(\hat{\lambda}_i - \bar{\lambda}_i)] \right\} =$$

$$= \sum_{k=1}^{n} B_{mk} [z_k + K_k(\hat{\lambda}_k - \bar{\lambda}_k)] + \sum_{k=N_1}^{N_2} F_{mk} \bar{\lambda}_k. \tag{6.6.99}$$

where $\mathbf{C} = [\mathbf{V} + \mathbf{Q}]^{-1}$ – inverse matrix; $\mathbf{Q}$ – matrix of order $(N_2 - N_1)^2$, determined by relationship

$$Q_{ik} = \begin{cases} K_i \delta_{ik}, & 1 < i, \ k < n, \\ 0, & i, \ k > n, \\ 0, & i, \ k < 1. \end{cases} \tag{6.6.100}$$

Finally, matrix $\mathbf{B}$ is determined in expanded form by relationship

$$B_{mk} = \begin{cases} \sum_{i=N_1}^{N_2} F_{mi} C_{ik}, & 1 < k < n, \\ 0, & \begin{cases} k < 1, \\ k > n. \end{cases} \end{cases} \tag{6.6.101}$$

From (6.6.101), taking into account values of C_{ik} and V_{ik}, we can obtain equation

$$B_{mk} + \sum_{i=1}^{n} B_{mi} K_i R_{ik} = \sum_{i=N_1}^{N_2} F_{mi} R_{ik}, \tag{6.6.102}$$

connecting discrete linear operator B_{ik} with the correlation function of parameter R_{ik} and the operator of linear functional F_{ik}.

Another presentation of solution of (6.6.99) has the form

$$\hat{v}_m = \sum_{k=1}^{n} H_{mk} z_k + \sum_{k=N_1}^{N_2} F_{mk} \bar{\lambda}_k, \tag{6.6.103}$$

where H_{mk} is related to B_{mk} by relationship

$$H_{mk} = B_{mk} + \sum_{i=k}^{n} B_{mi} K_i G_{ik}, \tag{6.6.104}$$

and G_{ik}, in turn, is determined through R_{ik} by equations (6.6.20) and (6.6.22), describing the meter of the actual parameter $\lambda(t)$.

Transition to continuous observation gives

$$\hat{\nu}(t_1) = \int_{t_0}^{t} b(t_1, s)[z(s) + \bar{K}(s)(\hat{\lambda}(s) - \overline{\lambda(s)})]\,ds +$$
$$+ \int_{T_1}^{T_2} F(t_1, s)\overline{\lambda(s)}\,ds, \qquad (6.6.105)$$

where the relationship between $z(t)$, $K(t)$ and $F(t, s)$ and z_k, K_k and F_{ik} in the light of what was slated above does not require explanation; $t_i = m\Delta + t_0$; $t = n\Delta + t_0$; and $b(t_i, t_j) = B_{ij}$ — a function, connected with $R(t, \tau)$ by an integral equation, analogous to (6.6.102):

$$b(t_1, \tau) + \int_{t_0}^{t} b(t_1, s) R(s, \tau)\,ds = \int_{T_1}^{T_2} F(t_1, s) R(s, \tau)\,ds$$
$$[b(t_1, \tau) = 0 \text{ when } \tau > t]. \qquad (6.6.106)$$

We shall interpret relationship (6.6.106) in the form of the block diagram of an optimum meter of the functional of the parameter. According to Fig. 6.21 this circuit, first, contains the complete circuit of an optimum meter of $\lambda(t)$, repeating the circuit of Fig. 6.13. From the adder, connecting the discriminator output to the internal loop of feedback in smoothing circuits, we make a tap to a linear filter with pulse response $b(t, s)$. We pass the function depicting the a priori mean value of the parameter through filter $F(t, s)$. Inasmuch as $\overline{\lambda(t)}$ is known beforehand, physically unrealizable elements here do not appear. Addition of the two formed output voltages gives the optimum estimate of the functional $\hat{\nu}(t)$.

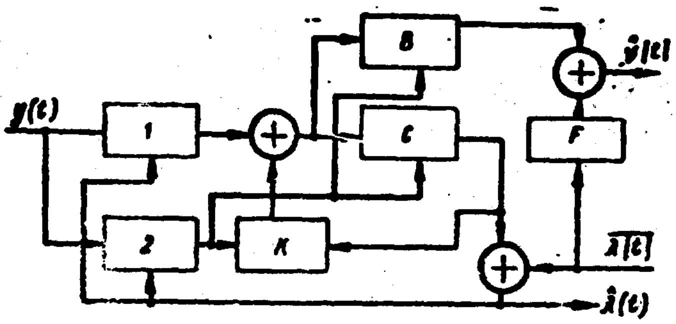

Fig. 6.21. Optimum meter of a linear functional on basis of the two-loop variant of the meter of a parameter: 1) discriminator; 2) accuracy unit; C — linear filter with pulse response $c(t, \tau)$; K — amplifier with gain factor $K(t)$; B — linear filter with pulse response $b(t, \tau)$; F — operator of the linear functional.

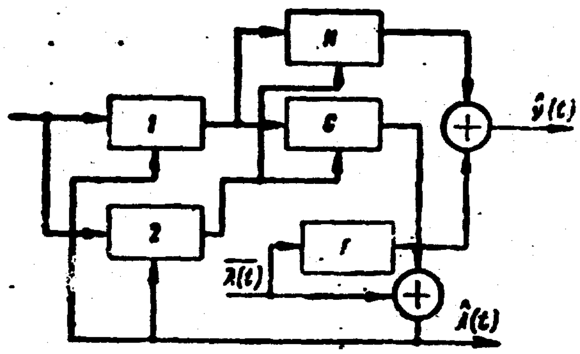

Fig. 6.22. Optimum meter of a linear functional on the basis of a single-loop variant of the meter of a parameter: 1) discriminator; 2) accuracy unit; G, H — linear filters with pulse responses $g(t, \tau)$, $h(t, \tau)$; F — operator of the linear functional.

Inasmuch as $b(t, s) \neq \int_{T_1}^{T_2} F(t, s)c(s, \tau)ds$, the assumption that linear functional $\int_{T_1}^{T_2} F(t, s)\hat{\lambda}(s)ds$ from the estimate of the parameter is not the optimum estimate of the linear functional $\int_{T_1}^{T_2} F(t, s)\lambda(s)ds$ of this parameter was confirmed.

Another modification of the optimum circuit is obtained by means of transition to continuous observation in relationship (6.6.103):

$$\hat{v}(t_1) = \int_{t_0}^{t} h(t_1, s)\, z(s)\, ds + \int_{T_1}^{T_2} F(t_1, s)\, \overline{\lambda(s)}\, ds. \tag{6.6.107}$$

It is conveniently combined with the single-loop variant of a meter of $\lambda(t)$ (Fig. 6.14). The complicated variant of a joint meter of $\lambda(t)$ and $\nu(t)$ is presented in Fig. 6.22. Output of the discriminator is processed by a linear filter H with pulse response $h(t, s)$ satisfying a relationship, analogous to (6.6.104):

$$h(t_1, s) = b(t_1, s) - \int_{\tau}^{t} b(t_1, s)\, K(s)\, g(s, \tau)\, ds, \tag{6.6.108}$$

where $g(t, \tau)$ is determined by the correlation function of the parameter according to equations (6.6.31) and (6.6.32).

For characterizing accuracy of measurement of the linear functional we can obtain the following formulas:

$$R_{\text{вых}}(t_1, t_2) = \int_{t_0}^{t} b(t_1, s)\, F(t_2, s)\, ds, \tag{6.6.109}$$

$$\sigma^2_{\text{вых}}(t_1) = \int_{t_0}^{t} b(t_1, s)\, F(t_1, s)\, ds.$$

The given results show that for optimum measurement of an arbitrary linear functional $\nu(t)$ of parameter $\lambda(t)$ it is sufficient to introduce in the optimum meter of the actual parameter two additional linear filters, to one of which there is a tap from the smoothing circuits, and to the other, a tap from the feed circuit feeding the mean value of $\overline{\lambda(t)}$. Characteristics of filters are determined by the form of the functional, by the correlation function of parameter $R(t, \tau)$ and by the accuracy characteristic of the discriminator $K(t)$.

In connection with this for every form of functional and correlation properties of $\lambda(t)$ determination of the form of filters constitutes a separate problem, the

concrete solution of which will be the subject of the corresponding paragraph in § 6.8. With respect to nonlinear processing of the input signal in discriminators and accuracy units for construction of optimum meters of functionals sufficient will be those results which in concrete cases are obtained later for different statistical properties of mixture $y(t)$ and codings in it of $\lambda(t)$.

6.6.10. Relationship of the Method of Synthesis with Other Branches of the Theory of Solutions

In conclusion we deem it useful to discuss briefly interconnections of the developed method of synthesis with other branches of the theory of statistical solutions. In the first place one should stress the link with the theory of statistical estimates, observed in two spheres: with respect to producible operations and with respect to accuracy characteristics. It is important to indicate that the discriminator of tracking meters constantly issues an efficient estimate of current mismatch $\varepsilon = \lambda - \hat{\lambda}$ between the true and measured values of the parameter. It is especially convenient to use this for a discrete signal. Disregarding parametric fluctuations, according to Paragraph 6.6.3, we have

$$z_k = \bar{K}_k(\varepsilon_k + \eta_k),$$

i.e., z_k with some proportionality factor

$$\bar{K}_k = -\frac{\partial^2}{\partial\lambda^2}\ln P_k(y|\lambda)$$

is equal to mismatch plus some fluctuating addition. Variance of addition is equal to $1/\bar{K}_k$, i.e., to variance of the efficient estimate of the constant parameter for the period of quantization. This proves the formulated proposition.

During continuous observation for proof of the same fact it is necessary to separate a certain segment Δ. Then it turns out that variance of the estimate of mismatch for that time is equal to $1/(\overline{K(t)}\Delta)$, i.e., again to the variance of the efficient estimate of a constant parameter.

However, for larger intervals of observation error of measurement of a varying quantity is not equal to $1/(\overline{K(t)}\Delta)$, and only monotonically decreases with growth of $\overline{K(t)}$. This is understandable, too, inasmuch as besides fluctuating ones it is necessary to track dynamic disturbances.

One more interrelationship with the theory of estimation will be revealed in § 6.8, when we consider measurement of a quantity, the law of change of which in turn depends on several constant coefficients [14].

We turn now to the relationship with the theory of linear filtration. It, obviously, follows from equations (6.6.31) and (6.6.53). They coincide with equations of Wiener filtration for separation of a signal from an additive mixture of it with white noise. Here, if K(t) is not subject to averaging, or if averaging d... eliminate the dependence of $\overline{K}(t)$ on time in the equivalent Wiener problem should consider interference in the form of white noise with variable intensity, and if $\overline{K}(t) = K$ does not depend on time, then white noise is stationary. The obtained result was a consequence of the assumption of Gaussian statistics of $\lambda(t)$, but let us note that the condition of normalness of noise at the output of the discriminator turns out to be unnecessary.

The shown reduction to the Wiener case is valid for a Gaussian parameter. In general, as in § 6.9 the example of a Markovian parameter will convincingly show, operations of smoothing are nonlinear. Furthermore, one should remember that relative to the input realization the meter remains an especially nonlinear device.

Comparison of the developed method of synthesis of a meter of a Markovian parameter with the method of R. L. Stratonovich later (§ 6.9) will show their great similarity. A peculiarity of the method developed above, besides the assumption of Gaussian distribution of $\lambda(t)$, is the clear division into primary and secondary operations of processing, the introduction of concepts of an optimum discriminator, accuracy unit and smoothing circuits. This gives the possibility of studying separately questions of construction of these devices depending upon properties, correspondingly, of the signal and the parameter coded in it.

We shall discuss now the problem of forming the a posteriori probability in the whole domain of definition of $\lambda(t)$. As we proved in Chapter 3, this operation is necessary in detection regime, when a priori information is scant. We assume the presence of a unit of optimum receivers, covering the whole region with small quantization. At the same time discriminators and accuracy units are devices, narrowly selective with respect to λ, and technically their use is preferable with low a posteriori inaccuracy.

The same unit of optimum receivers (estimator unit) is necessary in nonclosed circuits of meters, even when there is carried out transition to forming of few characteristics of the a posteriori probability. This shows the technical advantages of closed circuits, in which the actual estimate controls the region selected by the receiver. However, in conditions of large a priori variances or high-level noises,

where there will be frequent breakoffs of tracking, use of multichannel units of receivers is inevitable.

§ 6.7. Synthesis of Discriminators For Different Statistical Properties of Signals

Results obtained in § 6.6 shows that as a basic element of an optimum meter we include discriminators, which form magnitude $z(t)$ (or z_k), which is the measure of mismatch between the true and measured values of the parameter. In the same place we introduced the basic characteristic of a discriminator $\overline{K}(t)$ (or $\overline{K}_k$). Concretely, both functions are determined by the statistics of the mixture of the signal with noise at the input $y(t)$ and the method of encoding in it the parameter $\lambda(t)$. However, it is possible to reveal certain other general rules, allowing one to quickly find operations of a discriminator and its characteristic in the most interesting cases. In the present section we shall consider different forms of operations of a discriminator and explain characteristics of their accuracy. At the base of our classification, in distinction from subsequent chapters, we here place the form of statistics of $y(t)$, more exactly its useful component, depending on parameter $\lambda(t)$. Separately we consider cases of regular signals, signals with random phases, Gaussian signals, and so forth. Here, we always consider continuous observation, when the likelihood functional, operations of the discriminator and its characteristics are expressed, correspondingly, by formulas (6.6.35)-(6.6.37) and (6.6.44)-(6.6.46).

6.7.1. Regular Signals in Gaussian Noises

For simplicity we first consider the case of one signal $v(t, \lambda(t))$, taken against a background of white noise with spectral density N_0. The logarithm of the likelihood functional in an elementary segment of observation $(t - \Delta/2, t + \Delta/2)$ here has the form

$$L(y, \lambda) = C - \frac{1}{2N_0} \int_{t-\Delta/2}^{t+\Delta/2} [y(\tau) - v(\tau, \lambda(\tau))]^2 \, d\tau, \qquad (6.7.1)$$

where C — constant, depending neither on realization $y(t)$ nor the measured parameter $\lambda(t)$.

By comparison of (6.7.1) and (6.6.35) we prove that

$$l(\tau, \lambda(\tau), y(\tau)) = -\frac{1}{2N_0} [y(\tau) - v(\tau, \lambda(\tau))]^2, \qquad (6.7.2)$$

whence from (6.6.36) we have

$$z(t) = \frac{1}{N_0} \frac{\partial v(t, \hat{\lambda}(t))}{\partial \lambda} [y(t) - v(t, \hat{\lambda}(t))]. \qquad (6.7.3)$$

Formula (6.7.3) is illustrated in Fig. 6.23. For formation of $z(t)$ it is necessary to subtract from realization $y(t)$ a function describing a regular signal with the measured value of the parameter $\hat{\lambda}(t)$ and to multiply the result by the derivative of this function with respect to the parameter, also taken for the estimated value $\hat{\lambda}(t)$. For mismatch of $\lambda(t)$ and $\hat{\lambda}(t)$ subtraction of the regular part of the mixture will be incomplete and the remainder, correlating with $\frac{\partial}{\partial\lambda} v(t, \hat{\lambda}(t))$, will give at the output a quantity proportional to $\lambda - \hat{\lambda}$. As we see, operation of the discriminator here was purely inertialess and even linear relative to the input signal (but not with respect to difference $\lambda - \hat{\lambda}$).

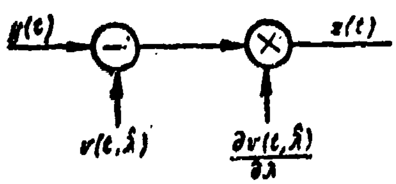

Fig. 6.23. Discriminator for a regular signal in white noise.

From (6.6.37) it is easy to prove that the characteristic of accuracy of discriminator $K = \overline{K(t)}$ is equal to

$$K=\lim_{T\to\infty}\frac{1}{TN_0}\int_0^T\left[\frac{\partial v(t,\lambda(t))}{\partial\lambda}\right]^2 dt. \qquad (6.7.4)$$

If $v(t, \lambda)$ for a fixed λ is a periodic function, then according to (6.6.45) it is possible to consider function (6.7.3) integrated over period of repetition T_r a discrete signal at the discriminator output:

$$z_k=\frac{1}{N_0}\int_{(k-1)T_r}^{kT_r}\frac{\partial v(\tau,\hat{\lambda}_k)}{\partial\lambda}\left[y(\tau)-v(\tau,\hat{\lambda}_k)\right]d\tau. \qquad (6.7.5)$$

Here, in relationship (6.7.4) there is no need to carry out passage to the limit; it is sufficient that T embrace a whole number of periods T_r:

$$K=\frac{1}{T_rN_0}\int_0^{T_r}\left[\frac{\partial v(t,\lambda)}{\partial\lambda}\right]^2 dt. \qquad (6.7.6)$$

According to (6.7.4) and (6.7.6) accuracy of a discriminator is determined by the ratio of the mean value of the square of the derivative of the regular signal with respect to the parameter to the spectral density of white noise. This result is well-known [19].

If we now generalize (6.7.1) for several input signals ${}^{(1)}y(t), \ldots, {}^{(m)}y(t)$, consisting of regular signals ${}^{(1)}v(t, \lambda), \ldots, {}^{(m)}v(t, \lambda)$ and generally correlated white noises ${}^{(1)}n(t), \ldots, {}^{(m)}n(t)$, so that

$${}^{(i)}y(t)={}^{(i)}v(t,\lambda)+{}^{(i)}n(t),$$

$$\int_{-\infty}^{+\infty}\overline{{}^{(i)}n(t)\,{}^{(j)}n(t+\tau)}\,d\tau=N_{ij}\quad(i,j=1,\ldots,m),$$

we can obtain function

$$l(\tau, \lambda, y) = -\frac{1}{2}\sum_{i,j=1}^{m} W_{ij}\,[{}^{(i)}y(t) - {}^{(i)}v(t,\lambda)]\,[{}^{(j)}y(t) - {}^{(j)}v(t,\lambda)] = \frac{1}{2}\,[\mathbf{y}(t) - \mathbf{v}(t,\lambda)]^{+}\,\mathbf{W}\,[\mathbf{y}(t) - \mathbf{v}(t,\lambda)],$$

where there are introduced column vectors $\mathbf{y}(t)$, $\mathbf{v}(t, \lambda)$ and matrix $\mathbf{W} = ||W_{ij}|| = \mathbf{N}^{-1}$, the inverse of $\mathbf{N}$. From this we have

$$z(t) = \sum_{j=1}^{m}\left(\sum_{i=1}^{m} W_{ij}\,\frac{\partial\,{}^{(i)}v(t,\lambda)}{\partial\lambda}\right)[{}^{(j)}y(t) - {}^{(j)}v(t,\lambda)]. \tag{6.7.7}$$

Modification of (6.7.7) for periodic $v(t, \lambda)$ is obvious. According to Fig. 6.24, illustrating relationship (6.7.7), from each input mixture there is subtracted the corresponding regular signal for $\lambda = \hat{\lambda}$, and then results are multiplied by linear combinations of derivatives with respect to the parameter of all regular signals and are added. For uncorrelated noises matrices $\mathbf{N}$ and $\mathbf{W}$ are diagonal, and the discriminator, according to (6.7.7), consists of m independent channels, similar to Fig. 6.23, which are summed with weights, inversely proportional to intensities of noises.

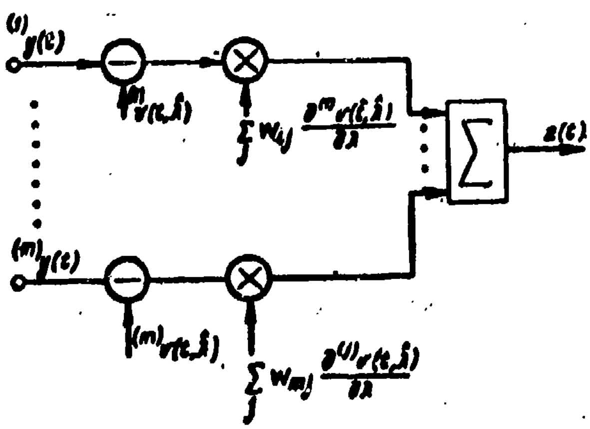

Fig. 6.24. Discriminator for a set of regular signals in white noise.

The accuracy characteristic of a discriminator for the considered case is characterized by the formula

$$K = \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{\partial}{\partial\lambda}\mathbf{v}^{+}(t,\lambda)\,\mathbf{W}\,\frac{\partial}{\partial\lambda}\mathbf{v}(t,\lambda)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_0^T \sum_{i,j=1}^{m} W_{ij}\,\frac{\partial\,{}^{(i)}v(t,\lambda)}{\partial\lambda}\,\frac{\partial\,{}^{(j)}v(t,\lambda)}{\partial\lambda}\,dt \tag{6.7.8}$$

and also is expressed through mean values of the derivative of separate signals with respect to the parameter.

Now we turn to the more complicated case of correlated Gaussian noises. Considering the input mixture unique, we have

$$L(y,\lambda) = C - \frac{1}{2}\iint_{t-\Delta/2}^{t+\Delta/2} W_{\text{п}}(t_1,t_2)\,[y(t_1) - v(t_1,\lambda)]\,[y(t_2) - v(t_2,\lambda)]\,dt_1\,dt_2, \tag{6.7.9}$$

where $W_\Pi(t_1, t_2)$ — a function, the inverse in interval Δ to the correlation function of noise $R_\Pi(t_1, t_2)$, i.e., satisfying equation

$$\int_{t-\Delta/2}^{t+\Delta/2} R_\Pi(t_1, s)\, W_\Pi(s, t_2)\, ds = \delta(t_1 - t_2).$$

It is implied that Δ exceeds the interval of correlation of interference (see § [illegible]).

Several transitions from (6.7.9) to the operation of a discriminator are possible. Questions of physical realizability of the obtained circuits are easy to avoid, if, using the symmetry of the expression under the integral sign in (6.7.9) for arguments t_1, t_2 we produce integration not over a square $t - \Delta/2 \leq t_1, t_2 \leq t + \Delta/2$, but over a triangle $t - \Delta/2 < t_1 < t + \Delta/2$, $t_1 < t_2 < t - \Delta/2$, and then double the result:

$$L(y, \lambda) = C - \int_{t-\Delta/2}^{t+\Delta/2} dt_1 \int_{t-\Delta/2}^{t_1} dt_2 W_\Pi(t_1, t_2)\,[y(t_1) - v(t_1, \lambda)]\,[y(t_2) - v(t_2, \lambda)].$$

Then differentiation with respect to λ and rejection of the outer integral gives the operation of the discriminator:

$$z(t) = \frac{\partial v(t; \hat{\lambda}(t))}{\partial \lambda} \int_{-\infty}^{t} W_\Pi(t_1, t_2)\,[y(t_2) - v(t_2; \hat{\lambda}(t))]\, dt_2 + [y(t) - v(t; \hat{\lambda}(t))] \int_{-\infty}^{t} W_\Pi(t_1, t_2) \frac{\partial v(t_2; \hat{\lambda}(t))}{\partial \lambda}\, dt_2. \qquad (6.7.10)$$

Inasmuch as the interval of observation Δ exceeds the interval of correlation of interferences, we directed the lower limit to $-\infty$. Operations, producible according to (6.7.10), are illustrated in Fig. 6.25. From the realization at the input, as above, there is subtracted the expected form of the signal, and the difference is passed through a linear filter with pulse response $W_\Pi(t, \tau)$ $(\tau < t)$ and is multiplied by the derivative of the expected form of the signal with respect to λ. Additionally this derivative is passed through the same filter and is multiplied by difference $y(t) - v(t; \hat{\lambda}(t))$. Both results of processing are added, forming the signal at the discriminator output. With

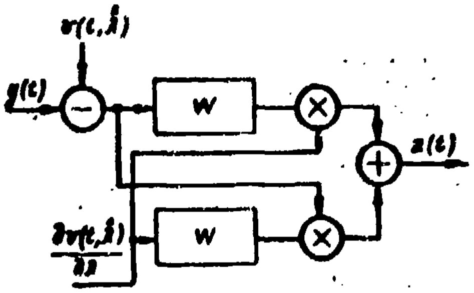

Fig. 6.25. Discriminator for a regular signal in correlated noise (1st variant). W — linear filters with pulse responses $W(t, \tau)$ $(t > \tau)$.

such method there are made no approximations during the transition to z(t).

Another formally simpler method of transition from (6.7.9) to operation of the discriminator is differentiation directly of (6.7.9) with respect to λ, which, taking into account symmetry of the arguments, gives

$$\frac{\partial L}{\partial \lambda} = -\iint\limits_{t-\Delta t/2}^{t+\Delta t/2} W_{\Pi}(t_1, t_2)\frac{\partial v(t_1, \lambda)}{\partial \lambda}[y(t_2) - v(t_2, \lambda)]\,dt_1 dt_2, \qquad (6.7.11)$$

and rejection of one of the integrals. Depending on what argument we choose for rejecting the integral in (6.7.11), it is possible to obtain different, but statistically equivalent expressions for operation of the discriminator:

$$z(t) = \frac{\partial v(t; \hat{\lambda}(t))}{\partial \lambda}\int\limits_{-\infty}^{+\infty}[y(\tau) - v(\tau, \hat{\lambda}(t))]\,W_{\Pi}(t, \tau)\,d\tau \qquad (6.7.12)$$

or

$$z(t) = [y(t) - v(t; \hat{\lambda}(t))]\int\limits_{-\infty}^{+\infty}\frac{\partial v(\tau, \hat{\lambda}(t))}{\partial \lambda}W_{\Pi}(t, \tau)\,d\tau, \qquad (6.7.13)$$

where limits of integration are expanded to $\pm\infty$ for reasons already explained.

Operations (6.7.12) and (6.7.13) are carried out by the circuits of Figures 6.26 and 6.27, respectively, which repeat two channels of the circuit of Fig. 6.25, with the exception of differences in properties of linear filters, appearing due to infinite limits of integration. In all cases filtration with weighting function $W_{\Pi}(t_1, t_2)$ has rejector properties with respect to interference, carrying out suppression of the most intense part of the spectrum of interference.

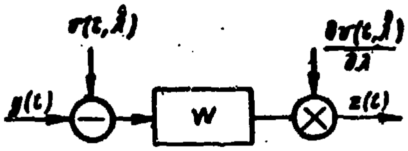

Fig. 6.26. Discriminator for a regular signal in correlated noise (2nd variant): W — linear filter with pulse response $W(t, \tau)$.

We shall explain this circumstance in greater detail, considering a periodic signal with period T_r, larger than the interval of correlation of interference. If we consider interference stationary $R_{\Pi}(t_1, t_2) = R_{\Pi}(t_1 - t_2)$, then $W_{\Pi}(t_1, t_2)$ one should seek also in the form $W_{\Pi}(t_1 - t_2)$, from which, pushing the limits of integration in (6.7.9) to $\pm\infty$, we have the Fourier transform from the inverse function $W_{\Pi}(t_1 - t_2)$ in the form

$$\tilde{W}_{\Pi}(\omega) = \int\limits_{-\infty}^{+\infty} W_{\Pi}(\tau)\,e^{-j\omega\tau}\,d\tau = \frac{1}{S_{\Pi}(\omega)}, \qquad (6.7.14)$$

where $S_п(\omega)$ — spectral density of interference.

The operation for the discriminator in the case of a periodic signal will be given directly by relationship (6.7.11), where limits embrace the period of repetition. From this, passing to Fourier transforms and using (6.7.14), we have

$$z_k = \mathrm{Re}\,\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{Y_k(\omega) - V(\omega, \hat{\lambda}_k)}{S_п(\omega)}\,\frac{\partial V^*(\omega, \hat{\lambda}_k)}{\partial \lambda}\,d\omega, \qquad (6.7.15)$$

where $Y_k(\omega)$, $V(\omega, \lambda)$ — Fourier transforms of functions $y(t)$, $v(t, \lambda)$ in the k-th period of the signal. According to (6.7.15), in those sections of the spectrum where interference is more intense, the transmission factor of the linear filter occupied in the circuit decreases. This also explains the rejector properties.

The schematic of the discriminator for this case is shown in Fig. 6.28.

It remains to clarify the question of physical realizability of operations (6.7.12), (6.7.13) and (6.7.15), which is placed under doubt by the infinite upper limits of integration in (6.7.12) and (6.7.13) or, which is the same, the presence in (6.7.15) of function $1/S_п(\omega)$, the Fourier original of which is symmetric with respect to zero time. The situation is eased by the fact that the interval of correlation of interference is usually considerably less than the interval of noticeable change of signal function $v(t; \lambda)$. In these conditions it is possible to replace the Fourier original $1/S_п(\omega)$ by a function obtained from it by shifting a quantity somewhat larger than the interval of correlation of interference, and after that cutting off the small remainder in negative time. Then we will obtain a signal at the discriminator output statistically equivalent to the optimum. It is easy to prove this by calculating the mean value or spectral density of the output signal. This quantity characterizes accuracy of measurement in the presence of correlated interference, and is equal to

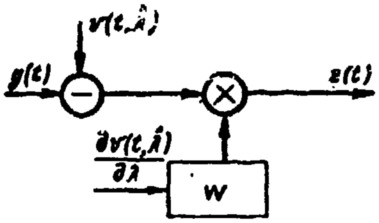

Fig. 6.27. Discriminator for a regular signal in correlated noise (3rd variant). W — linear filter with pulse response $W(t, \tau)$.

$$K = \lim_{T\to\infty}\frac{1}{T}\int\limits_0^T\int\limits_0^T\frac{\partial v(t_1, \lambda)}{\partial \lambda}\,\frac{\partial v(t_2, \lambda)}{\partial \lambda}\,W_п(t_1, t_2)\,dt_1\,dt_2, \qquad (6.7.16)$$

where in the case of a periodic signal one should reject the sign of the limit and equate T to the period of repetition T_r.

Although in subsequent chapters the case of a regular signal will not be

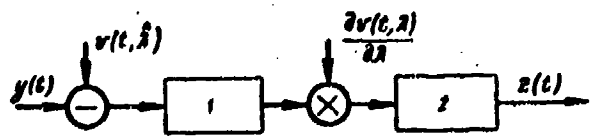

Fig. 6.28. Discriminator for a pulse regul[illegible] [illegible]ignal in correlated noise: 1) [illegible]ear filter with frequency response [illegible] (ω); 2) integrator over the period.

considered, results of the present paragraph will help us to comprehend certain laws governing construction of optimum circuits in cases more complicated and more interesting for practice.

6.7.2. Gaussian Coherent Signals in Gaussian Noises

First we shall consider that simple case when the carrier of the parameter is a single signal, received against a background of normal white noise n(t):

$$y(t) = E(t) u_a(t, \lambda) \cos[\omega_0 t + \psi(t, \lambda) + \varphi(t)] + n(t) = \\ = \mathrm{Re}\{E(t) e^{i\varphi(t)} u(t, \lambda) e^{i\omega_0 t}\} + n(t). \tag{6.7.17}$$

Here ω_0 – carrier frequency; E(t), $\varphi(t)$ – random amplitude and phase modulations, distributed correspondingly by Rayleigh and uniform law, so that E(t) cos $\varphi(t)$, E(t) sin $\varphi(t)$ are stationary normal independent random processes with correlation function $r(t_1 - t_2)$. Then, $u_a(t, \lambda)$, $\psi(t, \lambda)$ are regular modulations, depending on the measured magnitude of $\lambda(t)$; in (6.7.17) there is introduced the complex modulation factor

$$u(t, \lambda) = u_a(t, \lambda) e^{i\psi(t, \lambda)}, \tag{6.7.18}$$

reflecting immediately both functions $u_a(t; \lambda)$ and $\psi(t, \lambda)$. The correlation function of mixture y(t) according to Chapter I (Vol. 1) has the form

$$\overline{y(t_1) y(t_2)} = R(t_1, t_2) = \\ = \mathrm{Re}\{u(t_1, \lambda) r(t_1 - t_2) u^*(t_2, \lambda) e^{i\omega_0(t_1 - t_2)}\} + N_0 \delta(t_1 - t_2), \tag{6.7.19}$$

where

$$r(t_1 - t_2) = \frac{1}{2} \overline{E(t_1) e^{i\varphi(t_1)} E(t_2) e^{-i\varphi(t_2)}}.$$

According to formula (6.5.1), for construction of the likelihood functional for a normal process it is necessary to find function $W(t_1, t_2)$, the inverse of the correlation function, i.e., satisfying equation

$$\int_{t-\Delta/2}^{t+\Delta/2} W(t_1, s) R(s, t_2) ds = \delta(t_1 - t_2), \tag{6.7.20}$$

where integration is conducted over the elementary interval $(t - \Delta/2, t + \Delta/2)$, which we discussed in § 6.6. Similarly to how this was done in Chapter IV (Vol. 1), we find $W(t_1, t_2)$ in the form

$$W(t_1, t_2) = -\mathrm{Re}\left\{u(t_1,\lambda)\frac{w(t_1,t_2)}{N_0^2}u^*(t_2,\lambda)\,e^{i\omega_0(t_1-t_2)}\right\} + \frac{1}{N_0}\delta(t_1 - t_2), \tag{6.7.21}$$

whence for auxiliary function $w(t_1, t_2)$ we find equation

$$\frac{1}{2}\int_{t-\Delta t/2}^{t+\Delta t/2} r(t_1 - s)\frac{|u(s,\lambda)|^2}{N_0}w(s, t_2)\,ds + w(t_1, t_2) = r(t_1 - t_2). \tag{6.7.22}$$

As the first example of solution of equation (6.7.22) we shall consider a signal without amplitude modulation, when $u_a(t) = u_0 = \text{const}$. Then, inasmuch as the interval of integration exceeds the interval of correlation of fluctuations of the signal, determined by function $r(t)$, it is possible to consider limits of integration in (6.7.22) infinite and to seek the solution of $w(t_1, t_2)$ in the form $w(t_1 - t_2)$. The method of Fourier transformations in these conditions gives

$$\int_{-\infty}^{+\infty} w(t)\,e^{-i\omega t}\,dt \equiv \bar{w}(\omega) = \frac{S_c(\omega)}{1 + \frac{S_c(\omega)}{2N_0}} = \frac{P_c}{\Delta f_c}\,\frac{S_0(\omega)}{1 + hS_0(\omega)}. \tag{6.7.23}$$

Here as the condition of normalization we assume $u_a^2 = 1$, so that the correlation function of signal fluctuations $r(t)$ must be normalized in the form

$$r(0) = P_c, \tag{6.7.24}$$

where P_c — mean signal power.

Further, $S_c(\omega)$ — Fourier transform from $r(t)$; and $S_0(\omega) = S_c(\omega)/S_c(0)$ — normalized spectrum of fluctuations, related to the width of this spectrum Δf_c by relationship

$$\Delta f_c = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_0(\omega)\,d\omega. \tag{6.7.25}$$

Finally, quantity $h = P_c/2N_0\Delta f_c$, introduced in (6.7.23), is the signal-to-noise ratio encountered earlier (Chapter IV) in the theory of optimum reception of a coherent fluctuating signal.

Another interesting example of solution of equation (6.7.22), essentially reducible to the first one, is the case of rapid amplitude modulation. If we limit ourselves to modes of modulation with constant period T_r, this is equivalent to the condition of the interval of correlation of the signal greatly exceeding period T_r. In other words, the set of neighboring pulse trains fluctuate harmoniously. Then it is possible to divide the integral in (6.7.22) into the sum of integrals over separate periods, removing factors $r(t_1 - iT_r)$, $w(iT_r, t_2)$, constant in period T_r:

$$\int_{t-\Delta/2}^{t+\Delta/2} r(t_1-s)\frac{|u(s,\lambda)|^2}{N_0}w(s,t_2)\,ds \sim \sum_i r(t_1-iT_r)\times$$

$$\times w(iT_r,t_2)\int_{(i-1)T_r}^{iT_r}\frac{|u(s,\lambda)|^2}{N_0}ds. \qquad (6.7.26)$$

...ring integration over separate periods we consider condition of normalization

$$\frac{1}{T_r}\int_{(k-1)T_r}^{kT_r} u_a^2(s,\lambda)\,ds=\frac{1}{T_r}\int_{(k-1)T_r}^{kT_r}|u(s,\lambda)|^2\,ds=1, \qquad (6.7.27)$$

which is a generalization of the former normalization in the case of the presence of modulation. Passing then in (6.7.26) to an integral in the initial limits, as the solution of equation (6.7.22) we again have relationship (6.7.23).

Let us now turn to expression (1.4.5) for the derivative of the logarithm of the likelihood function $P(y(t)|\lambda)$ of a normal process $y(t)$, taken with respect to arbitrary parameter $\lambda(t)$. Using (6.7.21) and (6.7.23) and considering h independent of λ, we have

$$\frac{\partial}{\partial\lambda}\ln P(y|\lambda)=-\frac{1}{2N_0^2}\iint_{t-\Delta/2}^{t+\Delta/2} y(t_1)\,y(t_2)\,w(t_1-t_2)\times$$

$$\times\Big\{u_a(t_1,\lambda)\frac{\partial}{\partial\lambda}u_a(t_2,\lambda)\cos[\omega_0(t_1-t_2)+\psi(t_1,\lambda)-$$

$$-\psi(t_2,\lambda)]-u_a(t_1,\lambda)\,u_a(t_2,\lambda)\frac{\partial\psi(t_2,\lambda)}{\partial\lambda}\sin[\omega_0(t_1-t_2)+$$

$$+\psi(t_1,\lambda)-\psi(t_2,\lambda)]\Big\}\,dt_1dt_2. \qquad (6.7.28)$$

In order to carry out transition to the operation of the discriminator, according to relationship (6.6.36), it is necessary to present (6.7.28) in the form of a simple integral over the interval of observation. For this it is sufficient to assume that there exists such a function h(t), for which

$$\int_{t-\Delta/2}^{t+\Delta/2} h(\tau-t_1)\,h(\tau-t_2)\,d\tau=w(t_1-t_2)/2N_0, \qquad (6.7.29)$$

or, in frequency presentation, that this relationship is satisfied:

$$|H(i\omega)|^2=\frac{\tilde{w}(\omega)}{2N_0}=\frac{hS_0(\omega)}{1+hS_0(\omega)}, \qquad (6.7.30)$$

where $H(i\omega)$ — Fourier transform from $h(t)$.

Substitution of relationship (6.7.29) in (6.7.28) gives the possibility of

passing to the operation of the discriminator:

$$z(t)=\frac{1}{N_0}\int_{-\infty}^{+\infty} y(t_1)\,u_a(t_1;\ \hat{\lambda}(t))\cos[\omega_0 t_1+\psi(t_1;\ \hat{\lambda}(t))]\times$$

$$\times h(t-t_1)\,dt_1\int_{-\infty}^{+\infty} y(t_2)\Big[\frac{\partial}{\partial\lambda}u_a(t_2;\ \hat{\lambda}(t))\cos[\omega_0 t_2+\psi(t_2;\ \hat{\lambda}(t))]-$$

$$-u_a(t_2;\ \hat{\lambda}(t))\,\frac{\partial\psi(t_2;\ \hat{\lambda}(t))}{\partial\lambda}\sin[\omega_0 t_2+\psi(t_2;\ \hat{\lambda}(t))]\Big]\,h(t-t_2)\,dt_2+$$

$$+\frac{1}{N_0}\int_{-\infty}^{+\infty} y(t_1)\,u_a(t_1;\ \hat{\lambda}(t))\sin[\omega_0 t_1+\psi(t_1;\ \hat{\lambda}(t))]\,h(t-t_1)\,dt_1\times$$

$$\times\int_{-\infty}^{+\infty} y(t_2)\Big[\frac{\partial}{\partial\lambda}u_a(t_2;\ \hat{\lambda}(t))\sin[\omega_0 t_2+\psi(t_2;\ \hat{\lambda}(t))]+$$

$$+u_a(t_2;\ \hat{\lambda}(t))\,\frac{\partial\psi(t_2;\ \hat{\lambda}(t))}{\partial\lambda}\cos[\omega_0 t_2+\psi(t_2;\ \hat{\lambda}(t))]\Big]\,h(t-t_2)\,dt_2. \quad (6.7.31)$$

The operation represented by relationship (6.7.31) is illustrated in Fig. 6.29. Input signal $y(t)$ proceeds to two quadrature mixers, in one of which it is multiplied by sinusoidal oscillation of the expected frequency ω_0, modulated in phase by the law of phase (frequency) modulation with a value of the parameter equal to the measured, and in the second mixer it is multiplied by cosinusoidal oscillation of the expected frequency with the same modulation. Then there occurs multiplication of output signals of mixers by functions reflecting the expected form of amplitude modulation, the derivative of amplitude modulation, and the derivative of phase modulation with respect to parameter λ (see Fig. 6.29), all at the same measured value of $\lambda(t)$. There is conducted filtration of the obtained signals in low-frequency linear filters with pulse response $h(t)$, satisfying relationship (6.7.29) or (6.7.30). Then the obtained signals are mutually multiplied in multipliers, after which all output signals proceed to an adder, completing the circuit of the discriminator.

Although the given circuit is complicated, it is possible to easily explain the work of all its elements. Input mixers and multipliers are elements of so-called correlation processing, by which we usually mean multiplication by a signal of expected form with subsequent accumulation of some type. The presence of sine-cosine channels is explained by fluctuations of phase of the signal, making it impossible to predict its concrete value. In any case the signal correlates (completely or partially) with heterodyne signals, in quadrature shift relative to each other,

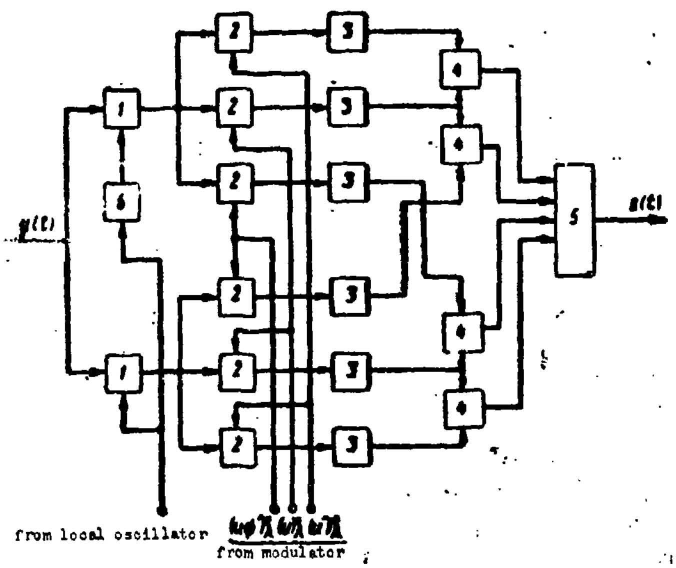

Fig. 6.29. Optimum discriminator for a fluctuating coherent signal (low-frequency variant): 1) quadrature mixers; 2) variable-gain amplifiers; 3) linear low-frequency filters; 4) multipliers; 5) adder; 6) $\pi/2$ phase shifter.

which also permits forming the output signal of the device.

In the corresponding circuit of an optimum receiver it would be sufficient further to have only elements of multiplication by the function of amplitude modulation, in the case of a pulse signal technically executed in the form of gated stages. In the circuit of the discriminator it is additionally necessary to use derivatives with respect to the parameter of amplitude and phase (frequency) modulations, taken for the measured value of parameter $\lambda(t)$. The useful signal appears at the output of amplifiers with a law of amplification in the form of derivatives only when there is mismatch between the measured and true values of λ. Here, the signal at the output of these amplifiers in sign and magnitude corresponds to difference $\lambda - \hat{\lambda}$, if this difference is small.

Then in the circuit there follow linear filters, whose pulse response is determined by the spectrum of fluctuations of signal and the signal-to-noise ratio. They carry out accumulation in time intervals which for a low level of the signal are equal to the correlation interval and upon increase of the signal decrease somewhat. Multiplication of signals carrying information about mismatch $\varepsilon = \lambda - \hat{\lambda}$ by signals carrying only information about the amplitude of the signal is produced in order to emphasize overshoots of the signal and to maximally filter off noises. Final addition helps to compensate phase fluctuations and unites effects caused by deviations of $\hat{\lambda}(t)$ from $\lambda(t)$ in channels sensitive to amplitude and phase modulation, respectively.

Physical realizability of operations (6.7.31), not obvious in view of the infinite upper limits of integration, is explained just as in Paragraph 6.7.1. The fact is that on an optimum filter there is imposed only condition (6.7.30) with

respect to the square of the modulus of its frequency response; the phase response can be selected arbitrarily, if, of course, there does not appear here delay comparable with the interval of change of $\lambda(t)$. Therefore, the first method of finding $h(t)$ is factoring function $|H(i\omega)|^2 = hS_0(\omega)/(1 + hS_0(\omega))$, i.e., its expansion into factors having zeroes and poles in the upper and lower half-planes, respectively, of complex variable ω. Then one of the factors will give the sought physically realizable frequency response. When there is a rational-fraction spectral density of fluctuations of signal $S_0(\omega)$ such a possibility, as shown in § 6.5, always exists.

A second method is direct determination of the Fourier original of function $\sqrt{hS_0(\omega)/(1 + hS_0(\omega))}$, which will give a physically unrealizable pulse response, and introduction then according to the method in Paragraph 6.7.1 of time delay an interval comparable with the duration of this response. Obviously, it is also possible to suggest other methods. All of them are equivalent in results and give filters of an approximately identical transmission band.

The given "low-frequency" interpretation of operations of the discriminator is not the only possible one. Another possible variant is transfer of operations of filtration to a certain intermediate frequency, allowing us to decrease the number of channels in processing (see Chapter IV, Vol. 1):

$$\begin{aligned} z(t) = \text{const} \int y(t_1)\, u_\lambda(t_1; \hat{\lambda}(t)) \cos[(\omega_0 + \omega_{\text{пр}}) t_1 + \\ + \varphi(t_1; \hat{\lambda}(t))]\, h(t - t_1) \cos \omega_{\text{пр}} (t - t_1)\, dt_1 \int y(t_2) \times \\ \times \Big[\frac{\partial}{\partial \lambda} u_\lambda(t_2, \hat{\lambda}(t)) \cos[(\omega_0 + \omega_{\text{пр}}) t_2 + \varphi(t_2; \hat{\lambda}(t))] + \\ + u_\lambda(t_2, \hat{\lambda}(t)) \frac{\partial}{\partial \lambda} \varphi(t_2; \hat{\lambda}(t)) \sin[(\omega_0 + \omega_{\text{пр}}) t_2 + \\ + \varphi(t_2; \hat{\lambda}(t))] \Big] h(t - t_2) \cos \omega_{\text{пр}} (t - t_2)\, dt_2. \end{aligned} \qquad (6.7.32)$$

Processing of $y(t)$ according to (6.7.32) should be conducted by the circuit of Fig. 6.30. At the input of the circuit there are installed two mixers to which there are fed heterodyne voltages with quadrature phase shift and the expected phase modulation. In view of noncoincidence of the middle frequency of heterodyne signals with the carrier frequency ω_0 simultaneously with "convolution" of phase (frequency) modulation the mixers shift the spectrum of the signal to intermediate frequency $\omega_{\text{пр}}$. Then signals are fed to variable-gain amplifiers. Laws of change of

amplification in them are $u_a(t, \hat{\lambda}) \frac{\partial u_a(t, \hat{\lambda})}{\partial \lambda}$, $u_a(t, \hat{\lambda}) \frac{\partial \psi(t, \hat{\lambda})}{\partial \lambda}$, respectively, i.e., are determined by the form of regular modulation of the signal. The mixers together with controlled amplifiers carry out correlation processing. In the circuit of Fig. 6.30 there occurs unification of outputs of amplifiers in two channels. ...ing in both channels is conducted by identical band pass filters, tuned to intermediate frequency $\omega_{\text{пр}}$ and being high-frequency equivalents of low-frequency optimum filters. Naturally, frequency shift of heterodyne voltage, as also of the face value of the intermediate frequency to which this shift is equal, is arbitrary and in no way follows from theory. During transition to (6.7.32) we pursued the goal of practical convenience of fulfillment of operations, where there was implied the circumstance that after mixing in subsequent filtration there does not participate a "mirror" signal.

As a result of bandpass filtration in the channel connected with the amplifiers, whose law of amplification is determined by the derivative of the modulating functions, there will form voltage proportional in amplitude to current mismatch ε and corresponding in phase to the sign of this difference. On the output of the other filter voltage depends little on ε and will form as it were a reference signal, multiplication of the differential signal wi' which, carried out in the phase detector, completes the operation of the discriminator. As a result there is formed low-frequency voltage $z(t)$, on the average proportional to $\lambda - \hat{\lambda}$ for small values of this quantity.

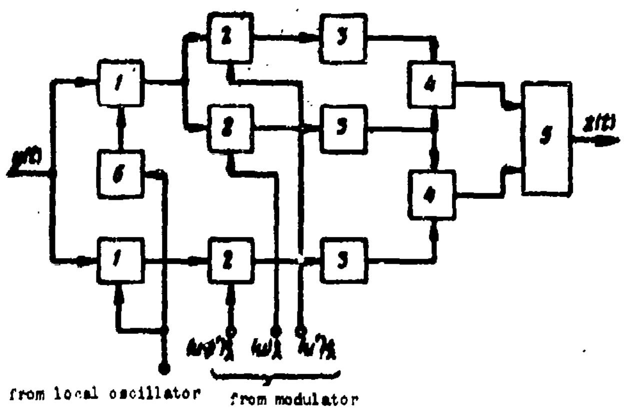

Fig. 6.30. Optimum discriminator for a fluctuating coherent signal (high-frequency variant): 1) quadrature mixers; 2) variable-gain amplifiers; 3) linear bandpass filters; 4) multipliers; 5) adder; 6) $\pi/2$ phase shifter.

Let us turn to characterization of accuracy of the discriminator. In view of the stationariness of fluctuations of the signal the accuracy is characterized by a constant $\overline{K(t)} = K$. In the most general case of a normal random process $y(t)$ by differentiation of the logarithm of the likelihood function and averaging it is possible to show that $\overline{K(t)}$ is expressed directly through the correlation function

of the signal $R(t_1, t_2)$ and it inverse $W(t_1, t_2)$ in the form

$$K = -\lim_{T\to\infty} \frac{1}{2T} \iint\limits_{-T/2}^{+T/2} \frac{\partial W(t_1, t_2; \lambda)}{\partial \lambda} \frac{\partial R(t_1, t_2; \lambda)}{\partial \lambda} \, dt_1 dt_2, \tag{6.7.33}$$

where limits for constant λ can be directed to $\pm\infty$ by virtue of the factors shown earlier.

Substituting in (6.7.33) functions $R(t_1, t_2)$ and $W(t_1, t_2)$ from (6.7.19) and (6.7.21), we obtain

$$\begin{aligned} K = \lim_{T\to\infty} \frac{1}{2T} \iint\limits_{-T/2}^{+T/2} \Big\{ & u_a^2(t_1, \lambda) \Big[\Big(\frac{\partial u_a(t_2, \lambda)}{\partial \lambda} \Big)^2 + \\ & + u_a^2(t_2, \lambda) \Big(\frac{\partial \psi(t_2, \lambda)}{\partial \lambda} \Big)^2 \Big] + u_a^2(t_2, \lambda) \Big[\Big(\frac{\partial u_a(t_1, \lambda)}{\partial \lambda} \Big)^2 + \\ & + u_a^2(t_1, \lambda) \Big(\frac{\partial \psi(t_1, \lambda)}{\partial \lambda} \Big)^2 \Big] + u_a^2(t_1, \lambda) u_a^2(t_2, \lambda) \frac{\partial \psi(t_1, \lambda)}{\partial \lambda} \times \\ & \times \frac{\partial \psi(t_2, \lambda)}{\partial \lambda} \Big\} r(t_1 - t_2) w(t_1 - t_2) \, dt_1 dt_2. \end{aligned} \tag{6.7.34}$$

Relationship (6.7.34) is valid in the very general case. If, however, on function $\psi(t, \lambda)$ there are imposed certain limitations, then this convenient formula, having more modest limits of applicability, is valid:

$$K = J_1(u) J_2(h, S_0), \tag{6.7.35}$$

where

$$\left. \begin{aligned} & J_1(u) = b_2 - b_1^2; \\ & b_2 = \lim_{T\to\infty} \frac{1}{T} \int\limits_{-T/2}^{+T/2} \Big[\Big(\frac{\partial u_a(t, \lambda)}{\partial \lambda} \Big)^2 + u_a^2(t, \lambda) \Big(\frac{\partial \psi(t, \lambda)}{\partial \lambda} \Big)^2 \Big] dt, \\ & b_1 = \lim_{T\to\infty} \frac{1}{T} \int\limits_{-T/2}^{+T/2} u_a^2(t, \lambda) \frac{\partial \psi(t, \lambda)}{\partial \lambda} dt, \end{aligned} \right\} \tag{6.7.36}$$

$$J_2(h, S_0) = \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} \frac{h^2 S_0^2(\omega)}{1 + h S_0(\omega)} d\omega. \tag{6.7.37}$$

Naturally, during periodic modulation passages to the limit in (6.7.34) and (6.7.36) are replaced by averaging over one period of modulation.

Let us discuss formula (6.7.35). It contains two factors, the first of which J_1 is wholly determined by the form of regular modulation, and the other J_2 is determined only by the form of spectral density of fluctuations of signal $S_0(\omega)$ and the signal-to-noise ratio h. In other words, J_1 is determined by the form of coding

λ in the signal, and J_2 is determined by power properties of the latter. Therefore, detailed discussion of J_1 can be postponed to subsequent chapters. For J_2 it is useful to indicate limiting values. For very large values of h we have

$$J_2 \approx 2h \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_0(\omega)\, d\omega = 2h\Delta f_c \quad (h \gg 1), \tag{6.7.38}$$

... for very small h

$$J_2 \approx 2h^2 \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_0^2(\omega)\, d\omega = 2h^2 k\Delta f_c \, (h \ll 1), \tag{6.7.39}$$

where k — constant, value of which is between one (for fast-dropping spectrum $S_0(\omega)$) and 1/2 (for a slow-dropping spectrum).

Thus, J_2 monotonically depends on signal-to-noise ratio h and is proportional to the width of the spectrum of fluctuations of the signal Δf_c.

The condition of invariability of the correlation function within limits of the period of modulation, assumed above, is not always observed in practice. Separate periods can be weakly correlated. The sphere of applicability of the developed methods can be considerably expanded if we assume that modulation has a pulse character with a sufficiently high off-duty factor, so that instead of u(t) we take

$$\sum_k u(t - kT_r) = \sum_k u_k(t - kT_r)\, e^{i\psi(t-kT_r)}, \tag{6.7.40}$$

where u(t) — complex law of modulation in a separate k-th period.

Correlation of signal fluctuations, divided by the period of repetition, we consider arbitrary, but within the limits of duration of each pulse train (or pack) we shall consider the correlation function and its inverse function constant, so that, for instance,

$$\int_0^{T_r} |u(t-\tau)|^2 r(t)\, dt = r(\tau) \int_0^{T_r} |u(t)|^2\, dt.$$

As before, there remains in force the condition that in the interval of correlation of the signal the measured parameter does not vary. The correlation function of the mixture of the signal with noise has the form

$$\tilde{R}(t_1, t_2) = \mathrm{Re}\left\{\sum_{k,j} u(t_1 - kT_r)\, r(t_1, t_2)\, u(t_2 - jT_r))\, e^{j\omega_0(t_1-t_2)}\right\} + \\ + N_0\delta(t_1 - t_2). \tag{6.7.41}$$

Function $W(t_1, t_2)$, the inverse of the correlation function in interval Δ, including many periods of repetition and intervals of correlation of the signal, we find in the form

$$W(t_1,t_2)=-\mathrm{Re}\left\{\sum_{k,j}u(t_1-kT_r)\frac{w(t_1,t_2)}{N_0^2}u(t_2-jT_r)e^{j\omega_d(t_1-t_2)}\right\}+$$
$$+\frac{1}{N_0}\delta(t_1-t_2). \qquad (6.7.42)$$

This gives an equation for $w(t_1, t_2)$:

$$\frac{1}{2}\int_{t-\Delta/2}^{t+\Delta/2}w(t_1, s)\frac{\sum_i u^2(s-iT_r)}{N_0}r(s-t_2)\,ds+w(t_1, t_2)=$$
$$=r(t_1-t_2). \qquad (6.7.43)$$

Considering the condition of a large off-duty factor, it is possible to produce integration over s, after which it is useful to quantize equation (6.7.43), i.e., to look for its solution $w(iT_r, jT_r) = w_{ij}$ at discrete points coinciding with moments of action of pulses, since only at these points will it affect the form of the final solution of (6.7.42). This gives a discrete analog of equation (6.7.22):

$$\frac{T_r}{2N_0}\sum_i w_{ij}r_{j-l}+w_{il}=r_{i-l}, \qquad (6.7.44)$$

where it is assumed that the energy of pulses does not depend on time, and condition (6.7.27) is satisfied. Directing the limits of summation in (6.7.44) to infinity and seeking a solution in the form $w_{il} = w_{i-l}$, it is easy to solve (6.7.44) by the method of discrete Fourier transformation:

$$\tilde{w}(\Omega)=\frac{S_{cD}(\Omega)}{1+\frac{T_r S_{cD}(\Omega)}{2N_0}}=\sum_{k=-\infty}^{+\infty}r_k\frac{S_{0D}(\Omega)}{1+h_D S_{0D}(\Omega)}. \qquad (6.7.45)$$

Here

$$h_D=\sum_{k=-\infty}^{+\infty}r_k\frac{T_r}{2N_0} \qquad (6.7.46)$$

– signal-to-noise ratio, analogous to h and changing into h as $T_r \to 0$, inasmuch as, here,

$$h_D\simeq\frac{\int_{-\infty}^{+\infty}r(\tau)\,d\tau}{2N_0}=\frac{S_c(0)}{2N_0}=h; \qquad (6.7.47)$$

Ω – dimensionless frequency, connected with usual frequency and the period of repetition by relationship

$$\Omega=\omega T_r;$$

then,

$$S_{cD}(\Omega)=\sum_{k=-\infty}^{+\infty} r_k e^{-ik\Omega}, \qquad (6.7.48)$$

$$\tilde{w}(\Omega)=\sum_{k=-\infty}^{+\infty} w_k e^{-ik\Omega}$$

.. screte Fourier transforms of the correlation function and the inverse function, and

$$S_{0D}(\Omega)=\frac{S_{cD}(\Omega)}{S_{cD}(0)}=\frac{S_{cD}(\Omega)}{\sum_k r_k}$$

— normalized discrete spectrum.

Let us expand w_i analogously to w(t) in (6.7.29):

$$\frac{T_r}{2N_0}\tilde{w}(\Omega)=\frac{h_D S_{0D}(\lambda)}{1+h_D S_{0D}(\lambda)}=|H_D(i\Omega)|^2 \qquad (6.7.49)$$

$$\frac{T_r}{2N_0} w_{i-j}=\sum_l h_{i-l} h_{j-l},$$

where h_i — a certain function of a discrete argument with Fourier transform $H_D(i\Omega)$.

Then, considering for simplicity the case when the signal-to-noise ratio h_D does not depend on parameter λ, we have a discrete analog of relationship (6.7.31):

$$\begin{gathered}
z(jT_r)=\frac{T_r}{N_0}\sum_l h_{j-l}\int_{(l-1)T_r}^{lT_r} y(t_1)\,u_a(t_1-lT_r;\hat{\lambda}_j)\cos[\omega_0 t_1+ \\
+\psi(t_1-lT_r;\hat{\lambda}_j)]\,dt_1\sum_l h_{j-l}\int_{(l-1)T_r}^{lT_r} y(t_2)\Big\{\frac{\partial}{\partial\lambda}u_a(t_2-lT_r;\hat{\lambda}_j)\times \\
\times\cos[\omega_0 t_2+\psi(t_2-lT_r;\hat{\lambda}_j)]-u_a(t_2-lT_r;\hat{\lambda}_j)\frac{\partial}{\partial\lambda}\psi(t_2- \\
-lT_r;\hat{\lambda}_j)\sin[\omega_0 t_2+\psi(t_2-lT_r;\hat{\lambda}_j)]\Big\}\,dt_2+ \\
+\frac{T_r}{N_0}\sum_l h_{j-l}\int_{(l-1)T_r}^{lT_r} y(t_1)\,u_a(t_1-lT_r;\hat{\lambda}_j)\sin[\omega_0 t+ \\
+\psi(t_1-lT_r;\hat{\lambda}_l)]\,dt_1\sum_l h_{j-l}\int_{(l-1)T_r}^{lT_r} y(t_2)\Big\{\frac{\partial}{\partial\lambda}u_a(t_2- \\
-lT_r;\hat{\lambda}_j)\sin[\omega_0 t_2+\psi(t_2-lT_r;\hat{\lambda}_j)]+u_a(t_2-lT_r;\hat{\lambda}_j)\times \\
\times\frac{\partial}{\partial\lambda}\psi(t_2-lT_r;\hat{\lambda}_j)\cos[\omega_0 t_2+\psi(t_2-lT_r;\hat{\lambda}_j)]\Big\}\,dt_2. \qquad (6.7.50)
\end{gathered}$$

Processing of the signal according to (6.7.50) obviously breaks down into an intraperiod and an interperiod part. The intraperiod processing completely repeats the first correlation part of processing in Fig. 6.29. Then, instead of low-frequency

filters there should follow integrators with drop, accumulating signals inside each period. Results of accumulation in discrete form are smoothed by discrete linear filters, realizing accumulation in the interval of fluctuations of the signal. Technically, a circuit starting with these filters can be imagined in the form of a specialized digital computer. Further processing, consisting in multiplication of the results of filtration and addition with each other, accurately repeats the operation of Fig. 6.29.

As before, let us assume transfer of operations (6.7.49) to intermediate frequency. This leads to a relationship, analogous to (6.7.32):

$$
\begin{aligned}
z(jT_r) = \mathrm{const} \int_{-\infty}^{+\infty} & y(t_1)\, u_a(t_1, \hat{\lambda}_j) \cos[(\omega_0 + \omega_{\text{пр}}) t_1 + \\
& + \psi(t_1, \hat{\lambda}_j)]\, h(t_j - t_1) \cos \omega_{\text{пр}} (t_j - t_1)\, dt_1 \int_{-\infty}^{+\infty} y(t_2) \times \\
& \times \Big[\frac{\partial}{\partial \lambda} u_a(t_2, \hat{\lambda}_j) \cos[(\omega_0 + \omega_{\text{пр}}) t_2 + \psi(t_2, \hat{\lambda}_j)] + \\
& + u_a(t_2, \hat{\lambda}_j) \frac{\partial \psi(t_2, \hat{\lambda}_j)}{\partial \lambda} \sin[(\omega_0 + \omega_{\text{пр}}) t_2 + \psi(t_2, \hat{\lambda}_j)] \Big] \times \\
& \times h(t_j - t_2) \cos \omega_{\text{пр}} (t_j - t_2)\, dt_2,
\end{aligned}
\qquad (6.7.51)
$$

where the envelope of pulse response h(t) at discrete moments $t = iT_r$ satisfies condition (6.7.48), and in the remaining moments inside the period is arbitrary. The block diagram describing relationship (6.7.51) simply repeats the diagram of Fig. 6.30.

We shall show that, although operations (6.7.50) and (6.7.51) also break down into intra- and interperiod operations, interperiod processing of the shown form cannot be transferred to the output of the discriminator, inasmuch as phase detection (multiplication) at the output is an operation which is especially nonlinear with respect to y(t).

For characterization of accuracy of measurement K in the same conditions as in case (6.7.36) we have expression

$$K = J_1(u)\, J_{2D}(h_D, S_{0D}), \qquad (6.7.52)$$

where factor J_1 has the former value (6.7.36), and $J_{2D}(h_D, S_{0D})$ is equal to

$$J_{2D} = \frac{1}{\pi T_r} \int_{-\infty}^{+\infty} \frac{h_D^2 S_{0D}^2(\Omega)}{1 + h_D S_{0D}(\Omega)}\, d\Omega. \qquad (6.7.53)$$

As factor J_2 from (6.7.37), J_{2D} monotonically depends on signal-to-noise ratio h_D. In the extreme case, when the correlation coupling between packages is very great, the range of values of Ω, where function $S_{0D}(\Omega)$ markedly differs from zero, is considerably less than interval $(-\pi, +\pi)$. Therefore, considering relationship (6.7.46), and also the fact that $\Omega = \omega T_r$ and $S_{0D}(\omega T_r) \to S_0(\omega)$ and directing to infi... limits of integration in (6.7.53), we note that this relationship ...s (6.7.37). The other extreme case of total absence of correlation will be considered in detail in the next paragraph.

Above we considered the case of one input mixture of signal with noise y(t). Let us try to generalize the preceding results to the case of many input mixtures ${}^{(l)}y(t)$ ($l = 1, 2, \ldots, m$), each of which can contain a whole series of components in general with different forms of modulation and correlation properties. An example of such a set of mixtures are outputs of a multiunit antenna system, in each of which there are included signals reflected from many targets and radiated by interference sources.

Here, carrier frequencies for all components of the mixture may differ. There is introduced a certain middle frequency ω_0, and phase modulations ${}^{(l)}\psi_k(t;\lambda)$ contain terms of form $(\omega_{0k} - \omega_0)t$, linearly increasing in time, reflecting differences of frequencies of separate components ω_k.

Let us assume the l-th mixture has the form

$$
\begin{aligned}
{}^{(l)}y(t) = \sum_{k=1}^{p_l} {}^{(l)}u_{ak}(t;\lambda(t))\,{}^{(l)}E_k(t)\cos[\omega_0 t + {}^{(l)}\psi_k(t;\lambda(t)) + \\
+ {}^{(l)}\varphi_k(t)] + {}^{(l)}n(t) \quad (l = 1, 2, \ldots, m).
\end{aligned} \tag{6.7.54}
$$

We introduce a complex nearly-diagonal matrix of coefficients of modulation U(t), the l-th diagonal element of which is column $\{{}^{(l)}u_1(t), \ldots, {}^{(l)}u_{p_l}(t)\}$, where ${}^{(l)}u_j(t) = {}^{(l)}u_{aj}(t)e^{i\,{}^{(l)}\psi_j(t)}$. Additionally we introduce complex column $\theta(t)$, the l-th subcolumn element of which is a column of complex elements

$$
{}^{(l)}\theta = \{{}^{(l)}E_1(t)\,e^{i\,{}^{(l)}\varphi_1(t)}, \ldots, {}^{(l)}E_{p_l}(t)\,e^{i\,{}^{(l)}\varphi_{p_l}(t)}\},
$$

reflecting random modulations. Then the whole set of mixtures (6.8.62) can be presented in matrix form

$$
\mathbf{y}(t) = \mathrm{Re}\{\mathbf{U}^{+}(t)\,\theta(t)\,e^{i\omega_0 t}\} + \mathbf{n}(t), \tag{6.7.55}
$$

where $\mathbf{y}(t)$, $\mathbf{n}(t)$ — simple column vectors of order $(m \times 1)$, composed of values of function ${}^{(i)}y(t)$ and ${}^{(i)}n(t)$ $(i = 1, \ldots, m)$. The correlation matrix of the whole set of mixtures (6.7.55) is obtained in the form

$$\mathbf{R}(t_1, t_2) = \overline{\mathbf{y}(t_1)\,\mathbf{y}^+(t_2)} = \mathrm{Re}\,\{\mathbf{U}^+(t_1)\,\mathbf{r}(t_1 - t_2) \times \\ \times \mathbf{U}^*(t_2)\,e^{i\omega_0(t_1 - t_2)}\} + \mathbf{N}\delta(t_1 - t_2),$$

where

$$\mathbf{r}(t_1 - t_2) = \frac{1}{2}\,\overline{\boldsymbol{\theta}^*(t_1)\,\boldsymbol{\theta}^+(t_2)}$$

— complex matrix function of order $\left(\sum_{l=1}^{m} p_l \times \sum_{l=1}^{m} p_l\right)$ with elements being submatrices ${}^{(lj)}\mathbf{r}(t_1 - t_2) = \|{}^{(lj)}r_{pq}(t_1 - t_2)\|$, where $r_{pq}(t_1 - t_2)$ is the function of cross-correlation of stationary random processes ${}^{(l)}E_p(t_1)\,{}^{\cos}_{\sin}\,{}^{(l)}\varphi_p(t_1)$ and ${}^{(j)}E_q(t_2)\,{}^{\cos}_{\sin}\,{}^{(j)}\varphi_q(t_2)$. Matrix $\mathbf{N}$ in (6.7.56) is the matrix of spectral densities of white noises, not necessarily diagonal, since white noises in channels may be interrelated. The likelihood functional of the whole set of mixture $\mathbf{y}(t)$ has a form analogous to that in the one-dimensional case:

$$P(\mathbf{y}\,|\,\boldsymbol{\lambda}) = C\exp\left\{-\frac{1}{2}\lim_{\Delta\to 0}\ln\det\mathbf{R} - \right. \\ \left. -\frac{1}{2}\iint\limits_{t-\Delta/2}^{t+\Delta/2}\mathbf{y}^+(t_1)\,\mathbf{W}(t_1, t_2)\,\mathbf{y}(t_2)\,dt_1 dt_2\right\},$$

where $\mathbf{W}(t_1, t_2)$ — matrix, the inverse of $\mathbf{R}(t_1, t_2)$, i.e., satisfying equation

$$\int\limits_{t-\Delta/2}^{t+\Delta/2}\mathbf{W}(t_1, s)\,\mathbf{R}(s, t_2)\,ds = \mathbf{I}\,\delta(t_1 - t_2). \qquad (6.7.58)$$

We shall look for $\mathbf{W}(t_1, t_2)$ in a form which is a multi-dimensional generalization of (6.7.21):

$$\mathbf{W}(t_1, t_2) = -\mathrm{Re}\,\{\mathbf{N}^{-1}\mathbf{U}^+(t_1)\,\mathbf{w}(t_1, t_2)\,\mathbf{U}^*(t_2)\,\mathbf{N}^{-1}e^{i\omega_0(t_1 - t_2)}\} + \\ + \mathbf{N}^{-1}\delta(t_1 - t_2).$$

Then from relationships (6.7.56), (6.7.58) and (6.7.59) we arrive at the following integral matrix equation for matrix function $\mathbf{w}(t_1, t_2)$, structurally simialr to $\mathbf{r}(t_1, t_2)$:

$$\frac{1}{2}\int\limits_{t-\Delta/2}^{t+\Delta/2}\mathbf{r}(t_1 - s)\,\mathbf{B}_0(s)\,\mathbf{w}(s, t_2)\,ds + \mathbf{w}(t_1, t_2) = \mathbf{r}(t_1, t_2). \qquad (6.7.60)$$

Here $\mathbf{B}_0(s) = \mathbf{U}^*(s)\mathbf{N}^{-1}\mathbf{U}^+(s)$ — complex matrix, the character of whose dependence on s is determined by the form of regular modulation of all components of the mixture, not only of amplitude, but also of phase (frequency) modulation. Comparison of

equations (6.7.60) and (6.7.21) shows their very complete analogy, from which there ensue the same cases of solutions as in the one-dimensional variant.

Considering, in particular, laws of modulation of all separate components of the mixture to be rapidly varying, it is possible to average $B_0(s)$ in time under the si[illegible] integral, introducing constant matrix

$$\bar{\mathbf{B}}_0 = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{U}^*(s) \mathbf{N}^{-1} \mathbf{U}^+(s)\, ds. \tag{6.7.61}$$

Then the Fourier transform of $\mathbf{w}(t_1, t_2) = \mathbf{w}(t_1 - t_2)$ will be expressed in the form

$$\tilde{\mathbf{w}}(\omega) = \left[\mathbf{I} + \frac{1}{2} \mathbf{S}_0(\omega) \bar{\mathbf{B}}_0 \right]^{-1} \mathbf{S}_0(\omega), \tag{6.7.62}$$

where $\mathbf{S}_0(\omega)$ — Fourier transform of matrix function $\mathbf{r}(t)$.*

On the assumption that power characteristics of signals do not depend on λ, we can record the operation of the discriminator in the following form, generalizing formula (6.7.52):

$$\begin{gathered} z(t) = A \sum_{i,\,j=1}^{m} \sum_{k=1}^{p_i} \sum_{l=1}^{p_j} \int_{t_0}^{t} {}^{(i)}x_k(\tau_1;\, \hat{\lambda}(t))\, {}^{(ij)}h_{kl}^{(1)}(t-\tau_1) \times \\ \times \cos\omega_{\text{пр}}(t-\tau_1)\, d\tau_1 \int_{t_0}^{t} \frac{\partial}{\partial\lambda}\, {}^{(j)}x_l(\tau_2;\, \hat{\lambda}(t))\, {}^{(ij)}h_{kl}^{(2)}(t-\tau_2) \times \\ \times \cos\omega_{\text{пр}}(t-\tau_2)\, d\tau_2. \end{gathered} \tag{6.7.63}$$

Here A — proportionality factor;

$${}^{(i)}x_k(t,\, \lambda) = {}^{(i)}y(t)\, {}^{(i)}a_{sk}(t,\, \lambda) \cos\left[(\omega_0 + \omega_{\text{пр}})t + {}^{(i)}\psi_k(t,\, \lambda)\right] -$$

result of correlation processing of the i-th mixture for the purpose of separating from it the k-th component of the signal, and envelopes of pulse responses ${}^{(ij)}h_{kl}^{(1)}(t)$ and ${}^{(ij)}h_{kl}^{(2)}(t)$ are selected physically realizable according to the rules shown [illegible], and in such a manner that we satisfy

$$\int_{-\infty}^{+\infty} {}^{(ij)}h_{kl}^{(1)}(t)\, e^{-i\omega t}\, dt \int_{-\infty}^{+\infty} {}^{(ij)}h_{kl}^{(2)}(t)\, e^{i\omega t}\, dt = {}^{(ij)}\tilde{w}_{kl}(\omega), \tag{6.7.64}$$

where ${}^{(ij)}\tilde{w}_{kl}(\omega)$ — element of complex matrix $\mathbf{w}(\omega)$, determined by relationship (6.7.[illegible]).

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*We recall that the Fourier transform of a matrix is the matrix of Fourier transforms of separate elements of the initial matrix.

Examples of optimum processing of several input signals will be given in subsequent chapters, and in view of the complexity of general case (6.7.63) we [illegible] description of the circuit of the discriminator here.

For K(t) in the most general case of a set of normal processes with [illegible] matrix $R(t_1, t_2)$ and its inverse $W(t_1, t_2)$ there is a formula, analogous to [illegible]:

$$K = -\lim_{T\to\infty}\frac{1}{2T}\iint\limits_{-T/2}^{+T/2}\operatorname{spur}\left\{\frac{\partial}{\partial\lambda}\mathbf{W}(t_1,t_2)\frac{\partial}{\partial\lambda}\mathbf{R}^+(t_1,t_2)\right\}dt_1dt_2, \qquad (6.7.[illegible])$$

where "spur" signifies the trace of the matrix.

In the considered case this formula, [illegible] relationships [illegible] and (6.7.59), leads to this expression [illegible] (6.7.38):

$$K = \lim_{T\to\infty}\operatorname{Re}\frac{1}{2T}\iint\limits_{-T/2}^{+T/2}\operatorname{spur}\{\mathbf{B}_2(t_1)\,\mathbf{w}(t_1-t_2)\,\bar{\mathbf{B}}_0\mathbf{r}(t_1-t_2)+$$
$$+\mathbf{B}_1(t_1)\,\mathbf{w}(t_1-t_2)\,\mathbf{B}^*{}_1(t_2)\,\mathbf{r}(t_1-t_2)\}\,dt_1dt_2, \qquad (6.7.[illegible])$$

where

$$\mathbf{B}_2(t) = \frac{\partial}{\partial\lambda}\mathbf{U}^*(t)\mathbf{N}^{-1}\frac{\partial}{\partial\lambda}\mathbf{U}^+(t); \qquad (6.7.[illegible])$$
$$\mathbf{B}_1(t) = \mathbf{U}^*(t)\mathbf{N}^{-1}\frac{\partial}{\partial\lambda}\mathbf{U}^+(t)$$

– matrix, structurally similar to matrix $\bar{B}$ [illegible]

$$\lim_{T\to\infty}\frac{1}{T}\int\limits_{-T/2}^{+T/2}\mathbf{B}_2(t)\,dt = \bar{\mathbf{B}}_2,$$
$$\lim_{T\to\infty}\frac{1}{T}\int\limits_{-T/2}^{+T/2}\mathbf{B}_1(t)\,dt = \bar{\mathbf{B}}_1, \qquad (6.7.[illegible])$$

then, taking into account [illegible] we simplify relationship (6.7.66)

$$K = \frac{1}{4\pi}\int\limits_{-\infty}^{+\infty}\operatorname{spur}\left\{\left[1+\frac{1}{2}\mathbf{S}_0(\omega)\bar{\mathbf{B}}_0\right]^{-1}\mathbf{S}_0(\omega)\,[\bar{\mathbf{B}}_0\mathbf{S}_c^+(\omega)\bar{\mathbf{B}}_2+\right.$$
$$\left.+\bar{\mathbf{B}}_1\mathbf{S}_c^+(\omega)\bar{\mathbf{B}}_1]\right\}d\omega.$$

Let us consider a simple example. Let us assume that every mixture contains only one signal component ($p_j = 1$) with power P_{0} [illegible], where these components are completely correlated, and white noises are independent, [illegible] that elements of matrices $\mathbf{B}_0$ and $\mathbf{N}$ have form

$$^{(ih)}S_0(\omega) = \frac{\sqrt{P_{0i}P_{0h}}}{\Delta f_0}S_0(\omega), \quad {}^{(ih)}N = N_0\delta_{ih}, \qquad (6.7.[illegible])$$

where Δf_0, N_0 – unique width of the spectrum of fluctuations of signals and spectral density of white noises.

In this simple case it is convenient to introduce column vector $\mathbf{p}=\{\sqrt{P_{c1}},\ldots,\sqrt{P_{cm}}\}$, as a result of which it is possible to express the matrix of spectral densities of signals in the form

$$\mathbf{S}_c(\omega)=\frac{S_c(\omega)}{\Delta f_c}\mathbf{p}\mathbf{p}^+.$$

Matrices $\overline{\mathbf{B}}_0$, $\overline{\mathbf{B}}_1$, $\overline{\mathbf{B}}_2$ here are diagonal. $\overline{\mathbf{B}}_0$ consists of elements $1/N_0$, inasmuch as by force of (6.7.70) it is necessary to consider that for all forms of regular modulation this normalization is executed

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2}|{}^{(i)}u(t)|^2\,dt=1,$$

and diagonal elements of $\overline{\mathbf{B}}_2$, $\overline{\mathbf{B}}_1$ contain elements

$$\frac{1}{N_0}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2}\left|\frac{\partial\,{}^{(i)}u(t,\lambda)}{\partial\lambda}\right|^2dt=$$

$$=\frac{1}{N_0}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2}\left\{\left(\frac{\partial\,{}^{(i)}u_a(t,\lambda)}{\partial\lambda}\right)^2+{}^{(i)}u_a^2(t,\lambda)\left(\frac{\partial\,{}^{(i)}\psi(t,\lambda)}{\partial\lambda}\right)^2\right\}dt,$$

$$\frac{1}{N_0}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2}{}^{(i)}u(t,\lambda)\,\frac{\partial\,{}^{(i)}u^*(t,\lambda)}{\partial\lambda}\,dt=$$

$$=\frac{1}{N_0}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\{{}^{(i)}u_a^2(t,\lambda)\,\frac{\partial\,{}^{(i)}\psi(t,\lambda)}{\partial\lambda}\right\}dt. \quad (6.7.71)$$

Finding in these conditions matrix $\left[\mathbf{I}+\frac{1}{2}\,\mathbf{S}_c(\omega)\times\overline{\mathbf{B}}_0\right]^{-1}\mathbf{S}_c(\omega)$ in the form $A\mathbf{p}\mathbf{p}^+$, we have

$$A=\frac{S_c(\omega)}{\Delta f_c}\,\frac{1}{1+h_\Sigma S_c(\omega)},$$

where

$$h_\Sigma=\sum_{k=1}^{m}P_{ck}/2N_0\Delta f_c. \quad (6.7.72)$$

Finally, by formula (6.7.69), taking into account relationships (6.7.71), we obtain

$$K=J_1(U)\,J_2(h_\Sigma,\,S_c). \quad (6.7.73)$$

Here

$$J_1(U)=b_2-b_1^2,$$

$$b_2 = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} \sum_{k=1}^{m} a_k \left[\left(\frac{\partial^{(k)} u_a(t,\lambda)}{\partial\lambda} \right)^2 + {}^{(k)}u_a^2(t,\lambda) \times \right.$$
$$\left. \times \left(\frac{\partial^{(k)} \psi(t,\lambda)}{\partial\lambda} \right)^2 \right] dt,$$
$$b_1 = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} \sum_{k=1}^{m} a_k^{(k)} u_a(t,\lambda) \frac{\partial^{(k)} \psi(t,\lambda)}{\partial\lambda} dt,$$
$$a_k = P_{ck} \Big/ \sum_{i=1}^{m} P_{ci},$$

and factor

$$J_2(h_\Sigma, S_0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{h_\Sigma^2 S_0^2(\omega)}{1 + h_\Sigma S_0(\omega)} d\omega \tag{6.7.75}$$

repeats J_2 from relationship (6.7.37) with replacement of the signal-to-noise ratio h by quantity h_Σ, given by formula (6.7.72).

According to (6.7.73) and (6.7.74) the general form of quantity K remains the same as in the case of one input mixture, but in factor $J_1(U)$, determined by coding of $\lambda(t)$ in regular modulations of the mixture, there is performed averaging of analogous coefficients for separate mixtures taking into account signal powers, and in factor J_2 there appears a new signal-to-noise ratio, which, in view of total correlation of useful signals in separate mixtures, is proportional to the ratio of total power of these signals to the spectral density of noise in one channel. This indicates that potential accuracy does not depend on the number of channels with identical noises levels into which total signal power is divided, if we consider the latter constant. With the same success we can consider more complicated examples.

Inverse matrix $\mathbf{W}(t_1, t_2)$ can be effectively found also for pulse signals. The case of uncorrelated pulses we will consider in Paragraph 6.7.3.

Presentation of input mixtures in the form of (6.7.54) has a wide field of application. In particular, as separate components there can be considered correlated unmodulated interferences, which on we touched in Paragraph 6.7.1. If, however, correlated interferences have a complicated form of modulation, which it is impossible to present in the form of (6.7.54) (for instance, passive interferences), then finding matrix functions $\mathbf{W}(t_1, t_2)$, the inverse of correlation functions, is complicated. Here methods developed in Chapter 4 for problems of detection may help.

All results given in this paragraph will be illustrated by numerous examples in subsequent chapters.

6.7.3. Independently Fluctuating Sendings. General Case of an Incoherent Signal

We shall now discuss in detail the case of independently fluctuating sendings of ... signal. It is the limiting case both for coherent, and for incoherent ...ation, if the interval of fluctuations of the signal turns out to be less than the period of repetition.

Passing in (6.7.45) to the limit $S_0(\Omega) \to 1$ and introducing the new signal-to-noise ratio

$$q = \lim_{S(\Omega)\to 1} h_D = P_0 T_r / 2N_0,$$

we have the following expression for the likelihood coefficient of a series of n sendings:

$$\Lambda(\lambda,\ \mathbf{y}) = \frac{1}{(1+q)^n} \prod_{k=1}^{n} \exp\left\{\frac{q}{1+q}\,\frac{|f_k(\mathbf{y},\lambda)|^2}{N_0 T_r}\right\}, \tag{6.7.76}$$

where

$$f_k(\mathbf{y},\lambda) = \int y(t)\, u(t - kT_r;\ \lambda)\, e^{i\omega_0 t}\, dt$$

— correlation integral, components of which we already met above.

Thus, (6.7.76) is broken up into the product of n independent likelihood coefficients for each of the periods. Therefore, according to § 6 for analysis of elementary operations of primary processing it is sufficient to study the likelihood coefficient of a single sending

$$\Lambda(\lambda,\ \mathbf{y}) = \frac{1}{1+q} \exp\left\{\frac{q}{1+q}\,\frac{|f(\mathbf{y},\lambda)|^2}{N_0 T_r}\right\}. \tag{6.7.77}$$

Inasmuch as in a wide class of cases intraperiod processing includes forming of two components of quantity $f(y, \lambda)$, it is useful to definitize the method of this forming and to explain the physical meaning of $f(\mathbf{y}, \lambda)$ and $|f(\mathbf{y}, \lambda)|^2$.

According to Fig. 6.31 for formation of two components of $f(\mathbf{y}, \lambda)$ input signal y(t) should be fed to two mixers with quadrature shift of high-frequency heterodyne voltages fed to them. Phase (frequency) modulation of the heterodyne signal should correspond to the expected, and the middle frequency should be exactly equal to the frequency of the received signal. Additionally there is produced multiplication of signal by the expected form of amplitude modulation, i.e., gating by a matched gate pulse. Then there is produced integration of the results of processing within the

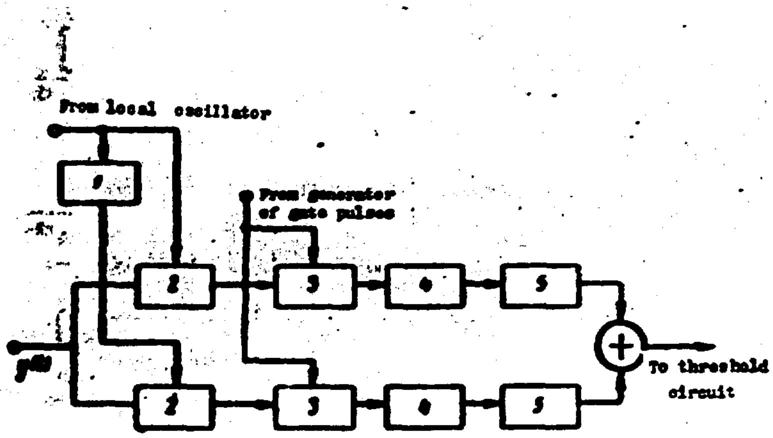

Fig. 6.31. Optimum receiver of an incoherent signal (correlation processing): 1) $\pi/2$ phase shifter; 2) quadrature mixers; 3) gated amplifiers; 4) integrators; 5) quadrature devices.

limits of a period (actually only within limits of duration of the sending). As already mentioned in Paragraph 6.7.2, such processing is called correlation, inasmuch as in the circuit there will be formed the coefficient of correlation between the received realization and the expected form of the signal. Shift to $|f(y, \lambda)|^2$ is carried out by squaring and addition of the two formed components of the correlation integral.

For the purpose of simplifying operations of forming $|f(y, \lambda)|^2$ we shall consider an expression for the output voltage of a certain bandpass filter with pulse response $h(t) \cos \omega_0 t$, to whose input there is fed signal $y(t)$. Then we have

$$\int y(\tau)\,h(t-\tau)\cos\omega_0(t-\tau)\,d\tau = \int y(\tau)\,h(t-\tau)\cos\omega_0\tau\,d\tau \times$$
$$\times\cos\omega_0 t + \int y(\tau)\,h(t-\tau)\sin\omega_0\tau\,d\tau\,\sin\omega_0 t =$$
$$= \sqrt{\left[\int y(\tau)\,h(t-\tau)\cos\omega_0\tau\,d\tau\right]^2 + \left[\int y(\tau)\,h(t-\tau)\sin\omega_0\tau\,d\tau\right]^2}\times$$
$$\times\sin\left[\omega_0 t + \chi(t)\right], \tag{6.7.78}$$

where

$$\operatorname{tg}\chi(t) = \frac{\int y(\tau)\,h(t-\tau)\sin\omega_0\tau\,d\tau}{\int y(\tau)\,h(t-\tau)\cos\omega_0\tau\,d\tau}.$$

Comparison of the expression for $|f(y, \lambda)|$ with the signal amplitude envelope, determined by relationship (6.7.78), shows that in the case of absence of regular phase (frequency) modulation ($\psi(t, \lambda) = 0$) quantity $|f(y, \lambda)|$ is proportional to the envelope of output voltage of a bandpass filter with pulse response

$$h(t)\cos\omega_0 t = u_a(t_0 - t)\cos\omega_0 t. \tag{6.7.79}$$

Such a filter in the literature usually has the name optimum, matched or conjugate filter. The last two terms seem more suitable to us, inasmuch as we arrive at linear filters in a number of other optimizing problems (Wiener filters, see Paragraph 6.5.5, filters for accumulation of fluctuations of signal, see Paragraph 6.7.2).

The square of the signal envelope at the output of a matched filter is taken

at moment t_0, on the average corresponding to the maximum value of the pulse envelope. Delay t_0, inserted in the pulse response, is arbitrary and can be selected in an interval of the order of pulse duration $u_a(t)$ for conformity to the principle of physical realizability. Considering (6.7.79), we can rewrite $|f(\mathbf{y}, \lambda)|^2$ in the form

$$|f(\mathbf{y},\lambda)|^2=\int_{-\infty}^{+\infty}\left[\int_{(k-1)T_r}^{kT_r} y(\tau)\,h(t-\tau;\lambda)\cos\omega_0(t-\tau)\,d\tau\right]^2\times \times v(t-kT_r-t_0)\,dt, \qquad (6.7.80)$$

where $v(t)$ — function in the form of a single peak near point $t = 0$ with duration considerably less than duration of pulse $u_a(t)$, but embracing a large or an integral number of periods of high-frequency filling of the signal. By this function, similar in properties to a Dirac δ-function, there is "snatched-off" the tip of the filtered and detected sending.

The block diagram of an optimum receiver, issuing quantity $|f(\mathbf{y}, \lambda)|^2$ according to relationship (6.7.80), is shown in Fig. 6.32 and consists of a matched filter, a square-law detector and a gated amplifier. Processing of the signal, expressed in form (6.7.80), usually carries in the literature the name of filtration.

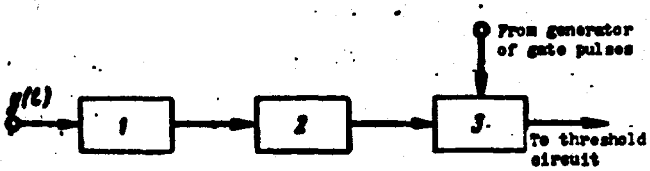

Fig. 6.32. Optimum receiver of an incoherent signal (filtration processing): 1) matched (shortening) filter; 2) square-law detector; 3) gated amplifier.

If the signal has phase (frequency) modulation, then next in the filtration method of processing will be insertion of a so-called shortening filter with a pulse response which is the inversion in time of the expected form of the phase-modulated signal with arbitrary initial phase of its high-frequency filling φ. This leads to ~~a somewhat complicated~~ modification of relationship (6.7.79):

$$h(t)\cos[\omega_0 t+\xi(t)]=u_a(t_0-t)\cos[\omega_0 t+\psi(t_0-t)+\varphi], \qquad (6.7.81)$$

where $\xi(t)$ — phase modulation.

To technically realize such a filter with complex forms of modulation is sometimes difficult [66]. Then, more preferable will be mixed correlation-filtration processing, which during formation of $|f(\mathbf{y}, \lambda)|^2$ consists of initial multiplication of $y(t)$ by heterodyne signal $\cos[(\omega_0 + \omega_{\text{пр}})t + \psi(t; \lambda)]$, leading to "convolution" of phase (frequency) modulation and transfer of the spectrum of the signal to a

certain intermediate frequency $\omega_{\text{пр}}$ analogously to how this was done in Paragraph 6.7.2. Then there is produced filtration of the signal by a bandpass filter tuned to frequency $\omega_{\text{пр}}$, whose pulse response envelope depicts only the amplitude modulation of the sending according to relationship (6.7.79), detection and gating. A circuit for such processing is presented in Fig. 6.33. Let us emphasize that correlation, filtration and mixed processings, conducted optimally, are absolutely equivalent in results during detection and measurement of parameters of a single sending.

If q does not depend on λ, the signal at the output of the discriminator is directly the derivative of $|f(y, \lambda)|^2$, taken for the measured value of $\hat{\lambda}$. Forming of this derivative, as also of $f(y, \lambda)$, can be produced by correlation, filtration or mixed methods.

In Fig. 6.34 there is presented a purely correlation method, which according to (6.7.77), can be expressed by the following formula:

$$\begin{aligned}
z(kT_r) = \text{const} \Big\{ & \int_{(k-1)T_r}^{kT_r} y(t_1)\, u_a(t_1 - kT_r;\ \hat{\lambda}_k) \cos[\omega_0 t_1 + \\
& + \psi(t_1 - kT_r;\ \hat{\lambda}_k)]\, dt_1 \int_{(k-1)T_r}^{kT_r} y(t_2) \Big[\frac{\partial}{\partial\lambda} u_a(t_2 - kT_r;\ \hat{\lambda}_k) \times \\
& \times \cos[\omega_0 t_2 + \psi(t_2 - kT_r;\ \hat{\lambda}_k)] - u_a(t_2 - kT_r;\ \hat{\lambda}_k) \times \\
& \times \frac{\partial}{\partial\lambda} \psi(t_2 - kT_r;\ \hat{\lambda}_k) \sin[\omega_0 t_2 + \psi(t_2 - kT_r;\ \hat{\lambda}_k)] \Big] dt_2 + \\
& + \int_{(k-1)T_r}^{kT_r} y(t_1)\, u_a(t_1 - kT_r;\ \hat{\lambda}_k) \sin[\omega_0 t_1 + \psi(t_1 - \\
& - kT_r;\ \hat{\lambda}_k)]\, dt_1 \int_{(k-1)T_r}^{kT_r} y(t_2) \Big[\frac{\partial}{\partial\lambda} u_a(t_2 - kT_r;\ \hat{\lambda}_k) \sin[\omega_0 t_2 + \\
& + \psi(t_2 - kT_r;\ \hat{\lambda}_k)] + u_a(t_2 - kT_r;\ \hat{\lambda}_k) \frac{\partial}{\partial\lambda} \psi(t_2 - \\
& - kT_r;\ \hat{\lambda}_k) \cos[\omega_0 t_2 + \psi(t_2 - kT_r;\ \hat{\lambda}_k)] \Big] dt_2 \Big\}.
\end{aligned} \qquad (6.7.82)$$

The circuit of Fig. 6.34 is very close to the circuit of Fig. 6.29. The first part of the circuit (processing in quadrature mixers and multiplication by functions u_a, $\frac{\partial u_a}{\partial\lambda}$ and $u_a \frac{\partial\psi}{\partial\lambda}$) completely repeats the circuit of Fig. 6.29. Further as filters there follow integrators over the period of the pulses, actually integrating inside

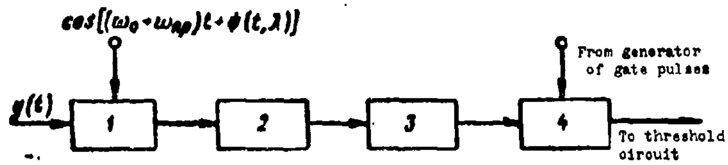

Fig. 6.33. Optimum receiver of an incoherent signal (filter processing): 1) mixer; 2) bandpass filter; 3) square-law detector; 4) gated amplifier.

the pulse. All the rest of the circuit — multipliers and adders — completely corresponds to Fig. 6.29. In other words, from processing in a discriminator of weakly correlated coherent pulses (see Paragraph 6.7.2) processing (6.7.82) differs only in the absence of a discrete interperiod accumulation before terminal multiplication. Thereby, processing (6.7.82) is limiting upon complete disappearance of correlation between periods.

Unfortunately, we cannot offer a sufficiently convenient circuit representation of purely filtration processing, if we are talking about absolutely exact fulfillment of the operation of taking the derivative of $|f(y, \lambda)|^2$. This is explained by the fact that parameters of a matched filter depend on λ, so that it is necessary to interpret in circuit terms a filter, the "derivative" of a matched one.

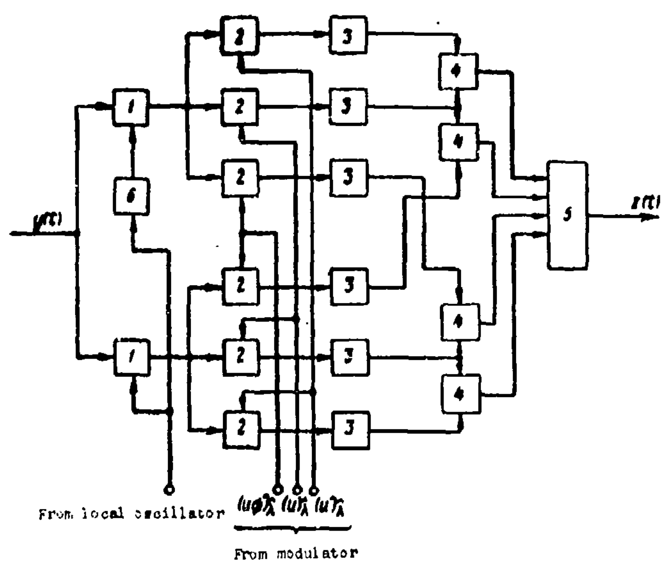

Fig. 6.34. Optimum discriminator for an incoherent signal (correlation processing): 1) quadrature mixers; 2) variable-gain amplifiers; 3) integrators; 4) multipliers; 5) adder; 6) $\pi/2$ phase shifter.

Without concretization of coding of the parameter this can hardly be done. If, however, we are talking about approximate realization of the operation in the form of two channels, detuned with respect to the parameter (see Paragraph 6.7.5), then as the discriminator there should be taken two circuits of type Fig. 6.32 or 6.33, detuned $\pm\Delta\lambda/2$ from the measured value of the parameter. In this case the filtration presentation of operations of the discriminator is obvious.

During calculation of the accuracy of a discriminator it is possible to use formula (6.7.52), where $J_1(u)$ is given by (6.7.36), and as J_{2D} we take the limit of

(6.7.53) as $S_{OD}(\omega) \to 1$:

$$J_{2D}(q,\ S_0) \to \frac{2q^2}{T_r(1+q)}.$$

Finally, we rewrite (6.7.52) in the form

$$K = \left\{ \frac{1}{T_r} \int_0^{T_r} \left[\frac{\partial u_a(t,\lambda)^2}{\partial\lambda} + u_a^2(t,\ \lambda) \left(\frac{\partial\psi(t,\lambda)}{\partial\lambda} \right)^2 \right] dt - \right.$$

$$\left. - \left[\frac{1}{T_r} \int_0^{T_r} u_a^2(t,\ \lambda) \frac{\partial\psi(t,\lambda)}{\partial\lambda}\, dt \right]^2 \right\} \frac{2q^2}{T_r(1+q)}, \qquad (6.7.83)$$

where T_r equals the period of repetition of sendings.

For independently fluctuating sendings there is possible a generalization of the preceding formulas to the case of correlated interference (with an interval of correlation, smaller than the repetition period). It turns out that in the period of repetition the functional of probability density as before is expressed by formula (6.7.77), but $|f(y,\ \lambda)|^2$ and q change their value:

$$q_\pi = q \left(1 - \frac{1}{2\pi T_r} \int_{-\infty}^{+\infty} |U(i\omega)|^2 \frac{h_\pi S_{0\pi}(\omega)}{1 + h_\pi S_{0\pi}(\omega)}\, d\omega \right), \qquad (6.7.84)$$

$$|f(y,\ \lambda)|^2 = \left[\iint_{(k-1)T_r}^{kT_r} y(t_1) W(t_1 - t_2) u_a(t_2 - kT_r;\ \lambda) \cos[\omega_0 t_2 + \right.$$

$$\left. + \psi(t_2 - kT_r;\ \lambda)]\, dt_1 dt_2 \right]^2 + \left[\iint_{(k-1)T_r}^{kT_r} y(t_1) W(t_1 - t_2) u_a(t_2 - \right.$$

$$\left. - kT_r;\ \lambda) \sin[\omega_0 t_2 + \psi(t_2 - kT_r;\ \lambda)]\, dt_1 dt_2 \right]^2 =$$

$$= \left| \int_{-\infty}^{+\infty} Y(\omega) \widetilde{W}(\omega) U^*(\omega - \omega)\, d\omega \right|^2. \qquad (6.7.85)$$

Here it is assumed that interference consists of white noise of spectral density N_0 and naturally correlated interference, the maximum of whose spectral density exceeds the density of white noise by a factor of h_π, and the normalized spectrum is equal to $S_{0\pi}(\omega)$ ($S_{0\pi}(0) = 1$). Further, $W(t_1 - t_2)$ — a function, the inverse of the correlation function of interferences of both types; q — signal-to-noise ratio in the absence of a correlated component of interference; $Y(\omega)$, $\widetilde{W}(\omega)$, $U(\omega)$ — Fourier transforms of functions of $y(t)$ (in the given period), $W(t_1 - t_2)$ and $u(t) = u_a(t)e^{i\psi(t)}$, respectively. Naturally, as $h_\pi \to 0$, according to (6.7.84), $q_\pi \to q$, and inasmuch as $W(t_1 - t_2) \to \frac{1}{N_0}\delta(t_1 - t_2)$, relationship (6.7.85) passes into (6.7.80).

A new processing element, appearing for correlated interference, is a rejector circuit with pulse response $W(t_1 - t_2)$, similar to that considered in Paragraph 6.7.1. Its minimum gain, equal to $[N_0(1 + h_{\pi})]^{-1}$ as compared to N_0^{-1} during white noise, corresponds to the frequency of the carrier of interference, and the rejection band grows with growth of h_{π}.

The formula for calculation of the accuracy characteristic here has the form

$$K = \frac{2q_{\pi}^2}{T_r(1+q_{\pi})}(b_2 + \operatorname{Re} b_1^2) - \frac{4q_{\pi}^3}{T_r(1+q_{\pi})^2}(\operatorname{Re} b_1)^2, \tag{6.7.86}$$

where

$$\left.\begin{aligned} b_2 &= \frac{1}{k_q}\frac{1}{2\pi T_r}\int\limits_{-\infty}^{+\infty}\frac{\left|\frac{\partial}{\partial\lambda}U(i\omega;\lambda)\right|^2}{1+h_{\pi}S_{0\pi}(\omega-\Delta)}\,d\omega;\\ b_1 &= \frac{1}{k_q}\frac{1}{2\pi T_r}\int\limits_{-\infty}^{+\infty}\frac{U^*(i\omega;\lambda)\frac{\partial}{\partial\lambda}U(i\omega;\lambda)}{1+h_{\pi}S_{0\pi}(\omega-\Delta)}\,d\omega;\\ k_q &= \frac{1}{2\pi T_r}\int\limits_{-\infty}^{+\infty}\frac{|U(i\omega;\lambda)|^2}{1+h_{\pi}S_{0\pi}(\omega-\Delta)}\,d\omega, \end{aligned}\right\} \tag{6.7.87}$$

k_q — factor of reduction of q_{π} in comparison with q;

$\Delta = \omega_{\pi} - \omega_0$ — separation by frequency of carriers of interference and the signal.

If we turn now to the general case of an incoherent signal against a background of white noise, it differs from the coherent appearance of uncontrollable (random) phase shifts θ_i in every period. The likelihood coefficient for this case can be found by introduction of these shifts in the expression for the likelihood coefficient of a coherent signal and by averaging all θ_i. In Chapter 5 (Vol. 1) it is shown that this averaging, besides the case of uncorrelated sendings, can be taken both for very large and very small noises. In all these cases the operation of detection is close to simple square accumulation of results of optimum intraperiod processing. Thus, from the point of view of operations of processing the whole analysis of construction of detection receivers and discriminators is also transferred to the case of an (correlated) incoherent signal, where it is more exact, the larger or smaller the q. This gives certain grounds to use the formulas derived with independent interperiods fluctuations as an approximation for the case of measurement of parameters of an arbitrary incoherent signal.

In completion we consider the case of many input signals ${}^{(i)}y(t)$ $(i = 1, \ldots, m)$, each of which may contain a whole series of useful components generally with different forms of modulation. We assume, however, that in all of them fluctuations

of the signal are independent from period to period, being "harmonious" within a period, and that interferences in all mixtures have the form of white noises. Construction of the functional of probability density then almost completely will repeat the analogous case of Paragraph 6.7.2. In view of the importance for applications we will repeat the sequence of these computations. The column of the set of mixtures analogously to (6.7.55) in the k-th period has the form

$$\mathbf{y}(t)=\mathrm{Re}\{\mathbf{U}^{+}(t)\theta e^{i\omega_0 t}\}+\mathbf{n}(t),$$

where $\mathbf{U}(t)$ — complex nearly-diagonal matrix of coefficients of regular modulation in the k-th period;

θ — complex column vector of random modulations (in distinction from Paragraph 6.7.2 it does not depend on time);

$\mathbf{n}(t)$ — column vector of white noises.

We have the correlation matrix of mixtures and its reciprocal matrix in the form

$$\mathbf{R}(t_1, t_2)=\mathrm{Re}\{\mathbf{U}^{+}(t_1)\mathbf{r}\mathbf{U}^{*}(t_2)e^{i\omega_0(t_1-t_2)}\}+\mathbf{N}\delta(t_1-t_2),$$

$$\mathbf{W}(t_1, t_2)=-\mathrm{Re}\{\mathbf{N}^{-1}\mathbf{U}^{+}(t_1)\mathbf{w}\mathbf{U}^{*}(t_2)\mathbf{N}^{-1}e^{i\omega_0(t_1-t_2)}\}+ \\ +\mathbf{N}^{-1}\delta(t_1-t_2),$$

where $\mathbf{r}$ — correlation matrix of order $\left\{\sum_{i=1}^{m} p_i\cdot\sum_{i=1}^{m} p_i\right\}$, reflecting relationships between all components of all mixtures;

$\mathbf{w}$ — a structurally similar matrix, which in this case is expressed in the form

$$\mathbf{w}=\left[\mathbf{1}+\frac{1}{2}\mathbf{r}\overline{\mathbf{B}}_0\right]^{-1}\mathbf{r};$$

$$\left(\overline{\mathbf{B}}_0=\frac{1}{T_r}\int_0^{T_r}\mathbf{U}^{*}(s)\mathbf{N}^{-1}\mathbf{U}(s)\,ds\right),$$

analogous to (6.7.62). From this we can obtain the operations of the detection receiver and discriminator.

Finally, quantity K by formula (6.7.65) can easily be reduced to a form analogous to (6.7.69):

$$K=\frac{1}{T_r}\,\mathrm{spur}\left\{\left[\mathbf{1}+\frac{1}{2}\mathbf{r}\overline{\mathbf{B}}_0\right]^{-1}\mathbf{r}\,[\overline{\mathbf{B}}_0\mathbf{r}\overline{\mathbf{B}}_2+\overline{\mathbf{B}}_1\mathbf{r}\overline{\mathbf{B}}_1]\right\}, \qquad (6.7.92)$$

where

$$\overline{\mathbf{B}}_1=\frac{1}{T_r}\int_0^{T_r}\mathbf{U}^{*}(s)\mathbf{N}^{-1}\frac{\partial}{\partial\lambda}\mathbf{U}^{+}(s)\,ds,$$

$$\overline{\mathbf{B}}_2=\frac{1}{T_r}\int_0^{T_r}\frac{\partial}{\partial\lambda}\mathbf{U}^{*}(s)\mathbf{N}^{-1}\frac{\partial}{\partial\lambda}\mathbf{U}^{+}(s)\,ds \qquad (6.7.93)$$

are matrices, reflecting regular modulation of components of mixtures ${}^{(1)}y(t)$.

In that particular case when the mixtures contain only one signal component, where these components are completely correlated, and white noises are independent, K again is expressed by formula (6.7.83), where J_2 is replaced by

$$J_2 = \frac{2q_\Sigma^2}{T_r(1+q_\Sigma)}, \qquad (6.7.94)$$

and $q_\Sigma = \frac{\sum_i P_{0i} T_r}{2N_0} = \frac{\sum_i S_i}{2N_0}$ – ratio of total signal energy in all channels to the intensity of white noise in one of the channels.

From the given examples it is clear that between accuracy characteristics of discriminators intended for signals of different nature there is great similarity. These interrelationships will be shown in more detail in subsequent chapters with examples of concrete coordinates.

6.7.4. Certain Forms of Non-Gaussian Signals in White Noises

Till now in examples of statistical properties of mixtures of signals with noises we almost always assumed Gaussian distribution of interference, which is usually not a limitation. Therefore it is basically interesting to study change of processing of signals for non-Gaussian statistics of the useful component of mixtures. From the example of one form of signals, taken against a background of white noises, we will show that deviation in the distribution of probabilities, even very considerable, does not lead to noticeable circuit changes of optimum discriminators.

We shall discuss independently fluctuating sendings of a signal, similar to those which we studied in Paragraph 6.7.3. If the phase in each sending is distributed evenly, then averaging with respect to phase and amplitude E of the likelihood functional of the signal against a background of normal white noise in one period leads to expression

$$P(y\,|\,\lambda) = C\int I_0\left(\frac{E}{N_0}\,|f(y,\lambda)|\right) P_t(E)\, e^{-\frac{E^2 T_r}{4N_0}}\, dE. \qquad (6.7.95)$$

It is useful to show the maximum possible number of regularities in general, and then to consider concrete examples. Expanding $I_0(x)$ in a Maclaurin series, convergent for any x, we have the logarithm of the likelihood function in the form

$$\ln P(\mathbf{y}|\lambda)=A_0+A_1|f(\mathbf{y},\lambda)|^2+A_2|f(\mathbf{y},\lambda)|^4+\ldots, \qquad (6.7.96)$$

where

$$A_1=\frac{1}{4N_0^2}\cdot\frac{M_1}{M_0},\quad A_2=\frac{1}{64N_0^4}\left[\frac{M_2}{M_0}-2\left(\frac{M_1}{M_0}\right)^2\right],\ldots$$

$$M_k=\int_0^\infty E^k P_0(E)\,e^{-\frac{E^2T_r}{4N_0}}\,dE, \qquad (6.7.97)$$

A_0 — constant.

According to (6.7.96) the logarithm of the likelihood function is a monotonically nondiminishing entire function of the square of the signal amplitude envelope at the output of a matched filter. If the level of the input signal is small, in expansion (6.7.96) it is sufficient to limit oneself to the first two terms, and the optimum receiver, and consequently also the discriminator, coincide with these for the case of a Gaussian signal. This circumstance is in force also in wider conditions and is intuitively intelligible in light of the fact that for a weak signal the mixture of it with noise is approximately normal for any properties of the signal. Unfortunately, similar general conclusions for high levels of the signal cannot be made.

Let us turn to the operation of the discriminator. According to (6.7.96)

$$z(kT_r)=[A_1+2A_2|f(\mathbf{y},\hat{\lambda}_k)|^2+\ldots]\frac{\partial}{\partial\lambda}|f(\mathbf{y},\hat{\lambda}_k)|^2=$$

$$=Q\{|f(\mathbf{y},\hat{\lambda}_k)|\}\,\frac{\partial}{\partial\lambda}|f(\mathbf{y},\hat{\lambda}_k)|^2. \qquad (6.7.98)$$

Thus, the operation of the discriminator for an arbitrary signal consists of reproduction of operations of a "Gaussian" discriminator, forming according to Paragraph 6.7.3 quantity $\frac{\partial}{\partial\lambda}|f(\mathbf{y},\lambda)|^2$, and of a "Gaussian" detector, forming quantity $|f(\mathbf{y},\lambda)|^2$. According to Fig. 6.35 the output signal of a "Gaussian" discriminator is multiplied by a positive function Q of $|f(\mathbf{y},\lambda)|$, concretely depending on statistical properties of the signal. As a result there will be formed the circuit of a discriminator for this signal. Inasmuch as characteristics of accuracy in the general case cannot be investigated, we consider certain examples.

a) Case of a nonfluctuating signal, when

$$P_0(E)=\delta(E-E_0).$$

Then according to (6.7.95),

$$\ln P(\mathbf{y}|\lambda)=C+\ln I_0\left(\frac{E_0}{N_0}|f(\mathbf{y},\lambda)|\right)$$

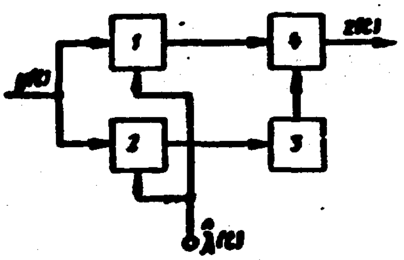

Fig. 6.35. Optimum discriminator for a "non-Gaussian" signal: 1) "Gaussian" discriminator; 2) optimum receiver forming $|f(y, \hat{\lambda})|^2$; 3) nonlinear inertialess converter; 4) multiplier.

and the operation of the discriminator will take form

$$z(kT_r)=\frac{I_1\left(\frac{E_0}{N_0}|f(y,\hat{\lambda}_k)|\right)}{I_0\left(\frac{E_0}{N_0}|f(y,\hat{\lambda}_k)|\right)}\frac{E_0}{2N_0|f(y,\hat{\lambda}_k)|}\frac{\partial|f(y,\hat{\lambda}_k)|^2}{\partial\lambda}. \quad (6.7.99)$$

Function

$$Q\{|f(y,\lambda)|\}=\frac{I_1\left(\frac{E_0}{N_0}|f|\right)}{I_0\left(\frac{E_0}{N_0}|f|\right)}\frac{E_0}{2N_0|f|}\approx\begin{cases}\left(\frac{E_0}{2N_0}\right)^2 & \text{when } \overline{|f|}<1,\\ \frac{E_0}{2N_0\overline{|f|}} & \text{when } \overline{|f|}>1\end{cases}$$

in this case is monotonically diminishing. For low-level signals, as already proved in general, it seeks a constant. For high-level signals, as it is easy to see from (6.7.99),

$$z(kT_r)=\text{const}\cdot\frac{\partial}{\partial\lambda}|f(y,\hat{\lambda}_k)|.$$

Operation of an optimum receiver in these conditions reduces to linear detection of the signal on the output of a matched filter.

If, however, we turn to analysis of optimum operations of the discriminator, then it is easy to prove that, instead of terminal ideal multipliers (phase detectors) contained in the "Gaussian" discriminator, here we have to take multipliers with clipping of an introduced signal which does not carry information about mismatch. Most phase detectors utilized in practice possess just such a property (see Chapter 2, Vol. 1). The physical explanation of the need for certain clipping of peaks of the input mixture as compared to the case of a Gaussian discriminator is the circumstance that the signal does not fluctuate, and considerable peaks can be formed only due to noises, suppressing the useful signal at separate moments.

The characteristic of accuracy of measurement can be easily obtained only for a high signal level, when

$$K\approx 2q\frac{1}{T_r}\int_0^{T_r}\left[\left(\frac{\partial a_0(t,\lambda)}{\partial\lambda}\right)^2+{}^{2}_{0}(t,\lambda)\left(\frac{\partial\varphi(t,\lambda)}{\partial\lambda}\right)^2\right]dt, \quad (6.7.100)$$

where $q = E_0^2T_r/4N_0$ – signal-to-noise ratio.

According to (6.7.100) K monotonically depends on the signal-to-noise ratio and on squares of the derivatives of modulating functions with respect to the measured quantity averaged over the period. We already met similar dependences above (Paragraphs 6.7.2 and 6.7.3).

b) The case of non-Gaussian distribution, having practical importance and presented in the form of the composition of several Rayleigh distributions for amplitude E:

$$P_e(E)=\sum_i p_i \frac{E}{P_{ei}} e^{-\frac{E^2}{2P_{ei}}}. \qquad (6.7.101)$$

Here p_i — weight of separate distribution $\left(\sum_i p_i=1\right)$. The likelihood functional, according to (6.7.90), is equal to

$$P(y|\lambda)=\sum_i \frac{p_i}{1+q_i} \exp\left\{\frac{q_i|f(y,\lambda)|^2}{N_0 T_r (1+q_i)}\right\}, \quad \left(q_i=\frac{P_{ci}T_r}{2N_0}\right), \qquad (6.7.102)$$

and operation of the discriminator is expressed by formula

$$z(kT_r)=\text{const}\frac{\sum_i \frac{p_i q_i}{(1+q_i)^2} \exp\left\{\frac{q_i|f(y,\hat{\lambda}_k)|^2}{N_0 T_r (1+q_i)}\right\}}{\sum_i \frac{p_i}{1+q_i} \exp\left\{\frac{q_i|f(y,\hat{\lambda}_k)|^2}{N_0 T_r (1+q_i)}\right\}} \times$$

$$\times \frac{\partial |f(y,\hat{\lambda}_k)|^2}{\partial \lambda}, \qquad (6.7.103)$$

where we assumed independence of q_i from λ. The coefficient in (6.7.103) with $\frac{\partial |f(y,\hat{\lambda})|^2}{\partial \lambda}$ with change of $f(y, \lambda)$ changes little, where members of series with small q_i play the prevailing role if for all distributions p_i in (6.7.101) they have the same order. In other words, the discriminator should be designed basically for a signal belonging to the distribution with the smallest q_i.

6.7.5. Approximate Methods of Construction of Optimum Discriminators. Method of Comparison of Performance of Circuits

By solution of the corresponding equations we found mathematical operations on the signal which optimum discriminators should execute. We also gave functional circuits corresponding to these operations. However, these circuits cannot always be practically realized exactly. Therefore, we give certain considerations about approximate technical realization of optimum discriminators.

The question reduces to approximate presentation of the first derivative of the logarithm of the likelihood function. The first natural method is replacement of the derivative by a finite difference. If exact operation of the discriminator is given by formula (6.6.36), the approximate operation will take form

$$z(t)\simeq\frac{1}{\Delta\lambda}\left[l\left(y;\hat{\lambda}(t)+\frac{\Delta\lambda}{2};t\right)-l\left(y;\hat{\lambda}(t)-\frac{\Delta\lambda}{2};t\right)\right], \qquad (6.7.104)$$

and for a discrete signal of the type studied in Paragraph 6.6.3 it will take form

$$z_k \simeq \frac{1}{\Delta\lambda}\Big[\int_{(k-1)T_r}^{kT_r} \frac{\partial}{\partial\lambda} l_k\Big(y;\, \hat{\lambda}_k + \frac{\Delta\lambda}{2};\, \tau\Big)d\tau - \\ - \int_{(k-1)T_r}^{kT_r} \frac{\partial}{\partial\lambda} l_k\Big(y;\, \hat{\lambda}_k - \frac{\Delta\lambda}{2};\, \tau\Big)d\tau\Big]. \qquad (6.7.105)$$

In essence the shown method reduces to forming a discriminator from two detection channels (without terminal integration or summation), detuned $\pm\Delta\lambda/2$ from the measured value of the parameter. Quantity $\hat{\lambda}$, as it were, is "forked." Sometimes it is technically inconvenient to have two channels immediately. Therefore, the second method of approximate realization of a discriminator is based on alternate detuning ("swinging") one detection channel near $\hat{\lambda}$. Naturally, this swinging should be faster than noticeable change of $\lambda(t)$.

Concrete values of the frequency and amplitude of swinging are selected from a series of theoretical and technical considerations and, not least of all, from consideration of maximum proximity to the potential limit of accuracy. Mathematically, swinging can be presented in the form

$$z(t) \simeq \frac{1}{T_{\kappa}} \int_{t-T_{\kappa}/2}^{t+T_{\kappa}/2} l\big(\tau;\, y;\, \hat{\lambda}(t) + f(\tau)\big)\, d\tau, \qquad (6.7.106)$$

where $f(t)$ — a function with period T_{κ} and amplitude, usually not exceeding the extent of the linear section of the discrimination characteristic of an optimum circuit.

As the most used forms of $f(t)$ we indicate a square or harmonic function.

All the shown approximations of the discriminator circuit, and also circuits, in which as compared to the optimum changes are experienced only by quantitative characteristics of separate elements, usually preserve the general idea of construction of the discriminator and therefore can be called quasi-optimum. The concept of quasi-optimality is imprecisely defined. Therefore, it is always necessary to have a quantitative comparison of these circuits, and also of any others which have appeared or will appear in practice for this function, with optimum circuits. Comparison of results of § 6.2 and § 6.6 shows that the basic characteristic of discriminators both optimum, and nonoptimal, is equivalent spectral density* (or variance of error of a

*Besides $S_{\text{экв}}$, we shall subsequently be interested in coefficient $S_{\text{пар}}$, characterizing parametric fluctuations, sometimes very noticeable. Then it is necessary to compare $S_{\text{пар}}$ with $S_{\text{пар опт}}$, corresponding to the optimum circuit.

unit measurement), designated correspondingly by $S_{экв}$ and $S_{экв\ опт} = 1/K(t)$. The coefficient of decrease of quality of the discriminator as compared to the optimum is introduced by relationship

$$\varkappa = \frac{S_{экв}}{S_{экв\ опт}}. \qquad (6.7.107)$$

In any case $\varkappa \geq 1$. If comparison of any constructed and practically convenient circuits with the optimum gives a small deviation $\varkappa$ from 1, then it is possible to express confidence that discovery of any new circuits will not improve performance indices of the radar and is therefore inexpedient.

§ 6.8. Synthesis of Smoothing Circuits and Resultant Accuracy of Measurements

In § 6.6 it was shown that optimum meters contain smoothing circuits. In the present paragraph without concretization of the physical nature of the measured quantity we obtain and analyze optimum smoothing circuits for different statistical properties of $\lambda(t)$. Pulse responses of these circuits, and also resultant error of measurements are determined by equations (6.6.31) and (6.6.32) or (6.6.53) and (6.6.54). Solutions of these equations depend on the correlation function of the parameter $R(t, \tau)$ and function $K(t)$, which depends on statistical properties of the input signal $y(t)$ and the method of encoding in it parameter $\lambda(t)$. We already indicated that for Gaussian statistics of λ (t) synthesis of smoothing circuits is analogous to synthesis of Wiener filters. Although there are solutions of the last problem in literature, the specific form of equivalent "interference" and presentation of the meter in the form of a closed tracking system permit us to obtain a series of new conclusions interesting for applications.

Below we shall consider parameters in the form of stationary random processes, processes with stationary increments, quasi-regular processes (linear combinations of known functions with random factors), and also mixed cases. We separately consider synthesis of smoothing filters for meters of linear functionals of parameters.

6.8.1. Parameter — Stationary Random Process

In a number of applications it is possible to consider the measured quantity a random stationary process. An example is steady fluctuations of speed in an aircraft Doppler meter, groundspeed meter, and so forth.

If the measured parameter is stationary, during an arbitrary time of observation equation (6.6.53) takes form

$$c(t, \tau)+K\int_{0}^{t} c(t, s) R(s-\tau) ds = R(t-\tau), \qquad (6.8.1)$$

i.e., turns out to be an integral equation with a nucleus, depending on the difference of the arguments. Methods of solution of such equations have been studied in principle repeatedly [19]. An exact solution can be obtained if spectral density of the parameter has the form

$$S(\omega)=\int_{-\infty}^{+\infty} R(\tau) e^{-i\omega\tau} d\tau=\frac{|Q_m(i\omega)|^2}{|P_n(i\omega)|^2}, \qquad (6.8.2)$$

where $Q_m(i\omega)$, $P_n(i\omega)$ — polynomials from $i\omega$ of degree m and $n > m$.

Such presentation of $S(\omega)$ is very general and embraces practically all real cases of stationary processes. Substituting in (6.8.1) function $R(t - \tau)$, expressed through $S(\omega)$, and applying to both parts of this equation differential operator $P_n(d/d\tau)\, P_n(-d/d\tau)$, taking into account (6.8.2) it is possible to show that equation (6.8.1) is equivalent to differential equation

$$P_n\left(\frac{d}{d\tau}\right)P_n\left(-\frac{d}{d\tau}\right)c(t,\tau)+Q_m\left(\frac{d}{d\tau}\right)Q_m\left(-\frac{d}{d\tau}\right)Kc(t,\tau)=$$
$$=Q_m\left(\frac{d}{d\tau}\right)Q_m\left(-\frac{d}{d\tau}\right)\delta(t-\tau). \qquad (6.8.3)$$

Additionally $c(t, \tau)$ should satisfy boundary conditions at the ends of the interval of observation $(0, t)$, which are obtained if we apply to both parts of (6.8.1) operators $(d^k/d\tau^k)P_n(d/d\tau)$ and $(d^k/d\tau^k)P_n(-d/d\tau)$ $(k = 0, 1, \ldots, n-1)$ and use certain theorems of the theory of functions of a complex variable:

$$\begin{cases}\dfrac{d^k}{d\tau^k}P_n\left(\dfrac{d}{d\tau}\right)c(t,\tau)+K\displaystyle\sum_{j=0}^{2m+k-n}A_j^{(k)}\frac{d^j}{d\tau^j}c(t,\tau)=0 & \text{when } \tau=0,\\[2ex] \dfrac{d^k}{d\tau^k}P_n\left(-\dfrac{d}{d\tau}\right)c(t,\tau)+K\displaystyle\sum_{j=0}^{2m+k-n}(-1)A_j^{(k)}\frac{d^j}{d\tau^j}\times \\ \times c(t,\tau)=0 & \text{when } \tau=t.\end{cases} \qquad (6.8.4)$$

Here $\sum_{j=0}^{2m+k-n} A_j^k (i\omega)^j$ — multinomial part of the fraction

$$|Q_m(i\omega)|^2(i\omega)^k P_n^{-1}(i\omega).$$

Time t in (6.8.3) and (6.8.4) is considered a fixed parameter.

Thus, solution of equation (6.8.1) is equivalent to finding Green's function of the corresponding linear differential equation with constant coefficients with boundary conditions (6.8.4). Solution of this problem is sometimes cumbersome, but does not present fundamental difficulties.

Certain simplification is attained when $m = 0$, $S(\omega) = |P_n(i\omega)|^2$. Boundary conditions here are simplified, since $A_j^{(k)} = 0$.

We shall consider as an example a parameter in the form of a stationary random process, which it is formally possible to consider formed by passage of white noise through an inertial link with time constant T:

$$R(t)=\sigma_\lambda^2 e^{-|t|/T}, \quad S(\omega)=\frac{2\sigma_\lambda^2 T}{1+(\omega T)^2}. \tag{6.8.5}$$

Note that in this case

$$P\left(\frac{d}{dt}\right)=\frac{1}{\sqrt{2\sigma_\lambda^2 T}}+\sqrt{\frac{T}{2\sigma_\lambda^2}}\,\frac{d}{dt},$$

then relationships (6.8.3) and (6.8.4) will take form

$$\begin{gathered}(1+2\sigma_\lambda^2 TK)c(t,\tau)+T^2\frac{d^2}{d\tau^2}c(t,\tau)=2\sigma_\lambda^2 T\delta(t-\tau),\\ \left[c(t,\tau)+T\frac{d}{d\tau}c(t,\tau)\right]_{\tau=0}=0,\\ \left[c(t,\tau)-T\frac{d}{d\tau}c(t,\tau)\right]_{\tau=t}=0.\end{gathered} \tag{6.8.6}$$

System (6.8.6) has the following solution:

$$c(t,\tau)=2\sigma_\lambda^2\frac{(\sqrt{1+\rho}+1)e^{\frac{\sqrt{1+\rho}\,\tau}{T}}+(\sqrt{1+\rho}-1)e^{-\frac{\sqrt{1+\rho}\,\tau}{T}}}{(\sqrt{1+\rho}+1)^2e^{\frac{\sqrt{1+\rho}\,t}{T}}-(\sqrt{1+\rho}-1)^2e^{-\frac{\sqrt{1+\rho}\,t}{T}}}, \tag{6.8.7}$$

where $\rho = 2\sigma_\lambda^2 TK$.

A circuit with pulse response (6.8.7) can be realized by a controlled variable-gain amplifier, an integrator and one more controlled amplifier, coupled in series (Fig. 6.36).

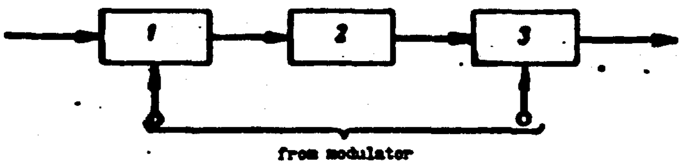

Fig. 6.36. Smoothing circuits for a stationary parameter with small observation time. 1, 3 — variable-gain amplifiers; 2 — integrator.

Very interesting is the case of a large time of observation, when $t/T \gg 1$. Then in view of attenuation of pulse response one should consider small differences between moments of application of input disturbances and reading $(t - \tau)/T \sim 1$, when simultaneously $t/T \gg 1$, $\tau/T \gg 1$. It is easy to prove that in these conditions

$$c(t,\tau)\to c(t-\tau)=\frac{2\sigma_0^2}{\sqrt{1+\rho}+1}\exp\left\{-\frac{\sqrt{1+\rho}}{T}(t-\tau)\right\}. \tag{6.8.8}$$

Thereby, in the limiting case pulse response $c(t, \tau)$ depends only on the difference of arguments and coincides with the response of a certain link of the first order with effective time constant

$$T_{эфф} = T/\sqrt{1+\rho}. \tag{6.8.9}$$

Seeking from (6.8.8) solution of equation (6.6.54) in the form $g(t - \tau)$, we have function

$$g(t-\tau) = \frac{2\sigma_0^2}{\sqrt{1+\rho}+1} \exp\left\{-\frac{(t-\tau)}{T}\right\} \tag{6.8.10}$$

with Fourier transform

$$G(i\omega) = \frac{2\sigma_0^2 T}{\sqrt{1+\rho}+1} \cdot \frac{1}{1+i\omega T}. \tag{6.8.11}$$

Thus, the single-loop variant of smoothing circuit should be a circuit with exactly that time constant which corresponds to the inertial link forming the parameter of white noise. Multiplying $G(i\omega)$ by the transmission factor of an optimum discriminator K, we note that the total gain factor in the loop of an optimum meter varies as

$$K_\Sigma = \frac{\rho}{\sqrt{1+\rho}+1} = \begin{cases} \rho/2 \text{ when } \rho \ll 1, \\ \sqrt{\rho} \text{ when } \rho \gg 1. \end{cases} \tag{6.8.12}$$

Finally, variance of resultant error of measurement of the parameter, according to (6.8.7), is equal to

$$\sigma^2_{изм}(t) = c(t,t) = $$
$$= 2\sigma_\lambda^2 \frac{(1+\sqrt{1+\rho})\, e^{\frac{\sqrt{1+\rho}}{T}t} + (\sqrt{1+\rho}-1)\, e^{-\frac{\sqrt{1+\rho}}{T}t}}{(1+\sqrt{1+\rho})^2 e^{\frac{\sqrt{1+\rho}}{T}t} - (\sqrt{1+\rho}-1)^2 e^{-\frac{\sqrt{1+\rho}}{T}t}} \to$$
$$\to \frac{2\sigma_\lambda^2}{1+\sqrt{1+\rho}}. \tag{6.8.13}$$

Quantity ρ in relationships (6.8.7)-(6.8.13) actually is the equivalent signal-to-noise ratio in the meter, inasmuch as it is equal to the ratio of the maximum value of spectral density of the parameter $2\sigma_\lambda^2 T$ to the equivalent spectral density of the discriminator $1/K$. The monotone character of the dependence of

maximum error in (6.8.13) on ρ does not require explanation. More interesting is the dependence of the overall gain factor in the loop on ρ. Physically the interconnection between K_Σ and ρ is explained by optimum selection of the transmission band of the system

$$\Delta f_{\ni\phi} = \frac{1}{2\pi}\int_0^\infty \left|\frac{KC(i\omega)}{KC(0)}\right|^2 d\omega = \frac{\sqrt{1+\rho}}{4T}. \tag{6.8.14}$$

For high levels of noise it is useful to narrow the bandwidth to a matched width $(\Delta f_{\ni\phi})_{corл} = 1/4T$ for maximum suppression of noises, sacrificing tracking of abrupt changes of the parameter. For low-level noises it is more useful to use a transmission band expanded by a factor of $\sqrt{1+\rho}$, within limits of which the level of spectral density of the parameter exceeds the level of the noises.

Especially important for applications is the case of a large time of observation, when the process of measurement "becomes steady" and its characteristics cease to depend on the initial conditions of measurement. To find limiting values of $c(t-\tau)$ and $g(t-\tau)$ for the case of a large time of observation in general is considerably more convenient by the method of factoring, given in § 6.5. For the given case there is performed factoring of function

$$1 + KS(\omega) = \Psi(i\omega)\Psi(-i\omega), \tag{6.8.15}$$

and the final result is recorded in the form

$$C(i\omega) = \frac{1}{\Psi(i\omega)}\left[\frac{S(\omega)}{\Psi(-i\omega)}\right]_+. \tag{6.8.16}$$

Here, the smoothing filter of the equivalent single-loop system has frequency response

$$G(i\omega) = C(i\omega)/(1 - KC(i\omega)), \tag{6.8.17}$$

which is found from equation (6.6.54) by the method of Fourier transforms.

Stationary error of measurement at the limit is constant and is expressed by a simple formula, based on the theory of Fourier integrals:

$$\sigma^2_{\text{вых}} = \lim_{\omega\to\infty} i\omega C(i\omega). \tag{6.8.18}$$

Especially convenient is the method of factoring for a rational-fractional spectral density of the parameter of form (6.8.2). Turning as an example again to the very simple case described by relationships (6.8.5) we have, concretely,

$$1+KS(\omega)=\frac{1+\rho+(\omega T)^2}{1+(\omega T)^2}=$$
$$=\frac{\sqrt{1+\rho}+i\omega T}{1+i\omega T}\cdot\frac{\sqrt{1+\rho}-i\omega T}{1-i\omega T}=\Psi(i\omega)\Psi(-i\omega).$$

Then from (6.8.16) we have

$$\left[\frac{S(\omega)}{\Psi(-i\omega)}\right]_+=\frac{\rho}{K}\left[\frac{1-i\omega T}{(\sqrt{1+\rho}-i\omega T)[1+(\omega T)^2]}\right]_+=$$
$$=\frac{\rho}{K(\sqrt{1+\rho}+1)}\left[\frac{1}{\sqrt{1+\rho}-i\omega T}+\frac{1}{1+i\omega T}\right]_+=$$
$$=\frac{\rho}{K(\sqrt{1+\rho}+1)}\,\frac{1}{1+i\omega T},$$

and, finally, the frequency response of the filter is equal to

$$C(i\omega)=\frac{\rho}{K\sqrt{1+\rho}\,(\sqrt{1+\rho}+1)}\cdot\frac{1}{1+\frac{i\omega T}{\sqrt{1+\rho}}},$$

which coincides with the Fourier transform of (6.8.8). Increase of the orders of polynomials $P_n(i\omega)$ and $Q_k(i\omega)$ in (6.8.2) leads to complication of the factoring procedure, consisting of finding roots of algebraic equations of a high order. In principle this is the same equation as for an arbitrary time of observation, but the limiting case nevertheless sharply simplifies further calculations freeing us from the need to allow for boundary conditions.

6.8.2. Parameter — Random Process with Stationary Increasements

In engineering practice in a number of cases it impossible even to approximately consider the measured quantity stationary: however, it is possible to consider stationary a certain derivative of the parameter. Thus, if acceleration of the object changes chaotically and steadily, the coordinate of it is a random process with a stationary second derivative. Such a type of random process we include in the class of processes with k-th stationary increments ($k = 0, 1, \ldots$) [59].

Mathematically a process with stationary increments of the k-th order is defined as process $x(t)$, statistical characteristics of whose k-th difference

$$\Delta_\tau^k x(t)=\sum_{m=0}^{k}\binom{k}{m}(-1)^m x(t-m\tau)$$

are not time-dependent. Formally, it is possible equate such a process to a process

with a stationary k-th derivative. To this class there belong stationary processes and processes obtained from them by application of integro-differential operators with constant coefficients. Correlation function $R(t, \tau)$ of a process with stationary increments is easily expressed through the correlation function of the stationary k-th derivative of $\rho(t - \tau)$ in the form

$$R(t, \tau) = \begin{cases} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 (t - t_1)^{k-1} (\tau - t_2)^{k-1} \rho(t_1 - t_2) & \text{when } k > 1, \\ \rho(t - \tau) & \text{when } k = 0. \end{cases} \tag{6.8.19}$$

This relationship with a stationary process permits us to introduce the concept of spectral density $S(\omega)$ of the process with stationary increments, determining it according to the formula

$$\rho(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{+i\omega\tau} \omega^{2k} S(\omega) d\omega. \tag{6.8.20}$$

To find function $c(t, \tau)$ from equation (6.6.53) with arbitrary observation time in principle it would have been possible to use the method of reduction to a differential equation with a special right member, absolutely analogous to that presented in Paragraph 6.8.1, if the spectral density of the process is a rational-fraction function of $i\omega$, i.e., can be presented in the form (6.8.2). We select a somewhat different way of presentation. By two important and very graphic examples we explain the method of reduction of equation (6.6.53) to differential equations of another type; with the help of the latter we effectively solve these particular problems; and then we expound a general method for obtaining limiting operators, i.e., smoothing filters for a large time of observation.

Let us start with the very simple example of a parameter in the form of a Wiener process, i.e., the integral of white noise. The correlation function of this process is equal to

$$R(t, \tau) = B_1 \min(t, \tau), \tag{6.8.21}$$

where B_1 = const is the spectral density of the initial white noise.

Substituting (6.8.21) in equation (6.6.53) for $t \geq \tau$, we have

$$c(t, \tau) + KB_1 \int_0^{\tau} c(t, s) s\, ds + KB_1 \tau \int_{\tau}^{t} c(t, s)\, ds = B_1 \tau, \tag{6.8.22}$$

and successive differentiation with respect to τ gives

$$\frac{\partial c(t,\tau)}{\partial \tau}+KB_1\int_{\tau}^{t} c(t,s)\,ds=B_1, \qquad (6.8.23)$$

$$\frac{\partial^2 c(t,\tau)}{\partial \tau^2}-KB_1 c(t,\tau)=0. \qquad (6.8.24)$$

The solution of equation (6.8.24), satisfying conditions (6.8.22) and (6.8.23) simultaneously has the form

$$c(t,\tau)=\sqrt{\frac{B_1}{K}}\,\frac{\operatorname{sh}\sqrt{B_1K}\,\tau}{\operatorname{ch}\sqrt{B_1K}\,t}. \qquad (6.8.25)$$

From this, according to (6.8.25) and (6.6.45), we have

$$\sigma^2_{out}(t)=\sqrt{\frac{B_1}{K}}\operatorname{th}\sqrt{B_1K}\,t. \qquad (6.8.26)$$

The last expression shows that for small times of observation variance of error of measurement is equal to variance of the a priori distribution of $R(t, t) =$ $= B_1T$, and with growth of t passes to a stationary value $\sqrt{B_1/K}$, monotonically depending on intensity B_1 of white noise, from which the parameter is formed, and the equivalent spectral density of an optimum discriminator $1/K$. The process of becoming steady is faster, the larger the product B_1K.

Function $c(t, \tau)$ for large t, τ also has limiting value

$$c(t,\tau)\to c(t-\tau)=\sqrt{\frac{B_1}{K}}\,e^{-\sqrt{B_1K}(t-\tau)},$$

corresponding to a RC-circuit with time constant $(B_1K)^{-1/2}$, which follows from the form of the frequency response

$$C(i\omega)=\int_{-\infty}^{+\infty} c(t-\tau)\,e^{-i\omega(t-\tau)}\,d(t-\tau)=\frac{\sqrt{B_1/K}}{\sqrt{B_1K}+i\omega}. \qquad (6.8.27)$$

Pulse response of an open loop $g(t, \tau)$ is determined by equation (6.6.54), and for $c(t, \tau)$, given by formula (6.8.25), it is equal to

$$g(t,\tau)=\sqrt{\frac{B_1}{K}}\operatorname{th}\tau, \qquad (6.8.28)$$

i.e., in general, does not depend on t. According to (6.8.28) smoothing consists of amplification of the output signal of the discriminator by an amplifier with gain factor, varying according to the law (6.8.28), and subsequently by integration.

In view of the a priori known continuity of parameter $\lambda(t)$ and zero initial error of measurement the output signal of the discriminator $z(t)$ for $\tau < (B_1 K)^{-1/2}$ is transmitted with a small, but increasing amplification. The build-up is explained by gradual increase of variance of $\lambda(t)$. When $\tau \gg (B_1 K)^{-1/2}$ smoothing passes into constant amplification and to integration.

It is interesting to note that a smoothing circuit with one integrator is very often used in practice. Theoretical consideration shows that without taking into account the transient regime such a smoothing filter is optimum for parameter $\lambda(t)$ in the form of the first integral of white noise. The magnitude of the overall gain factor in open loop $K\sqrt{B_1/K} = \sqrt{KB_1}$ here depends on the intensity of parameter B_1, and also on the statistics of the input mixture of the signal with noise (through K). This result, according to Paragraph 6.8.1, is natural.

Let us consider now a process in the form of the second integral of white noise, the correlation function of which is equal to

$$R(t,\tau) = \begin{cases} B_2 \dfrac{\tau^2 (3t-\tau)}{6} & \text{when } \tau < t, \\ B_2 \dfrac{t^2 (3\tau - t)}{6} & \text{when } t < \tau. \end{cases} \tag{6.8.29}$$

Successively differentiating with respect to τ equation (6.6.53), in which there is placed the value of $R(t, \tau)$ from (6.8.29), we have differential equation

$$\frac{\partial^4 c(t,\tau)}{\partial \tau^4} + a_2^4 c(t,\tau) = 0, \quad (a_2 = (B_2 K)^{1/4}) \tag{6.8.30}$$

with four integro-differential subsidiary conditions:

$$\begin{cases} c^*(t,\tau) + a_2^4 \displaystyle\int_0^\tau c^*(t,s) \frac{s^2(3\tau - s)}{6} ds + \\ \quad + a_2^4 \displaystyle\int_\tau^t c^*(t,s) \frac{\tau^2 (3s-\tau)}{6} ds = a_2^4 \frac{\tau^2(3t-\tau)}{6}, \\ \dfrac{\partial c^*(t,\tau)}{\partial \tau} + a_2^4 \displaystyle\int_0^\tau c^*(t,s) \frac{s^2 ds}{2} + \\ \quad + a_2^4 \displaystyle\int_\tau^t c^*(t,s) \frac{2\tau s - \tau^2}{2} ds = a_2^4 \frac{2t\tau - \tau^2}{2}, \\ \dfrac{\partial^2 c^*(t,\tau)}{\partial \tau^2} + a_2^4 \displaystyle\int_\tau^t c(t,s)(s-\tau)\, ds = a_2^4 (t-\tau), \\ \dfrac{\partial^3 c^*(t,\tau)}{\partial \tau^3} - a_2^4 \displaystyle\int_\tau^t c^*(t,s)\, ds = -a_2^4, \end{cases}$$

where

$$c^*(t, \tau) = Kc(t, \tau).$$

Finally, we have a solution in the form

$$c^*(t,\tau) = 2\sqrt{2}a_2 \left[\frac{\operatorname{ch}\frac{a_2 t}{\sqrt{2}} \operatorname{sh}\frac{a_2 \tau}{\sqrt{2}} \cos\frac{a_2(t-\tau)}{\sqrt{2}}}{\operatorname{ch}\sqrt{2}a_2 t + 2 + \cos\sqrt{2}a_2 t} - \right.$$

$$\left. - \frac{\cos\frac{a_2 t}{\sqrt{2}} \sin\frac{a_2 t}{\sqrt{2}} \operatorname{ch}\frac{a_2(t-\tau)}{\sqrt{2}}}{\operatorname{ch}\sqrt{2}a_2 t + 2 + \cos\sqrt{2}a_2 t} \right] \tag{6.8.31}$$

Variance of error of measurement is equal to

$$\sigma^2_{изм}(t) = \frac{\sqrt{2}a_2}{K} \frac{\operatorname{sh}\sqrt{2}a_2 t - \sin\sqrt{2}a_2 t}{\operatorname{ch}\sqrt{2}a_2 t + 2 + \cos\sqrt{2}a_2 t}. \tag{6.8.32}$$

In particular, for large and small time of measurement

$$\sigma^2_{изм}(t) = \begin{cases} B_2 t^3/3 \text{ when } t < (B_2 K)^{-1/4}, \\ \sqrt{2}(B_2/K^3)^{1/4} \text{ when } t > (B_2 K)^{-1/4}. \end{cases} \tag{6.8.33}$$

As follows from (6.8.33), for a small time observation variance of error of measurement, as also in the preceding example, increases as the a priori variance of parameter $R(t, \tau)$ from (6.8.29). The fact is that in the first moments inertial smoothing elements have not yet accumulated a signal, which testifies to the appearing error of measurement. For a large time of observation variance of error seeks, as before, a stationary value, monotonically depending on spectral density B_2 of white noise, from which the parameters was formed, and on the equivalent spectral density of the discriminator $1/K$.

It is interesting to note that for large time of observation $c^*(t, \tau) = Kc(t, \tau)$ passes to

$$Kc(t,\tau) \to \frac{a_2}{\sqrt{2}} e^{-\frac{a_2(t-\tau)}{\sqrt{2}}} \cos\frac{a_2(t-\tau)}{\sqrt{2}}, \tag{6.8.34}$$

i.e., describes a circuit with constant parameters and frequency response

$$KC(i\omega) = \frac{(i\omega)\sqrt{2}a_2 + a_2^2}{(i\omega)^2 + (i\omega)\sqrt{2}a_2 + a_2^2}. \tag{6.8.35}$$

In both the examples given, in spite of the nonstationariness of the parameter, operators of smoothing, analogously to Paragraph 6.8.1, belong to circuits with constant parameters.

The physical expalnation of this is that during measurement of a process with stationary increments there is established dynamic equilibrium between the effect of the growth of uncertainty of the value of the parameter due to its random changes and the effect of more precise definition of its value due to the arrival of new data. These mutually opposite effects follow from relationships (6.5.35) and will be touched on again in § 6.9.

It is interesting that transmission of the mixture of a process with a stationary first or second derivative with white noise through a circuit with a rational-fraction frequency response of type (6.8.27) or (6.8.35), in which the order of the polynomial of the numerator is less than the order of the polynomial of the denominator by one, ensures minimum necessary difference of orders of these polynomials for smoothing white noise and simultaneously gives at the output a random process with a stationary derivative, i.e., a process, similar in properties to the measured quantity. This is especially graphic for a system of the first order, at whose output there will be formed a random process with spectral density B_1/ω^2, equal in accuracy to the spectral density of the tracked quantity.

It is natural to assume that for any processes with stationary derivatives there exists limiting operators of smoothing, belonging to circuits with a constant parameters. Therefore, we generalize the algorithm of finding limiting operators, based on factoring, to parameters in the form of random processes with stationary increments. Let us substitute in equation (6.6.53) correlation function $R(t, \tau)$, expressed according to (6.8.19), and direct in this expression $t_0 \to \infty$, seeking a solution in the form $c(t, \tau) = c(t - \tau)$. Then, applying to both parts of the equation Fourier transformation and using relationship (6.8.20), one can prove that the solution of equation (6.6.53) in this case is expressed by the same formula (6.8.16) as for stationary processes. The pulse response of the smoothing filter of a single-loop circuit $g(t, \tau) = g(t - \tau)$ is determined through the Fourier transformation with the help of relationship (6.8.17), and maximum error of measurement is given by formula (6.8.18).

It is necessary only to stipulate that factoring of (6.8.15) in this case is based on presentation of the factor of form ω^{2k} in the denominator of spectral density $S(\omega)$ in the form $(i\omega)^k(-i\omega)^k$, where $1/i\omega$ is formally considered a pole in the upper half-plane, and $1/(-i\omega)$ is considered a pole in the lower half-plane of complex variable ω.

For the purpose of illustration of the method of factoring we again consider

a parameter in the form of the first and second integrals of white noise. In the first case spectral density of the parameter according (6.8.19) should be defined as $S(\omega) = B_1/\omega^2$. Here,

$$\Psi(i\omega)\Psi(-i\omega)=1+KS(\omega)=\frac{\sqrt{B_1K}+i\omega}{i\omega}\,\frac{\sqrt{B_1K}-i\omega}{-i\omega},$$

and in accordance with (6.8.16)-(6.8.18) we have

$$C(i\omega)=\frac{i\omega}{\sqrt{B_1K}+i\omega}\left[\frac{B_1(-i\omega)}{\omega^2(\sqrt{B_1K}+i\omega)}\right]_+=\sqrt{\frac{B_1}{K}}\,\frac{1}{\sqrt{B_1K}+i\omega},$$

$$G(i\omega)=\sqrt{\frac{B_1}{K}}\,\frac{1}{i\omega},\quad \sigma^2_{\text{вых}}=\sqrt{B_1/K}, \tag{6.8.36}$$

i.e., results, already analyzed by us above [see (6.8.26)-(6.8.28)].

For the second example we have $S(\omega) = B_2/\omega^4$ and factoring

$$\Psi(i\omega)\Psi(-i\omega)=\frac{\left[a_2\frac{1+i}{\sqrt{2}}+i\omega\right]\left[a_2\frac{1-i}{\sqrt{2}}+i\omega\right]}{(i\omega)^2}\times$$
$$\times\frac{\left[a_2\frac{1+i}{\sqrt{2}}-i\omega\right]\left[a_2\frac{1-i}{\sqrt{2}}-i\omega\right]}{(-i\omega)^2},$$

where again $a_2 = (B_2K)^{1/4}$. From this we finally have a result, coinciding with (6.8.35). A most interesting circumstance is revealed when by formulas (6.8.17) and (6.8.35) we find the filter of smoothing in a single-loop system. It turns out that

$$G(i\omega)=\sqrt{\frac{B_2}{K}}\,\frac{1+(4/KB_2)^{1/4}\,i\omega}{(i\omega)^2}, \tag{6.8.37}$$

i.e., the filter is an ideal double integrator with a correcting RC-circuit [illegible] time constant $\tau_{кор} = (4/KB_2)^{1/4}$ (Fig. 6.37). A filter with frequency [illegible] of type (6.8.37) is, probably, the most widespread in practice: however the conditions of its optimality revealed here are by no means universal. We recall that these conditions require presence in the parameter of the first time derivative and a chaotically varying stationary second derivative, which it is possible to approximate by white noise.

Analogously we can obtain a solution of equations (6.6.53) and (6.6.54) also for a parameter in the form of the integral of any n-th order of white noise.

In particular, for the third order we have

$$\left.\begin{aligned} KC(i\omega) &= \frac{2a_3(i\omega)^2 + 2a_3^2(i\omega) + a_3^3}{(i\omega)^3 + 2a_3(i\omega)^2 + 2a_3^2(i\omega) + a_3^3}, \\ KG(i\omega) &= \sqrt{KB_3}\cdot\frac{1 + 2(i\omega/a_3) + 2(i\omega/a_3)^2}{(i\omega)^3}, \\ \sigma^2_{min} &= 2\left(\frac{B_3}{K^5}\right)^{1/6}, \\ a_3 &= (KB_3)^{1/6}. \end{aligned}\right\} \qquad (6.8.38)$$

Calculation of an ever higher order of smoothness of parameter $\lambda(t)$, as follows from (6.8.36)-(6.8.38), leads to even greater complication of smoothing circuits. In general they contain n integrators and circuits of correction of the (n - 1)-th order, parameters of which depend on $a_n = (KB_n)^{1/2n}$. The gain factor of an open loop taking into account the gain factor of the discriminator in all cases is proportional to $\sqrt{KB_n}$, which qualitatively coincides with the case of a stationary parameter and is dictated by the same factors.

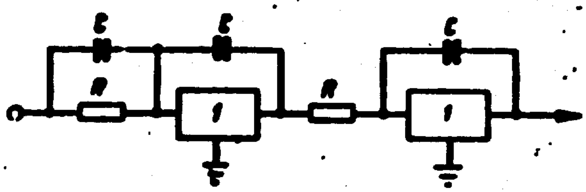

Fig. 6.37. Block diagram of smoothing circuits for a parameter with stationary increments. 1 — high-gain amplifiers.

All synthesized filters provide stability to the closed system. In particular, in a system of the second order, according to (6.8.36), there are provided conditions of stability close to the threshold between aperiodic and oscillatory regimes. This general circumstance is characteristic for all optimum filters knowingly giving a limited response to disturbances. It is convenient in that the question of stability during synthesis again arises only when there arises the necessity of departing in the meter from characteristics which are in the statistical sense optimum.

Let us give, finally, an expression for variance of stationary error of measurement of the parameter in the form of the n-th integral of white noise of intensity B_n:

$$\sigma^2_{min} = \frac{1}{\sin\frac{\pi}{2n}}\left(\frac{B_n}{K^{2n-1}}\right)^{1/2n} \underset{n\gg 1}{\rightarrow} \frac{2n}{\pi}\left(\frac{B_n}{K^{2n-1}}\right)^{1/2n}. \qquad (6.8.39)$$

It is easy to prove that in particular cases (n = 1, 2, 3) from (6.8.39) there ensue the corresponding formulas, given above. As follows from formula (6.8.39), the potential mean square error with growth of smoothing of the parameter all the less depends on the intensity of the parameter (B_n is taken to the 1/4n power), and

$\sigma^2_{BЫX}$ as total error is close in value to its fluctuating component. However, to guarantee this result we should correctly select the band of transmission of the closed system by selection of the corresponding gain factor.

6.8.3. Parameter — Linear Combination of Known Functions with Random Factors

In a broad class of practical applications, for instance during measurement of coordinates of bodies flying by the laws of ballistics, the character of change of $\lambda(t)$ is known with an accuracy of certain parameters $\alpha_1, \alpha_2, \ldots, \alpha_m$, which are not time-dependent, the values of which by virtue of initial conditions of observation one should consider random. It is possible to present this dependence in the form

$$\lambda(t) = F(t; a_1, \ldots, a_m). \tag{6.8.40}$$

We consider that

$$a_k = a_k + \mu_k, \tag{6.8.41}$$

where a_k – mean value;

μ_k – small normally distributed deviation from the mean value.

Then function (6.8.40) can be approximately presented in the form

$$\lambda(t) \approx \overline{\lambda(t)} + \sum_{k=1}^{m} \mu_k f_k(t), \tag{6.8.42}$$

where

$$\overline{\lambda(t)} = F(t; a_1, \ldots, a_m), \quad f_k(t) = \frac{\partial F(t; a_1, \ldots, a_m)}{\partial a_k}$$

are known functions.

According to (6.8.42) parameter $\lambda(t)$ then is a normal random process. Its mean value is equal to $F(t; a_1, \ldots, a_m)$, and the correlation function is determined by the obvious relationship

$$R(t_1, t_2) = \sum_{i,k=1}^{m} \mu_{ik} f_i(t_1) f_k(t_2) = \mathbf{f}^{+}(t_1) \mathbf{M} \mathbf{f}(t_2), \tag{6.8.43}$$

where $\mathbf{M} = \|M_{ik}\|\ \|\overline{\mu_i \mu_k}\|$ – matrix of mixed moments of quantities μ_i $(i, k = 1, \ldots, m)$;

$\mathbf{f}(t) = \{(ft_1), \ldots, f_m(t)\}$ – column vector with elements $f_k(t) = \partial/\partial\alpha_k\ F(t;$ $a_1, \ldots, a_m)$;

sign "+" signifies transposition.

In general, a parameter of form (6.8.42) is degenerate random process, completely determined by an m-dimensional distribution of probabilities, so that a reciprocal matrix $\mathbf{R}^{-1}$ for any $n > m$ (n – number of moments of observation) does not exist for it.

This formal difficulty, allowing one to doubt the applicability in the given case of results obtained in § 6.6, can easily be by passed if we add to $\lambda(t)$ an arbitrarily small nondegenerate random process $\lambda_1(t) = \varepsilon x(t)$. Then the correlation function of process $\lambda(t) + \varepsilon x(t)$ allows inversion; the distribution is nondegenerate for any $n > m$, and in equation (6.6.53) there will be contained correlation function $R(t_1, t_2) + \varepsilon^2 R_x(t_1, t_2)$. Directing ε to zero, we obtain an equation in which there enters correlation function (6.8.43).

We start to find smoothing circuits with the very simple case of $m = 1$:

$$\lambda(t) = \overline{\lambda(t)} + \mu f(t), \tag{6.8.44}$$

where $\overline{\lambda(t)}$ and $f(t)$ — known functions;

μ — normally distributed quantity, for which $\overline{\mu} = 0$, $\overline{\mu^2} = \sigma_0^2$, so that $R(t, \tau) = \sigma_0^2 f(t) f(\tau)$.

Equation (6.6.53) takes form

$$c(t, \tau) + K\sigma_0^2 \int_{t_0}^{t} c(t, s) f(s) f(\tau)\, ds = \sigma_0^2 f(t) f(\tau), \tag{6.8.45}$$

i.e., is an integral equation with a degenerate kernel. Seeking its solution in the form $c(t, \tau) = \varphi(t) f(\tau)$, it is simple to obtain finally

$$c(t, \tau) = \frac{\sigma_0^2 f(t) f(\tau)}{1 + K\sigma_0^2 \int_{t_0}^{t} f^2(s)\, ds}, \tag{6.8.46}$$

from which error of measurement $\sigma^2_{\text{BHX}}(t)$ is equal to

$$\sigma^2_{\text{вых}}(t) = \frac{\sigma_0^2 f^2(t)}{1 + K\sigma_0^2 \int_{t_0}^{t} f^2(s)\, ds}, \tag{6.8.47}$$

and when the value of the parameter does not grow faster than the power of t, it always seeks zero as $t \to \infty$. For instance, when

$$f(t) = \sum_{j=1}^{m} a_j t^j$$

we have

$$\sigma^2_{\text{вых}}(t) \xrightarrow[t \to \infty]{} \frac{2m+1}{Kt}. \tag{6.8.48}$$

The tendency of error of measurement to zero is a specific peculiarity of the considered case and is caused by the fact that the law of change of $\lambda(t)$ is known with an accuracy of a constant factor μ. The problem of filtration actually

consists of measurement (estimation) of this constant factor. From the theory of estimation we know (§ 6.6) that measurement of a constant parameter can be performed with error tending to zero with increase of the number of measurements, where during measurement of the coefficient of an m-th order parabola mixed with white noise with spectral density 1/K the variance of error in the absence of any a priori data about this coefficient is exactly (2m + 1)/(Kt).

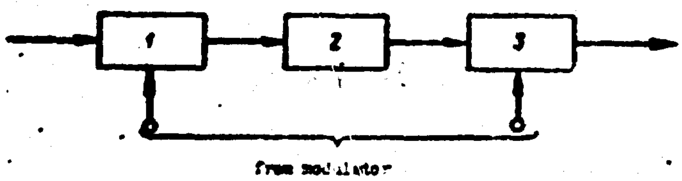

Fig. 6.38. Smoothing circuits for a quasi-regular parameter (single-loop variant of a meter): 1, 3 — variable-gain amplifiers; 2 — integrator.

Seeking a solution of equation (6.6.54) in the form $g(t, \tau) = f(t)\xi(\tau)$ for $c(t, \tau)$, determined according to (6.8.46), we have

$$g(t,\tau)=\frac{\sigma_0^2 f(t) f(\tau)}{1+K\sigma_0^2\int\limits_{t_0}^{\tau} f^2(s)\,ds}. \qquad (6.8.49)$$

According to (6.8.49) the smoothing filter of a single-loop system possesses variable parameters for any time of observation. Smoothing (Fig. 6.38) consists of multiplication of the output signal of the discriminator by function $\sigma_0^2 f(t)/\left[1+K\sigma_0^2\int_{t_0}^{t} f^2(s)\,ds\right]$, subsequent integration and multiplication by function $f(t)$. The output signal of the integrator is the current estimate of parameter μ and is used as a factor for the law of change of parameter $f(t)$, generated in the circuit. After introduction of $\overline{\lambda(t)}$ there will form estimate $\hat{\lambda}(t)$.

Characteristic for a filter with characteristic (6.8.49) is gradual ... of discriminator output or, in other words, "freezing" of the estimated coefficient, occurring faster, the less the equivalent spectral density 1/K. This is explained by the fact that the law of change of $\lambda(t)$ is known a priori so that the finite interval of the realization permits us with good accuracy to predict behavior of parameter $\lambda(t)$ in all subsequent moments of time. Upon the expiration of this interval error of measurement already becomes small, and only specially noticeable peaks of output voltage of the discriminator correct estimate μ.

With equal success smoothing can be conducted by a double-loop system of the type of Fig. 6.13, where pulse response of the filter c(t, τ) is given by relationship (6.8.46). As Fig. 6.39 illustrates, in this case output voltage of the adder of the loop of internal coupling which is approximately an additive mixture of the measured quantity and white noise, is multiplied by function f(t), and then is integrated.

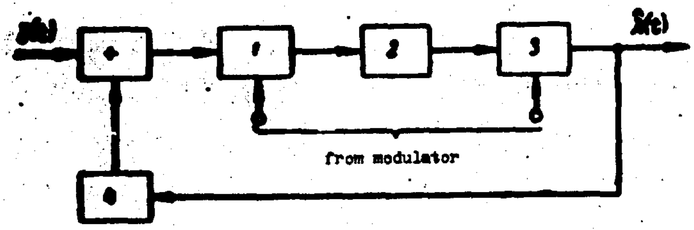

Fig. 6.39. Smoothing circuits for a quasi-regular parameter (double-loop variant of a meter): 1, 3 — variable-gain amplifiers; 2 — integrator; 4 — constant amplifier.

Thereby there will be formed correlation between the input (for the system of smoothing) realization and the expected form of "signal." Further processing reduces to multiplication of the formed quantity by function $f(t)\Big/\left[1+K\sigma_0^2\int_{t_0}^{t} f^2(s)\,ds\right]$, which simultaneously normalizes the output voltage of the integrator, compensating for the effect of constant accumulation, and forms the measured quantity. Both methods of smoothing are absolutely equivalent.

Now let us turn to the general case of process (6.8.42) for an arbitrary m. Substituting (6.8.43) in (6.6.53), we again have an equation with a degenerate kernel, solution of which is best sought in the form

$$c(t,\tau)=\boldsymbol{\varphi}^{+}(t)\,\mathbf{f}(\tau). \qquad (6.8.50)$$

As a result we have

$$c(t,\tau)=\mathbf{f}^{+}(t)\left[\mathbf{M}^{-1}+K\mathbf{U}(t)\right]^{-1}\mathbf{f}(\tau), \qquad (6.8.51)$$

where the square (m × m)-order matrix U(t) is determined by expression

$$\mathbf{U}(t)=\|U_{ik}(t)\|=\left\|\int_{t_0}^{t} f_i(s)\,f_k(s)\,ds\right\|. \qquad (6.8.52)$$

Expression (6.8.51) is a natural generalization of relationship (6.8.47). Analogously we have generalizations for pulse response of the filter and variance of

error:

$$g(t,\tau)=\mathbf{f}^{+}(t)[\mathbf{M}^{-1}+K\mathbf{U}(\tau)]^{-1}\mathbf{f}(\tau), \qquad (6.8.53)$$

$$\sigma^{2}_{\text{виз}}(t)=\mathbf{f}^{+}(t)[\mathbf{M}^{-1}+K\mathbf{U}(t)]^{-1}\mathbf{f}(t). \qquad (6.8.54)$$

According to (6.8.53) smoothing circuits in the single-loop variant consist of m parallel channels. In the k-th channel there is produced multiplication of $z(t)$ by functions which depend on all $f_i(t)$ ($i = 1, 2, \ldots, m$) and on K, M_{ik}; then results of multiplication are integrated, as a result of which there are formed estimates of m unknown coefficients. Then, from the estimates there is formed the value of the parameter according to formula (6.8.42).

The shown method of smoothing has the following peculiarities:

1. In it both the a priori accuracy of measurement of the unknown factor and also accuracy properties of an optimum discriminator, processing newly arriving data, are accounted for.

2. Any curvature of the functions which depict change of parameter $\lambda(t)$ in time is accounted for. For the purpose of minimizing errors during smoothing of especially nonlinear functions we use known laws of change of parameter $f_k(t)$ without any increase of the number of series-coupled integrating networks.

3. With increase of the time of observation the operators of smoothing and also the potential error of measurement all the less depend on a priori conditions, seeking magnitudes

$$g(t,\tau)\to\frac{1}{K}\mathbf{f}^{+}(t)\mathbf{U}^{-1}(\tau)\mathbf{f}(\tau), \qquad (6.8.55)$$

$$c(t,\tau)\to\frac{1}{K}\mathbf{f}^{+}(t)\mathbf{U}^{-1}(t)\mathbf{f}(\tau), \qquad (6.8.56)$$

$$\sigma^{2}_{\text{виз}}(t)\to\frac{1}{K}\mathbf{f}^{+}(t)\mathbf{U}^{-1}(t)\mathbf{f}(t). \qquad (6.8.57)$$

Formula (6.8.57) is met in literature devoted to measurements of coordinates of ballistic targets [14]. Here, the measured quantity is usually considered the constant parameters of the trajectory, but not the actual coordinates and they apply the theory of the maximum likelihood estimate, considering a priori information about the parameters generally absent. Let us explain this in greater detail.

We assume that radio processing in the discriminator is considered assigned, and it is assumed that the object of optimization is mixture $x(t)$ of the law of motion $\lambda(t, \mu)$, depending on constant parameters μ, and noise disturbances $n(t)$ with Guassian distribution. As results of §6.6 show, assuming linear work of the

discriminator and ignoring parametric fluctuations, by addition of the discriminator output (divided by its gain factor) to the measured value of the parameter it is indeed possible to obtain voltage having the properties of an additive mixture of equivalent "signal" and "noise." In conditions of § 6.6 noise is white, so that its distribution by virtue of inertia of subsequent circuits can be considered Gaussian. Then the likelihood functional of μ is recorded in the form

$$P(x(t)|\mu)=C\exp\left\{-\frac{K}{L^2}\int_{t_0}^{t}[x(\tau)-\lambda(\tau,\mu)]^2d\tau\right\}.$$

If the a priori distribution of μ has large variance, and for $\lambda(t;\mu)$ expansion (6.8.43) is approximately valid, then for determination of μ we have a system of linear equations of maximum likelihood

$$\sum_{j=1}^{m}\mu_j\int_{t_0}^{t}f_j(\tau)f_i(\tau)d\tau=\int_{t_0}^{t}x(\tau)f_i(\tau)d\tau. \qquad (6.8.58)$$

Solution of the system in matrix representation has the form

$$\hat{\mathbf{M}}=\mathbf{U}^{-1}(t)\int_{t_0}^{t}\mathbf{f}(\tau)x(\tau)d\tau, \qquad (6.8.59)$$

by which it is possible, according to (6.8.43), to express the value of $\lambda(t,\mu)$:

$$\hat{\lambda}(t,\mu)=\mathbf{f}^{+}(t)\mu=\int_{t_0}^{t}\left[\frac{1}{K}\mathbf{f}^{+}(t)\mathbf{U}^{-1}(t)\mathbf{f}(\tau)\right]Kx(\tau)d\tau. \qquad (6.8.60)$$

Comparing (6.8.56) with (6.8.60), we note that for a large time of observation the operator of smoothing studied earlier coincides with the operator obtained by the maximum likelihood method. If we have the a priori distribution of μ, the estimates should be formed by the method of maximum a posteriori probability. Analogously to how we obtained a solution for (6.8.60), we can obtain a solution coinciding with (6.8.51) for any time of observation. This circumstance established one more interconnection between the theory of optimum filtration and the theory of the maximum likelihood estimate.

The given results on measurement of coordinates of objects with a ballistic law of motion do not give, however, an exhaustive solution of the problem. The fact is that when accounting for different disturbing factors the differential equations describing variation of the parameters in time become nonlinear, and an exact law of motion in the form of elementary functions cannot be obtained from them. In these conditions finding of functions $\overline{\lambda(t)}$ and $f_i(t)$ is hampered, and simultaneously with the problem of smoothing one should somehow solve the problem of current

integration of equations of motion. Simple rules of carrying out the operation, especially in complicated cases, are absent here, and the important problem of synthesis of a convenient algorithm of smoothing, ensuring minimum errors and easily realized technically, remains open. Furthermore, complication of such algorithms occurs due to the need for accounting for additional components of errors (see § 6.2). Regardless, however, of the complexity in construction of the algorithm of smoothing, sometimes hindering direct use of the above mentioned results, the latter nevertheless can also be used in complicated cases if we are talking about calculation of errors of measurement. In this calculation less accuracy of calculation than during synthesis of smoothing circuits is permissible, so that the equation of motion can be integrated approximately, after which we conduct expansion of type (6.8.43) and find errors of measurement by formula (6.8.57).

The considered case of a quasi-regular parameter easily permits us to study, too, the case of self-tuning of smoothing circuits for correction of unequal accuracy of measurements, or which is the same, the influence of parametric fluctuations. Let us analyze the most simple case of one unknown parameter. We note that equation (6.8.45) in variable K(t) is replaced according to (6.6.31) by

$$c(t, \tau)+\sigma_0^2\int_{t_0}^{t} c(t, s)f(s)K(s)\,ds\,f(\tau)=\sigma_0^2 f(t)f(\tau). \qquad (6.8.61)$$

As an equation with degenerate kernel, it is easily solved:

$$c(t, \tau)=\frac{\sigma_0^2 f(t)f(\tau)}{1+\sigma_0^2\int_{t_0}^{t} f^2(s)K(s)\,ds}, \qquad (6.8.62)$$

from which, using equation (6.6.32), we obtain

$$g(t, \tau)=\frac{\sigma_0^2 f(t)f(\tau)}{1+\sigma_0^2\int_{t_0}^{\tau} f^2(s)K(s)\,ds}. \qquad (6.8.63)$$

Error of measurement should be calculated according to (6.6.43) by averaging (6.8.62) over random variables, determined by input signal y(t) the random factor. We note that randomness of $c(t, \tau)$ is connected only with function $\xi(t)$, which is a random of modulation factor K(t):

$$K(t)=K(1+\xi(t)), \quad K=\overline{K(t)}.$$

According to § 6.6 $\xi(t)$ has the properties of white noise:

$$\overline{\xi(t_1)\xi(t_2)}=\varkappa\delta(t_1-t_2). \qquad (6.8.64)$$

Considering fluctuations of the gain factor of the discriminator small, it is possible to expand (6.8.62) in powers of $\xi(t)$ and after averaging to obtain

$$\sigma^2_{\text{smx}}(t) \approx \frac{\sigma_0^2 f^2(t)}{1+\sigma_0^2 K \int_{t_0}^{t} f^2(s)\,ds}\left\{1+\frac{\sigma_0^4 \varkappa K^2 \int_{t_0}^{t} f^4(s)\,ds}{\left[1+\sigma_0^2 K \int_{t_0}^{t} f^2(s)\,ds\right]^2}\right\}, \tag{6.8.65}$$

which for a large time of observation gives

$$\sigma^2_{\text{smx}}(t) \approx \frac{f^2(t)}{K \int_{t_0}^{t} f^2(s)\,ds}\left[1+\frac{\varkappa \int_{t_0}^{t} f^4(s)\,ds}{\left(\int_{t_0}^{t} f^2(s)\,ds\right)^2}\right]. \tag{6.8.66}$$

Consequently, parametric fluctuations increase error of measurement. However, this increase is much more considerable if we do not allow for these fluctuations during synthesis of the circuit, when they, in fact exist. (Let us remember that in the past these fluctuations were not accounted for either during synthesis or during calculation of accuracy.) Then calculation of accuracy of measurement conducted by the method of § 6.2 gives

$$\sigma^2_{\text{smx}}(t) \approx \frac{\sigma_0^2 f^2(t)}{1+\sigma_0^2 K \int_{t_0}^{t} f^2(s)\,ds}\left[1+\frac{\sigma_0^4 \varkappa K \int_{t_0}^{t} f^4(s)\,ds}{1+\sigma_0^2 K \int_{t_0}^{t} f^2(s)\,ds}\right] \tag{6.8.67}$$

or for a large time of observation

$$\sigma^2_{\text{smx}}(t) \approx \frac{f^2(t)}{K \int_{t_0}^{t} f^2(s)\,ds}\left[1+\frac{\sigma_0^2 \varkappa K \int_{t_0}^{t} f^4(s)\,ds}{\int_{t_0}^{t} f^2(s)\,ds}\right]. \tag{6.8.68}$$

Inasmuch as for a constant or little-varying dimensionless function f(t) integrals

$$\frac{1}{t-t_0}\int_{t_0}^{t} f^2(s)\,ds, \quad \frac{1}{t-t_0}\int_{t_0}^{t} f^4(s)\,ds$$

are quantities of the same order, comparison of the second component in brackets in relationships (6.8.66) and (6.8.68) shows that in the case of allowance for parametric fluctuations during synthesis this addition decreases approximately according to the law $\varkappa/(t - t_0)$, and in the case of disregard it passes to a constant limit, approximately equal to $\sigma_0^2 \varkappa K$. Remembering that $\varkappa$ and 1/K are equal to the spectral densities of the nondimensional modulation coefficient of the gain factor and equivalent input noise of an optimum discriminator, respectively, and producing

multiplication and division of $\sigma_0^2 \varkappa K$ by a certain transmission band Δf, we prove that

$$\sigma_0^2 \varkappa K = \frac{\sigma_{\mu}^2 \sigma_{\text{мод}}^2}{\sigma_{\text{экв}}^2}, \tag{6.8.69}$$

i.e., increase of error of measurement is determined by the ratio of a priori variance μ to the variance of equivalent noise, multiplied by the variance of the modulation coefficient of the gain factor of the discriminator. In the case of large signal-to-noise ratios the magnitude of (6.8.69) will be rather noticeable. Thereby, from this simple example we proved that disregard of parametric fluctuations during synthesis, i.e., rejection of an accuracy unit and self-tuning of smoothing circuits, leads to growth of errors of measurement, in general, very undesirable. This justifies the continuing interest in parametric fluctuations in the present and subsequent chapters.

6.8.4. Mixed Cases

In the most general case the measured quantity contains components of different origin, possessing therefore various correlation properties. Let us study measurement of a parameter in the form of the sum of a nondegenerate (purely random) process, for concreteness, of a stationary (see Paragraph 6.8.1) and a quasi-regular random process in the form of a linear combination of known functions with random factors (see Paragraph 6.8.3). In principle the results obtained can be generalized to more complicated cases.

Writing the correlation function of the stationary and quasi-regular processes in the form $r(t_1 - t_2)$ and $R(t_1, t_2) = \mathbf{f}^+(t_1)\mathbf{M}\mathbf{f}(t_2)$, respectively, we have for pulse response of the smoothing filter of a double-loop variant of a meter the integral equation

$$\begin{aligned} c(t, \tau) + K\int_{t_0}^{t} c(t, s)\left[\mathbf{f}^+(s)\mathbf{M}\mathbf{f}(\tau) + r(s-\tau)\right] ds = \\ = \mathbf{f}^+(t)\mathbf{M}\mathbf{f}(\tau) + r(t-\tau). \end{aligned} \tag{6.8.70}$$

Let us assume that the solution of (6.8.70) can be presented in the form

$$c(t, \tau) = c_0(t, \tau) + c_1(t, \tau), \tag{6.8.71}$$

where $c_0(t, \tau)$ repeats the solution of the corresponding equation for a stationary process (see Paragraph 6.8.1):

$$c_0(t, \tau) + K\int_{t_0}^{t} c_0(t, s)\, r(s-\tau)\, ds = r(t-\tau). \tag{6.8.72}$$

Methods of solution of this equation were studied in detail above. Then for $c_1(t, \tau)$ we have equation

$$c_1(t, \tau)+K\int_{t_0}^{t} c_1(t, s)[\mathbf{f}^+(s)M\mathbf{f}(\tau)+r(s-\tau)]\,ds=$$
$$=\mathbf{f}^+(t)M\mathbf{f}(\tau)-K\int_{t_0}^{t} c_0(t, s)\mathbf{f}(s)\,ds\,M\mathbf{f}(\tau). \tag{6.8.73}$$

Let us assume that variance of the purely random component of the parameter is less than the variance of the quasi-regular component so that in equation (6.8.73) it is possible to seek out solution by the method of successive approximations:

$$c_1(t, \tau)=c_{10}(t, \tau)+c_{11}(t, \tau)+\dots \tag{6.8.74}$$

Consequently, for the first approximation we have equation

$$c_{10}(t, \tau)+K\int_{t_0}^{t} c_{10}(t, s)\mathbf{f}^+(s)M\mathbf{f}(\tau)\,ds=[\mathbf{f}(t)-\boldsymbol{\varphi}(t)]^+M\mathbf{f}(\tau), \tag{6.8.75}$$

where

$$\boldsymbol{\varphi}(t)=K\int_{t_0}^{t} c_0(t, s)\mathbf{f}(s)\,ds \tag{6.8.76}$$

is a column vector which is the result of processing of column vector f(s) by a linear filter-operator, designed for a purely random parameter.

For the following approximations we have equations of the same kind:

$$c_{1k}(t, \tau)+K\int_{t_0}^{t} c_{1k}(t, s)\mathbf{f}^+(s)M\mathbf{f}(\tau)\,ds=-K\int_{t_0}^{t} c_{1(k-1)}(t, s)r(s-\tau)\,ds \quad (k=1, 2, \dots). \tag{6.8.77}$$

Seeking the solution of equation (6.8.75), analogously to Paragraph 6.8.3, in the form $\mathbf{a}^+(t)\mathbf{f}(\tau)$, we have

$$c_{10}(t, \tau)=[\mathbf{f}(t)-\boldsymbol{\varphi}(t)]^+[M^{-1}+K\mathbf{U}(t)]^{-1}\mathbf{f}(\tau), \tag{6.8.78}$$

where $\mathbf{U}(t)$, $\boldsymbol{\varphi}(t)$ are expressed by relationships (6.8.52) and (6.8.76), respectively.

For further approximation it is necessary to introduce into consideration quantity $\psi(t, \tau)$, equal to

$$\psi(t, \tau)=K\int_{t_0}^{t} c_{10}(t, s)r(s-\tau)\,ds=[\mathbf{f}(t)-\boldsymbol{\varphi}(t)]^+[M^{-1}+K\mathbf{U}(t)]^{-1}K\int_{t_0}^{t}\mathbf{f}(s)r(s-\tau)\,ds.$$

Then equation (6.8.77) for k = 1 will take form

$$c_{11}(t, \tau)+K\int_{t_0}^{t} c_{11}(t, s)\mathbf{f}^+(s)M\mathbf{f}(\tau)\,ds=-\psi(t, \tau). \tag{6.8.79}$$

Seeking its solution in the form

$$c_{11}(t,\ \tau)=\mathbf{g}^{+}(t)\,\mathbf{f}(\tau)-\psi(t,\ \tau),$$

as a result we have

$$\begin{aligned} c_{11}(t,\ \tau)=K\int_{t_0}^{t}\psi(t,s)\,\mathbf{f}^{+}(s)\,ds\,[\mathbf{M}^{-1}+K\mathbf{U}(t)]^{-1}\mathbf{f}(\tau)-\psi(t,\tau)= \\ =[\mathbf{f}(t)-\boldsymbol{\varphi}(t)]^{+}[\mathbf{M}^{-1}+K\mathbf{U}(t)]^{-1}\times \\ \times\Big\{K\int_{t_0}^{t}\int_{t_0}^{t}\mathbf{f}(s_1)\,r(s_1-s_2)\,\mathbf{f}^{+}(s_2)\,ds_1 ds_2\times \\ \times[\mathbf{M}^{-1}+K\mathbf{U}(t)]^{-1}\mathbf{f}(\tau)-\int_{t_0}^{t}\mathbf{f}(s)\,r(s-\tau)\,ds\Big\}. \end{aligned} \qquad (6.8.80)$$

Let us explain the physical meaning of the obtained solution. This is easy to do in the next approximation:

$$c(t,\ \tau)\approx c_0(t,\ \tau)+c_{10}(t,\ \tau). \qquad (6.8.81)$$

According to relationships (6.8.72) and (6.8.75) the smoothing filter consists in this case of two basic channels (Fig. 6.40). The first exactly repeats the smoothing filter for a stationary parameter. The second repeats the smoothing filter for a quasi-regular parameter in that part which concerns formation of estimates of unknown random factors. However, during forming of a linear combination of functions with measured values of coefficients known functions $f_i(t)$ decrease by quantities equal to the output responses of the first channel upon feeding $f_i(t)$ to its input. This is explained by the fact that the linear combination of known functions to some extent passes through the channel for tracking the purely random process, causing necessity for correction. Relationships between $\mathbf{f}(t)$ and $\boldsymbol{\varphi}(t)$ may differ. If the channel for tracking the stationary parameter is wide-band, it to a considerable extent also tracks the linear combination of functions, so that estimated values of factors will be used only for compensation of dynamic errors of the first channel. In the case of narrow bandwidth of the first channel its influence on the tracking channel for the quasi-regular parameter is slight, so that each of the smoothing channels is designed for its own component.

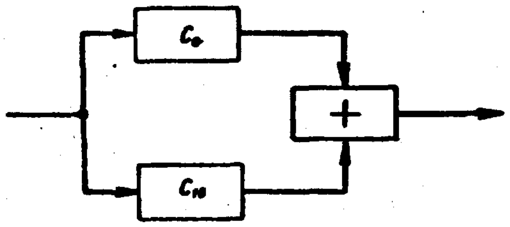

Fig. 6.40. Smoothing circuits for parameter in the form of a mixture of a purely random and a quasi-regular parameter: C_0 — channel for tracking the purely random parameter: C_{10} — channel for tracking the quasi-regular parameter.

Successive approximations to pulse response $c_{1k}(t, \tau)$ $(k = 1, 2, \ldots)$ give a more precise definition, ever better compensating the influence of the two tracking channels on one another. Usually these additions play a small role. This is conveniently estimated in the case of a rapidly varying stationary component, when this approximation is permissible:

$$r(t-s)=\frac{\sigma_\lambda^2}{\Delta f_\lambda}\delta(t-s).$$

Formulas (6.8.75) and (6.8.81) in these conditions give

$$c_{10}(t, \tau)=[\mathbf{f}(t)-\boldsymbol{\varphi}(t)]^+[\mathbf{M}^{-1}+K\mathbf{U}(t)]^{-1}\mathbf{f}(\tau), \tag{6.8.82}$$

$$c_{11}(t, \tau)=-[\mathbf{f}(t)-\boldsymbol{\varphi}(t)]^+[\mathbf{M}^{-1}+K\mathbf{U}(t)]^{-1}\times \times\left\{\frac{K\sigma_\lambda^2}{\Delta f_\lambda}[\mathbf{I}+\mathbf{M}K\mathbf{U}]^{-1}\right\}\mathbf{f}(\tau). \tag{6.8.83}$$

As it is easy to prove by comparing (6.8.82) and (6.8.83) the order of $c_{11}(t, \tau)$ with respect to $c_{10}(t, \tau)$ is characterized by matrix

$$\frac{K\sigma_\lambda^2}{\Delta f_\lambda}[\mathbf{I}+K\mathbf{M}\mathbf{U}(t)]^{-1}\rightarrow\frac{\sigma_\lambda^2}{\Delta f_\lambda}\mathbf{M}^{-1}\mathbf{U}^{-1}(t).$$

For a constant or slowing varying parameter analogously to Paragraph 6.8.3 it is possible to consider matrix $\frac{1}{t-t_0}\mathbf{U}(t)=\left\|\frac{1}{t-t_0}\int\limits_{t_0}^{t}f_i(s)f_j(s)\,ds\right\|$ little time-dependent. Then for $\Delta f_\lambda(t-t_0)\rightarrow\infty$ we have $\sigma_\lambda^2\mathbf{M}^{-1}\left[\frac{1}{t-t_0}\mathbf{U}(t)\right]^{-1}\times(\Delta f_\lambda(t-t_0))^{-1}\rightarrow 0,$ i.e., the influence of c_{11} is practically disregarded.

In particular, for one unknown factor with variance σ_0^2

$$\frac{c_{11}(t, \tau)}{c_{10}(t, \tau)}=-\frac{K\sigma_\lambda^2/\Delta f_\lambda}{1+K\sigma_0^2\int\limits_{t_0}^{t}f^2(s)\,ds}\approx$$

$$\approx\frac{\sigma_\lambda^2/\sigma_0^2}{\Delta f_\lambda(t-t_0)\left[\frac{1}{t-t_0}\int\limits_{t_0}^{t}f^2(s)\,ds\right]}\rightarrow 0 \quad \text{as} \quad t\rightarrow\infty.$$

Inasmuch as here $c_0(t, \tau) = c_0(t - \tau)$ is also a rapidly varying function

$$\int\limits_{t_0}^{t}c_0(t, s)f(s)\,ds\approx f(t)\int\limits_{t_0}^{t}c_0(\tau)\,d\tau,$$

and then the channel for tracking the regular part of the parameter has response

$$c_{20}(t,\tau)=\frac{\sigma_0^2 f(t) f(\tau)}{1+K\sigma_0^2\int\limits_{t_0}^{t} f^2(s)\,ds}\left[1-K\int\limits_0^\infty c_0(\tau)\,d\tau\right],$$

differing from the case of absence of a stationary part only by factor $1-K\int\limits_0^\infty c_0(\tau)\,d\tau$. Its meaning becomes completely clear if we consider that the integral is the value of the transmission factor of the filter for a stationary parameter at zero frequency.

Error of measurement according to (6.8.82) and (6.8.83) is expressed in the general formula

$$\sigma^2_{\text{вых}}(t)=c_0(t,t)+\left[\mathbf{f}(t)-K\int\limits_{t_0}^{t}c_0(t,s)\,\mathbf{f}(s)\,ds\right]^{+}\times$$
$$\times[\mathbf{M}^{-1}+K\mathbf{U}(t)]^{-1}\left\{1+K^2\int\limits_{t_0}^{t}\int\limits_{t_0}^{t}\mathbf{f}(s_1)r(s_1-s_2)\,\mathbf{f}^{+}(s_2)\,ds_1ds_2\times\right.$$
$$\left.\times[\mathbf{M}^{-1}+K\mathbf{U}(t)]^{-1}\right\}\left[\mathbf{f}(t)-K\int\limits_{t_0}^{t}c_0(t,s)\,\mathbf{f}(s)\,ds\right],$$

(6.8.84)

valid in that approximation when for pulse response we use relationship (6.8.81). In particular, for rapidly varying $r(t_1 - t_2)$

$$\sigma^2_{\text{вых}}(t)=c_0(t,t)+[\mathbf{f}(t)-\boldsymbol{\varphi}(t)]^{+}[\mathbf{M}^{-1}+K\mathbf{U}(t)]^{-1}\times$$
$$\times\left\{1+\frac{K\sigma_\lambda^2}{\Delta f_\lambda}K\mathbf{U}(t)[\mathbf{M}^{-1}+K\mathbf{U}(t)]^{-1}\right\}[\mathbf{f}(t)-\boldsymbol{\varphi}(t)]\to$$
$$\to c_0(t,t)+(1-\alpha(t))^2\left(1+\frac{K\sigma_\lambda^2}{\Delta f_\lambda}\right)\sigma_0^2(t),$$

where $\alpha(t)=\int\limits_{t_0}^{t}c_0(t,s)\,ds$; $\sigma_0^2(t)$ – variance of measurement of the quasi-regular part of the parameter, if it was measured separately.

Inasmuch as $(1-\alpha^2)\left(1+\frac{K\sigma_\pi^2}{\Delta f_\pi}\right)\to 1$, then in this case variance of error is equal to the sum of variances during measurement separately of the two components of the parameter.

6.8.5. Smoothing Circuits for Linear Functionals of Parameters

In § 6.6 it was shown that during measurement of linear functionals of a parameter primary processing of the input mixture of signal and noise, carried out by

discriminator, remains the same as during measurement of the actual parameter. Only there are added two special filters to the smoothing circuits. One of them will form the assigned functional from the a priori mean value of the parameters, and the second is used for processing data from the discriminator output. The latter filter is the only interesting element. Its concrete form depends on what variant of meter of the basic parameter we are calculating single or double-loop. Pulse responses in these two cases are determined by relationships (6.6.106) and (6.6.108), respectively. Basically below we consider the case of a double-loop system. Naturally we consider a correlation function of the same type as in Paragraphs 6.8.1 - 6.8.3.

a) If the parameter is a stationary process, and the linear functional also possesses stationary properties, i.e., can be presented in the form

$$\mu(t)=\int_{-\infty}^{+\infty}F(t-\tau)\lambda(\tau)\,d\tau,$$

where t -- the last moment of observation, equation (6.6.106) for a sufficiently large time of observation will take form ($b(t, \tau) = b(t - \tau)$):

$$b(t)+K\int_{0}^{\infty}b(t-\tau)R(\tau)\,d\tau=\int_{-\infty}^{+\infty}F(t-\tau)R(\tau)\,d\tau \quad (t>0). \tag{6.8.85}$$

As also in the case of measurement of the actual parameter, equation (6.8.85) is an integral equation of the Wiener-Hopf type, solution of which is found by the factoring method. Considering again

$$1+KS(\omega)=\Psi(i\omega)\Psi(-i\omega),$$

we have the Fourier transform of function b(t) in the form

$$B(i\omega)=\frac{1}{\Psi(i\omega)}\left[\frac{\breve{F}(i\omega)S(\omega)}{\Psi(-i\omega)}\right]_{+}= =\frac{1}{2\pi\Psi(i\omega)}\int_{0}^{\infty}e^{-i\omega t}dt\int_{-\infty}^{+\infty}\frac{\breve{F}(iu)S(u)}{\Psi(-iu)}e^{iut}du, \tag{6.8.86}$$

where

$$\breve{F}(i\omega)=\int_{-\infty}^{+\infty}F(t)e^{-i\omega t}dt.$$

Formula (6.8.86) for finding the limiting operator is completely preserved if the parameter is a process with stationary increments. Here, during factoring of $S(\omega)$ one should remember the generalizations of Paragraph 6.8.2. Accuracy of measurement of the parameter in both cases is expressed according to (6.6.109) by a formula analogous to (6.8.18):

$$\sigma^2_{\text{вых}} = \lim_{i\omega\to\infty} [i\omega B(i\omega) \tilde{F}(i\omega)]. \qquad (6.8.87)$$

As an example we shall consider measurement of the first derivative of the second integral of white noise, where

$$S(\omega) = \frac{B_2}{\omega^4}, \quad \tilde{F}(i\omega) = i\omega.$$

From formula (6.8.86) we have

$$B(i\omega) = \sqrt{\frac{B_2}{K}} \frac{i\omega}{(i\omega)^2 + \sqrt{2}(i\omega)a_2 + a_2^2}, \quad a_2^2 = KB_2.$$

Very graphic in this case is the form of the filter for forming of the functional in a single-loop system:

$$H(i\omega) = B(i\omega)[1 + KG(i\omega)] = \frac{\sqrt{B_2/K}}{i\omega}.$$

The smoothing filter here has the same form as in the case of measurement of a Wiener process by a single-loop system.

b) If the parameter is a linear combination of functions with random factors, the problem is simply solved without any assumptions about the form of the functional (it can be of any form).

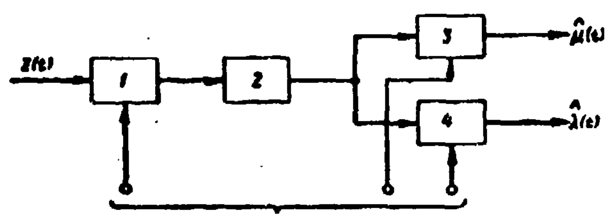

Fig. 6.41. Smoothing circuits for a linear functional of a quasi-regular parameter (in the single-loop variant): 1, 3, 4 — variable-gain amplifiers; 2 — integrator.

Equation (6.6.105) takes in this case the form

$$b(t_1, \tau) + K\int_{t_0}^{t} b(t_1, s)\mathfrak{f}^+(s)\,ds\mathbf{M}\mathfrak{f}(\tau) = \int_{T_1}^{T_2} F(t_1, s)\mathfrak{f}^+(s)\,ds\mathbf{M}\mathfrak{f}(\tau).$$

Seeking its solution in the form $b(t_1, \tau) = g(t_1, t)\mathfrak{f}(\tau)$, we finally have for $b(t_1, \tau)$ and $h(t_1, \tau)$

$$b(t_1, \tau) = \int_{T_1}^{T_2} F(t_1, s)\mathfrak{f}^+(s)\,ds[\mathbf{M}^{-1} + K\mathbf{U}(t)]^{-1}\mathfrak{f}(\tau), \qquad (6.8.88)$$

$$h(t_1, \tau) = \int_{T_1}^{T_2} F(t_1, s)\mathfrak{f}^+(s)\,ds[\mathbf{M}^{-1} + K\mathbf{U}(\tau)]^{-1}\mathfrak{f}(\tau), \qquad (6.8.89)$$

where U(t) is expressed by formula (6.8.52).

According to (6.8.88) and (6.8.89) the circuit for formation of estimates of proportionality factors remains the same as for measurement of the actual parameter, but then the formed set of estimates is multiplied, not by original functions $f_i(t)$, but by functions $f_i(t)$, processed by linear operator F(t, s), characterizing the linear functional. These operations are illustrated for one coefficient μ in the single-loop variant in Fig. 6.41.

Error of measurement accordin to formula (6.6.109) is equal to

$$\sigma^2_{\text{вых}}(t_1, t) = \int_{T_1}^{T_2} F(t_1, s)\,\mathbf{f}^+(s)\,ds\,[\mathbf{M}^{-1} + K\mathbf{U}(t)]^{-1}\mathbf{f}(t). \qquad (6.8.90)$$

In the particular case of a linearly varying parameter

$$\lambda(t) = Vt, \quad \overline{V^2} = \sigma_1^2$$

and an advance (delay) operator

$$\mu(t_1) = Vt_1 \quad (t_1 \neq t),$$

we have

$$\sigma^2_{\text{вых}}(t_1, t) = \frac{\sigma_1^2 t_1^2}{1 + K\sigma_1^2 t^3/3} \to \frac{3t_1^2}{Kt^3} = \frac{3}{Kt}\left(\frac{t_1}{t}\right)^2 \quad \text{as} \quad t \to \infty.$$

Thus, error by a factor of approximately t_1/t exceeds error of measurement of the current value of the parameter.

§ 6.9. Synthesis of Meters with a Markovian Parameter and with Limited Knowledge of the Statistics of the Parameters

In §§ 6.6-6.8 we considered questions of synthesis of radar meters for a Gaussian distribution of the measured quantity $\lambda(t)$. This case is very interesting, but is not all-embracing. Therefore results obtained with other assumptions about the statistics of $\lambda(t)$, and also with limited knowledge of the statistics attract our attention. In the class of completely assigned a priori distributions besides the Gaussian case in the literature there is considered the case of synthesis for Markovian parameters, which we touched on in § 6.5. For the case of limited a priori knowledge investigations are extremely scanty (see, for instance, [62]). Below we give certain results on the synthesis of meters with a Markovian parameter and with limited knowledge of statistics of the parameter, generalizing and supplementing the known works. Results of solution with limited statistics are far from complete and can only serve as a basis for further investigations.

6.9.1. Optimum Meter of a Markovian Parameter

Passing to synthesis of a meter of a Markovian parameter, it would have been possible to compose its a priori distribution for all moments of observation and, multiplying by the Gaussian approximation of likelihood function (6.6.3), find the conditional mathematical expectation. More convenient, however, will be methods, based on the results of Paragraph 6.5.6.

We shall start with the case of discrete observation (or of an incoherent pulse signal), when it is permissible to use relationship (6.5.24). Assuming low a posteriori inaccuracy we approximate the a posteriori probability density of the value of $\lambda(t)$ at time t_n of the Gaussian curve

$$\tilde{P}_n(\lambda_n)=\frac{1}{\sqrt{2\pi}\sigma_{\text{вых}\,n}}\exp\left\{-\frac{(\lambda_n-\hat{\lambda}_n)^2}{2\sigma^2_{\text{вых}\,n}}\right\}, \qquad (6.9.1)$$

where $\sigma^2_{\text{BHX}\,n}$ — a posteriori variance at the n-th moment;

$\hat{\lambda}_n$ — point of maximum a posteriori probability, according to results of Paragraphs 6.5.3 and 6.5.6 coinciding with the current optimum estimate.

We also take the Gaussian approximation for likelihood function

$$P(y_k\,|\,\lambda_k)=P(y_k\,|\,\check{\lambda}_k)\exp\left\{-\frac{(\lambda_k-\check{\lambda}_k)^2}{2\sigma_k^2}\right\}, \qquad (6.9.2)$$

where, analogously to (6.6.4), we expand the maximum likelihood of $\check{\lambda}_k$ at the current

moment, and

$$\sigma_k^2 = \left[-\frac{\partial^2 L(\check{\lambda}_k)}{\partial \lambda^2}\right]^{-1} \tag{6.9.3}$$

— variance of error of the k-th measurement.

Finally, we shall consider the parameter a Markovian process of diffusion type, for which in small time intervals Δ the transition function is expressed by a Gaussian curve

$$W(\lambda_{k+1} | \lambda_k) = \frac{1}{\sqrt{2\pi}\sigma(\lambda_k)} \exp\left\{-\frac{[\lambda_{k+1} - \lambda_k - a(\lambda_k)]^2}{2\sigma_\lambda^2(\lambda_k)}\right\}, \tag{6.9.4}$$

where $a(\lambda_k)$, $\sigma_\lambda^2(\lambda_k)$ are determined through the coefficients of drift $A(\lambda)$ and diffusion $B(\lambda)$ in the form

$$a(\lambda) = A(\lambda)\Delta, \quad \sigma_\lambda^2(\lambda) = B(\lambda)\Delta, \quad \Delta = t_{k+1} - t_k$$

and have the physical meaning of systematic bias and variance of random change of the parameter in the interval Δ between two successive measurements. In view of the low a posteriori inaccuracy, functions $A(\lambda)$ and $B(\lambda)$ vary little within the bounds of the width of the a posteriori peak, so that it is possible to consider $B(\hat{\lambda}) \approx B(\lambda)$ and

$$A(\lambda) \approx A(\hat{\lambda}) + A'(\hat{\lambda})(\lambda - \hat{\lambda}). \tag{6.9.5}$$

The essential difference from the case of a Gaussian parameter is the fact that the Gaussian approximation for the transition function $W(\lambda|\mu)$ is valid only for small intervals between the moments of precise definition of values of $\lambda(t)$ and that variance and the mean value of $W(\lambda|\mu)$ even in the Gaussian approximation depend on the concrete value of the measured quantity. Substituting in these conditions (6.9.1), (6.9.2) and (6.9.4) in (6.5.24), integrating and equating coefficients for different powers of λ_{n+1} in logarithms of the right and left members, we obtain

$$\frac{\hat{\lambda}_{n+1}}{\sigma^2_{\text{вмх}(n+1)}} = \frac{\hat{\lambda}_n + a(\hat{\lambda}_n)}{Q^2(\hat{\lambda}_n)\sigma^2_{\text{вмх}\,n} + \sigma_\lambda^2(\hat{\lambda}_n)} + \frac{\check{\lambda}_{n+1}}{\sigma^2_{n+1}}, \tag{6.9.6}$$

$$\frac{1}{\sigma^2_{\text{вмх}(n+1)}} = \frac{1}{Q^2(\hat{\lambda}_n)\sigma^2_{\text{вмх}\,n} + \sigma_\lambda^2(\hat{\lambda}_n)} + \frac{1}{\sigma^2_{n+1}}. \tag{6.9.7}$$

As (6.9.6) shows, the (n + 1)-th estimate will be formed by the weighted addition of the n-th estimate, anticipated one step with the help of component $a(\hat{\lambda}_n)$, and the result of the newly performed measurement $\check{\lambda}_{n+1}$. Weights are determined for the first component by the a priori variance at the n-th moment, increased by the

variance of the expected measurement of the parameter, and for the second component – by variance of the (n + 1)-th measurement. According to (9.6.7) the sum of these weights determines variance of the a posteriori distribution in the (n + 1)-th step. We intentionally did not make a distinction between variance, calculated according to (6.9.6) and (6.9.7) and true a posteriori variance, since with an optimum procedure of measurement they coincide with high accuracy. Here, it is possible to consider $\sigma^2_\lambda(\hat{\lambda}) = \sigma^2_\lambda(\lambda_И)$, $Q(\hat{\lambda}) = Q(\lambda_И)$, where $\lambda_И$ – true value of the parameter.

It remains only to explain the appearance in front of the expression for variance of measurement on the preceding step $\sigma_{\text{ВЫХ } n}$ of factor $Q^2(\hat{\lambda}_n)$, where

$$Q(\hat{\lambda}_n) = 1 + \frac{\partial a(\hat{\lambda}_n)}{\partial \lambda_n} = \frac{\partial}{\partial \lambda_n}(\lambda_n + A_n(\lambda_n)\Delta)\Big|_{\lambda_n=\hat{\lambda}_n} \tag{6.9.8}$$

is a partial derivative of the value of the parameter extrapolated to the (n + 1)-th moment with respect to the value of the parameter at the n-th moment. This factor is needed because systematic change of parameter $A(\lambda)\Delta$ depends on λ, and depending upon its value inside the peak of the a posteriori probability parameter it will obtain various mean increments in the interval of extrapolation. Change of variance of the expected value is proportional to variance in the preceding step. Variance increases if to the large coordinate there corresponds (algebraically) high speed, since this leads to ever-increasing dispersion, and conversely. Formally, we calculate increment $\Delta\lambda_n$ to increment $\Delta\lambda_{n+1}$ with the help of partial derivative (6.9.8).

Now, analogously to (6.6.3), we expand the likelihood function at a point of estimation anticipated one step $\hat{\lambda}_n + a(\hat{\lambda}_n)$ [before carrying out the (n + 1)-th measurement no more successful approximation to the value of the parameter λ_{n+1} exists]:

$$P(y_{n+1} \mid \lambda_{n+1}) = P(y_{n+1}) \mid \hat{\lambda}_n + a(\hat{\lambda}_n)) \times \\ \times \exp\left\{z_{n+1}(\lambda_{n+1} - \hat{\lambda}_n - a(\hat{\lambda}_n)) - \frac{(\lambda_{n+1} - \hat{\lambda}_n - a(\hat{\lambda}_n))^2}{2\sigma_n^2}\right\}. \tag{6.9.9}$$

Here,

$$z_{n+1} = \frac{\partial}{\partial \lambda} L(\hat{\lambda}_n + a(\hat{\lambda}_n)) \tag{6.9.10}$$

is the output signal of an optimum discriminator, and

$$\sigma_n^2 = \left[-\frac{\partial^2 L(\hat{\lambda}_n + a(\hat{\lambda}_n))}{\partial \lambda^2}\right]^{-1} \tag{6.9.11}$$

is again variance of the n-th measurement, practically coinciding with (6.9.3). Substituting (6.9.1), (6.9.4), (6.9.5) and (6.9.6) in (6.5.24) and performing

calculations, we have

$$\hat{\lambda}_{n+1}=\hat{\lambda}_n+a(\hat{\lambda}_n)+\sigma^2_{\text{вых}(n+1)}z_{n+1}, \tag{6.9.12}$$

where $\sigma^2_{\text{BHX}\ (n+1)}$ is given by relationship (6.9.7). The algorithm of formation of the (n + 1)-th estimate in this case is still simpler: to the expected value we add the output signal of the discriminator with a weight equal to the current a posteriori variance. The more exactly smoothed the data, the less we use the new information z_{n+1}, on the average proportional to current mismatch.

In both considered cases circuits of estimate formation are simple and possess the convenient quality that in the next step, besides the result of the new unit measurement, it is necessary to know only two magnitudes stored in the memory from the preceding measurement: the former result and its variance. In every step of the operation measurements are conducted by uniform recursion formulas. This is a result of the assumption that the measured parameter is a Markovian process of the first order.

Let us consider how the above mentioned relationships change with transition to continuous observation. Remembering the definition of the output signal z(t) of an optimum "continuous discriminator" (6.6.26), taking into account (6.9.4) we can rewrite (6.9.12) in the form

$$[\hat{\lambda}(t+\Delta)-\hat{\lambda}(t)]/\Delta=A(\hat{\lambda}(t))+\sigma^2_{\text{вых}}(t+\Delta)\frac{1}{\Delta}\int_t^{t+\Delta}z(\tau)\,d\tau,$$

from which, passing to the limit $\Delta\to 0$, we have

$$\frac{d\hat{\lambda}(t)}{dt}=A(\hat{\lambda}(t))+\sigma^2_{\text{вых}}(t)\,z(t). \tag{6.9.13}$$

Considering relationship (6.9.4) and the fact that $\sigma_n^{-2}=K(t_n)\Delta$, it is possible, analogously, to transform (6.9.7):

$$\frac{d\sigma^2_{\text{вых}}(t)}{dt}=B(\hat{\lambda}(t))-K(t)\,\sigma^4_{\text{вых}}(t)-A'(\hat{\lambda}(t))\,\sigma^2_{\text{вых}}(t). \tag{6.9.14}$$

The first components in (6.9.13) and (6.9.14) are the a priori known components of the rate of change of the parameter and variance of its random component, respectively. Second components reflect the influence on the estimate and its variance of newly arriving data. Let us note that in (6.9.14) the first component is always positive, and the second is always negative. Conditions with constant (or close to constant) variance occur with approximate equality of these two components, since the third component normally has little influence. Then the newly arriving data

approximately compensate the decrease in accuracy due to change of the parameter.

The set of relationships (6.9.13) and (6.9.14) is simulated by the block diagram of Fig. 6.42. The input realization of y(t) is fed to discriminator 1 and accuracy unit 2, the functions of which were already explained. Smoothing is produced by rather complex circuits, in which it is possible to separate two basic groups of elements (divided by the dotted line).

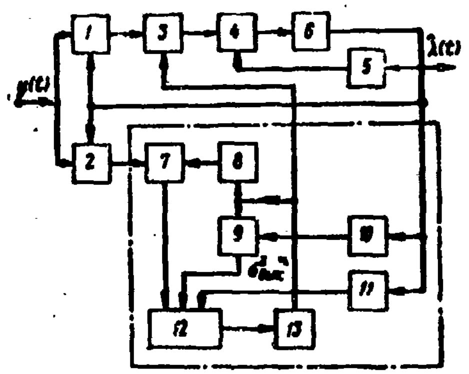

Fig. 6.42. Closed-loop optimum meter for a Markovian parameter: 1 — discriminator; 2 — accuracy unit; 3, 7, 9 — multipliers; 4, 12 — adders; 5, 10, 11 — nonlinear converters; 6, 13 — integrators; 8 — square-law generator.

The first group smooths estimate $\hat{\lambda}(t)$ and consists of multiplier 3, adder 4, a nonlinear converter 5 with characteristic $A(\lambda)$ and integrator 6, directly carrying out accumulation. The second group smooths the measure of a posteriori variance $\sigma^2_{\text{вых}}(t)$ and consists of multipliers 7, 9, square-law generator 8, nonlinear converters 10, 11 with characteristics $A'(\lambda)$ and $B(\lambda)$, adder 12 and integrator 13. The second group plays the auxiliary role of controlling the gain factor in 3 during smoothing of the estimate. The measured value $\hat{\lambda}(t)$ is fed to 1 and 2 to maintain selection. As we see, in distinction from a Gaussian parameter, the smoothing networks are nonlinear, but the general idea of smoothing in a closed loop is preserved here.

Transition to the continuous case in relationship (6.9.6) will lead to equation

$$\frac{d\hat{\lambda}(t)}{dt} = A(\hat{\lambda}(t)) + \sigma^2_{\text{вых}}(t)\, K(t)\, [\check{\lambda}(t) - \hat{\lambda}(t)]. \qquad (6.9.15)$$

Operations, determined by (6.9.13) and (6.9.15), can be performed by the block diagram of Fig. 6.43. The nonlinear estimator unit 1 issues the maximum likelihood estimate $\hat{\lambda}(t)$. Further circuits again are divided into two groups. The first, intended for smoothing the estimate, consists of subtractor 2, two multipliers 3, 4, adder 5, nonlinear converter 6 and integrator 7. Unit 3 controls gain in the smoothing loop taking into account current unequal-accuracy of measurements, taking with greater weight the segments of the realization with a high signal level. Unit 4, as in the scheme of Fig. 6.42, decreases gain with increase of resultant accuracy. In everything else this group of elements, as also the second group, smoothing the a posteriori variance, repeats part of the scheme of Fig. 6.42. It is important only to note

that closing of the meter by the estimator unit is not produced. This again permits us to call such a variant of a meter open-loop.

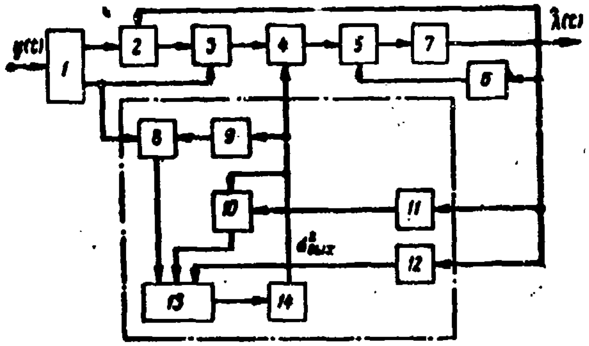

Fig. 6.43. Open-loop optimum meter for a Markovian parameter: 1 — estimator unit; 2 — subtractor; 3, 4, 8, 10 — multipliers; 5, 13 — adders; 6, 11, 12 — nonlinear converters; 7, 14 — integrators; 9 — square-law generator.

Comparing (6.9.15) and (6.9.14) with (6.5.35) and considering that according to definitions $l'(\lambda_0, t) = z(t)$, $l''(\lambda_0, t) = -K(t)$, we with interest prove the coincidence of these equations. Although we did not use diffusion equations, the equations for characteristics of the a posteriori distribution during continuous observation were the same as in the works of R. L. Stratonovich [16, 17], by virtue of using the same approximations.

It would have been possible to continue our investigation here and consider a parameter in the form of a Markovian process of the k-th order. However, this leads to a problem so similar to measurement of several Markovian parameters that it is better to postpone its investigation until Chapter XII.

6.9.2. Synthesis of a Meter with Statistics Unknown. Minimax Solution.

The theory of statistical solutions usually has something to do with two extreme cases — complete a priori knowledge and complete a priori ignorance. The first of these cases was considered in detail above. In the last case synthesis of optimum resolvers is carried out on the basis of the minimax principle.

In reference to the problem of synthesis of an optimum meter we will consider as the case of complete a priori ignorance such a case, when with respect to function $\lambda(t)$ we know only that it remains practically constant for certain intervals of time Δ. We have here no information about the law of change of the measured parameter from one discrete value λ_i to another λ_{i+1}.

The minimax solution is constructed in the following way. For every possible estimate $\hat{\lambda}(\mathbf{y})$ we determine conditional risk as a function of λ, its maximum for all λ, and then select such an estimate $\hat{\lambda} = \hat{\lambda}_M$, for which this maximum would be minimum. The minimax estimate is determined by relationship

$$r(\hat{\lambda}_M, \lambda) = \min_{\hat{\lambda}} \max_{\lambda} r(\hat{\lambda}, \lambda). \tag{6.9.16}$$

The character of its optimality consists in the fact that it minimizes the maximum value of conditional mathematical expectation of the loss function.

At present direct methods of construction of minimax solutions have not been developed, and finding them is based on Walde's [?] theorem [63]. The minimax solution is the Bayes solution relative to a certain a priori distribution, which gives a maximum magnitude of Bayes risk, i.e., satisfies relationship

$$R(\hat{\boldsymbol{\lambda}}_{б}^{*},\ P_{0}^{*})=\max_{(P_0)} R[\hat{\boldsymbol{\lambda}}_{б}(P_{0}),\ P_{0}], \qquad (6.9.17)$$

where $\hat{\boldsymbol{\lambda}}_{б}(P_0)$, $\hat{\boldsymbol{\lambda}}_{б}^{*}$ are the Bayes estimates with respect to a priori distributions $P_0(\boldsymbol{\lambda})$ and $P_0^*(\boldsymbol{\lambda})$, and gives conditional risk with a quantity not depending on $\boldsymbol{\lambda}$ for all values of $\boldsymbol{\lambda}$ which have, according to distribution $P_0^*(\boldsymbol{\lambda})$, non-zero a priori probability. The actual distribution with density $P_0^*(\boldsymbol{\lambda})$ is called the least preferable, and $\hat{\boldsymbol{\lambda}}_{б}^{*}(\mathbf{y}) = \hat{\boldsymbol{\lambda}}_{M}(\mathbf{y})$ is the minimax estimate, satisfying (6.9.17). This theorem, especially its second part, has fundamental importance, although its effective use for producing minimax solutions is rather difficult.

Let us consider one important case. Let us assume that the loss function is simple (6.5.5). Let us consider the Bayes estimate for a "uniform" a priori distribution. Mean risk in this case will be defined as

$$R(\hat{\boldsymbol{\lambda}},\ P_0)=C-\int P(\mathbf{y}\,|\,\hat{\boldsymbol{\lambda}})\,d\mathbf{y}, \qquad (6.9.18)$$

from which it follows that the optimum estimate $\hat{\lambda}(\mathbf{y})$ is that value of $\boldsymbol{\lambda}$ which turns likelihood function $P(\mathbf{y}|\boldsymbol{\lambda})$ into a maximum. Thus, we arrive at the maximum likelihood estimate for vector parameter $\boldsymbol{\lambda}$, which with observance of conditions of analyticity is determined from the system of likelihood equations

$$\frac{\partial}{\partial\lambda_i}P(\mathbf{y}\,|\,\boldsymbol{\lambda})=0,\ \ (i=1,\ \ldots,\ n).$$

If, then, the conditional risk $r(\hat{\lambda},\lambda)$ for the maximum likelihood estimate turns out not to depend on $\boldsymbol{\lambda}$, this estimate on the basis of the Walde theorem is the minimax estimate. It is not difficult to prove that the sufficient condition of this is the possibility of presenting the likelihood function in the form

$$P(\mathbf{y}|\boldsymbol{\lambda})=F(\boldsymbol{\lambda}-\boldsymbol{\lambda}_0)\,F_1(\mathbf{y}), \qquad (6.9.19)$$

where $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}_0(\mathbf{y})$;

$F(\mathbf{x})$ – an even positive function of the vector argument;

$F_1(\mathbf{y})$ – a function integrable with respect to $\mathbf{y}$, which by virtue of $P(\mathbf{y}|\boldsymbol{\lambda}) \geq 0$ and $F(\boldsymbol{\lambda} - \boldsymbol{\lambda}_0) \geq 0$ is nonnegative.

Then the maximum likelihood estimate is equal to $\hat{\boldsymbol{\lambda}}_M(\mathbf{y}) = \boldsymbol{\lambda}_0(\mathbf{y})$. Furthermore, vector $\boldsymbol{\lambda}_0(\mathbf{y})$ is the minimax estimate for any symmetric loss function, which is proved analogously to Paragraph 6.5.3.

Thus, the minimax solution of the problem of filtration under certain conditions consists of formation of a set of maximum likelihood estimates for values of λ_i at intervals of constancy of measured parameters Δ. Condition (6.9.19) leads actually to the existence of efficient estimates for parameters λ_i at intervals of time $[(t_0 + (i - 1)\Delta, t_0 + i\Delta)]$ of duration Δ. Practically, of course, it is sufficient to obtain approximate efficiency and, correspondingly, the possibility of presenting the likelihood function in the form of (6.9.19) with a satisfactory degree of approximation.

Let us consider one useful example. Let us assume that at the resolver input there is an additive mixture of the useful parameter $\lambda(t)$ and interference $n(t)$:

$$y(t) = \lambda(t) + n(t),$$

and the loss function is quadratic. Such a mixture can be considered the output signal of the discriminator of the meter with which we sum the measured value of the parameter. Conditions of the validity of such a presentation were explained above. Then the optimum estimate of vector $\boldsymbol{\lambda}$ for any a priori distribution has the form

$$\hat{\boldsymbol{\lambda}} = \frac{\int \boldsymbol{\lambda} P(y|\boldsymbol{\lambda}) P_0(\boldsymbol{\lambda}) d\boldsymbol{\lambda}}{\int P(\mathbf{y}\,|\,\boldsymbol{\lambda}) P_0(\boldsymbol{\lambda}) d\boldsymbol{\lambda}} = \frac{\int \boldsymbol{\lambda} P(y - \boldsymbol{\lambda}) P_0(\boldsymbol{\lambda}) d\boldsymbol{\lambda}}{\int P(y - \boldsymbol{\lambda}) P_0(\boldsymbol{\lambda}) d\boldsymbol{\lambda}}, \tag{6.9.20}$$

where $P(\mathbf{y}|\boldsymbol{\lambda}) = P(\mathbf{y} - \boldsymbol{\lambda})$ — density of distribution of vector $\mathbf{n}$.

Let us assume that a priori distribution $P_0(\boldsymbol{\lambda})$ is "uniform" in the whole infinite region of existence of $\boldsymbol{\lambda}$. Then, replacing the variable of integration in (6.9.20) $\mathbf{y} - \boldsymbol{\lambda} = \mathbf{n}$, we obtain

$$\hat{\boldsymbol{\lambda}} = \mathbf{y} - \frac{\int \mathbf{n} P(\mathbf{n})\, d\mathbf{n}}{\int P(\mathbf{n})\, d\mathbf{n}} = \mathbf{y} - \bar{\mathbf{n}}, \tag{6.9.21}$$

i.e., estimate $\hat{\lambda}$, with accuracy of a constant component not depending on $\mathbf{y}$, coincides with signal $\mathbf{y}$. Conditional risk for estimate (6.9.21) with uniform distribution is equal to

$$r(\hat{\boldsymbol{\lambda}}, \boldsymbol{\lambda}) = \int (\hat{\boldsymbol{\lambda}} - \boldsymbol{\lambda})^+ (\hat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}) P(\mathbf{y} - \boldsymbol{\lambda})\, d\mathbf{y} = \\ = \int (\mathbf{n} - \bar{\mathbf{n}})^+ (\mathbf{n} - \bar{\mathbf{n}}) P(\mathbf{n})\, d\mathbf{n} = \mathrm{spur}\,\mathbf{r},$$

(where $\mathbf{r}$ — correlation matrix of interference) and turns out not to depend on $\boldsymbol{\lambda}$. Consequently, the solution of (6.9.21) is the minimax solution of the problem of filtration of signal $\lambda(t)$ from an additive mixture with interference $n(t)$.

The obtained solution has a clear physical meaning. Actually, in the absence of any a priori information about $\lambda(t)$, realization of the input signal as estimate $\hat{\lambda}(t)$ will be the best that can be offered. Error of measurement here, as it should be, is simply equal to variance of interference $n(t)$. Using the a priori assumed constancy of $\lambda(t)$ in intervals of length Δ, it is possible to decrease error of measurement somewhat. Nevertheless, the conclusion that in the absence of any information about trajectory properties of the target effective smoothing cannot be realized remains valid, and error of unit measurements are wholly recalculated into resultant errors.

6.9.3. Description of a Parameter with Limited Knowledge of Its Statistics

Assignment of a multi-dimensional distribution of probabilities of $\lambda(t)$ requires sufficiently detailed statistical evidence about the law of change of target position data. Frequently in practice such information is inaccessible; however certain statistical characteristics of $\lambda(t)$ nevertheless exist. Such a case is intermediate between the above-considered cases of complete a priori knowledge and complete a priori ignorance. With incomplete statistical description there appears a very great variety of cases, which in general reduce to the fact that we assign certain limitations on characteristics of $\lambda(t)$. The latter permit us to determine a certain set Λ of permissible functions $\lambda(t)$ and subsequently in problems of analysis and synthesis to limit ourselves to consideration of only such $\lambda(t)$ as belong to this set $(\lambda(t) \in \Lambda)$. These limitations can be given in statistical form, and also in normal form.

We first consider the first case.

The simplest characteristic of $\lambda(t)$ is mathematical expectation $\overline{\lambda(t)}$. In most problems it is naturally assumed assigned. A more detailed description will be attained if we assign the variance at any moment of time and the moments of highest order right up to assignment of a one-dimensional law of distribution of probabilities. Further precise definition of the behavior of $\lambda(t)$ occurs with assignment, for instance, of variances of derivatives of $\lambda(t)$ up to a certain order at any moment of time. All these characteristics are more or less available in practice.

Very important both from the practical point of view, and also from the point of view of theory, is the case when there is assigned the correlation function of $\lambda(t)$. As we shall subsequently prove, under certain conditions of its assignment it is already sufficient for synthesis of an optimum meter. A still more detailed

description of $\lambda(t)$ is attained with assignment of moment functions of higher order. Limitations of this type in general permit us to assign a set Λ of permissible realizations of $\lambda(t)$ as the set described by the a priori distribution of probabilities $P_0(\lambda)$, obeying conditions

$$\int P_0(\lambda)\,d\lambda=1, \quad \int \mathbf{f}(\lambda)P_0(\lambda)\,d\lambda=\mu, \tag{6.9.22}$$

and otherwise being arbitrary. In (6.9.22) integrals are taken over the whole range of values of λ: $\mathbf{f}(\lambda)$ — certain multicomponent function of λ; μ — the assigned value of the same structure. In the particular case when we are assigned variance of $\lambda(t)$ at any moment of time t_i, we take $f_i(\lambda) = \lambda_i^2$, $\mu_i = \sigma_i^2$, and $\overline{\lambda}_i$ for simplicity is considered equal to zero. With assignment of the correlation matrix we assume

$$f_{ij}=\lambda_i\lambda_j, \quad \mu_{ij}=R_{ij} \ (i, j=1, \ldots, n), \tag{6.9.23}$$

with assignment of moments of the third order $f_{ijk} = \lambda_i\lambda_j\lambda_k$, $\mu_{ijk} = \lambda_i\lambda_j\lambda_k$, etc.

Other examples of partial statistical description are cases when multi-dimensional laws of distribution of probabilities are assigned with an accuracy to certain unknown parameters, or there are assigned distributions of probabilities of lowest orders — first, second, etc. Of special importance is the a priori assumption of a Markovian character of change of coordinates without indication of the form of the initial distribution and transition probabilities. Actually, movement of radar targets is described by second order differential equations, so that the assumption of a Markovian process of the second order corresponds to ignorance of the concrete form of the equation determining the form of transition probabilities, and characterizes that minimum a priori information which we always have.

Sometimes statistical evidence about the character of change of $\lambda(t)$ is completely lacking. So that the problem of measurement has meaning, it is necessary to impose on $\lambda(t)$ some limitations, determining the set Λ of permissible realizations of $\lambda(t)$. The simplest limitation of this type is assignment of a certain interval of time Δ, during which $\lambda(t)$ remains practically constant, so that $|\lambda(i + \Delta) - \lambda(i)| \ll \lambda(t)$ (see Paragraph 6.9.2). Analogous intervals can be assigned with respect to derivative $\frac{d\lambda(t)}{dt}$, etc. More detailed limitation is obtained if we know that $\lambda(t)$ and certain of its derivatives are limited by certain limits of variation, i.e.,

$$f_k(t)<\frac{d^k\lambda}{dt^k}<\varphi_k(t) \ (k=0, 1, 2, \ldots), \tag{6.9.24}$$

where $f_k(t)$, $\varphi_k(t)$ — assigned functions.

Such a method of assignment is frequently used in practice. From it, in particular, we can obtain necessary intervals of constancy Δ for λ (t) and a certain quantity of derivatives.

In a number of problems there is possible still more detailed assignment of $\lambda(t)$. For instance, during measurement of coordinates of a ballistic target it is possible to assign $\lambda(t)$ with the help of known functions depending on certain unknown parameters, which in distinction from § 6.8 have an unknown probability distribution. Equivalent to this case is assignment of $\lambda(t)$ with the help of differential equations depending on a series of unknown parameters (for instance, unknown initial conditions). The difference is only in the fact that frequently solution of these equations and explicit expressions for $\lambda(t)$ cannot be obtained.

6.9.4. Synthesis of a Meter with Limited Knowledge of the Statistics

Along with the Bayes solution, giving as it were an absolute optimum, of extraordinary interest from the practical point of view is solution of the problem of synthesis of a meter, optimum in conditions of limited a priori knowledge. Problems of this type in the theory of statistical solutions have generally not been formulated. A certain exception is the Wiener theory of filtration and its nonlinear generalization in the works of Zadeh [15], in which the optimum operator of filtration is found when there are assigned only correlation functions of $\lambda(t)$ and $n(t)$; however this operator is considered beforehand to belong to an assigned class — linear in Wiener theory — and a definite type of nonlinear ones in [15]. This last limitation naturally causes criticism, since it remains vague how close the synthesized circuits are to indeed optimum operators which one should obtain from the general theory of solutions without preliminary assignment of their structure. Certain results relative to solution of problem of synthesis with limited a priori knowledge are contained in [62]. Here, along with them, we will try to develop a general formulation and consider other particular cases.

Thus, let us assume that about $\hat{\lambda}(t)$ we have certain a priori statistical evidence, for instance of the type considered in Paragraph 6.9.4, which permits us to assign the set of permissible realizations of $\lambda(t)$. This set is described by an a priori probability distribution with density $P_0(\lambda)$, belonging to a given class of distributions, which we designate by P_0. A particular case is the total absence of statistical evidence, when P_0 is the set of all normalized, positive, integrable in an infinite range, values λ of the function.

The rule for finding estimates $\hat{\lambda}(\mathbf{y})$ in this case can be ontained by proper generalization of the rule based on the least preferred distribution. Here, we naturally consider that the least preferred distribution also obeys the limitations, and we seek it among distributions of class $\mathbf{P}_0$. The general rule for selection of the optimum solution can be formulated as follows:

$$R(\hat{\lambda}_6^*, P_0^*) = \max_{(P_0 \in \mathbf{P}_0)} \min_{\hat{\lambda}} R_6(\hat{\lambda}_6(P_0), P_0), \tag{6.9.25}$$

where $P_0(\lambda)$ — distribution, belonging to the assigned class $\mathbf{P}_0(P_0 \in \mathbf{P}_0)$;

$P_0^*(\lambda)$ — least preferred distribution of this class;

$\hat{\lambda}_6^*$ — Bayes estimate with respect to $P^*(\lambda)$.

During construction of the solution of equation (6.9.25) we first for any density $P_0(\lambda) \in \mathbf{P}_0$ find the Bayes estimate $\hat{\lambda} = \hat{\lambda}(P_0)$, which minimizes mean risk and is a functional of P_0. Then by selection of function $P_0 = P_0^*$ we maximize mean risk, calculated for this estimate. This function P_0^* is the density of the least preferred distribution from class P_0. Substituting it in the expression for estimate $\hat{\lambda}(P_0)$, we obtain the optimum solution $\hat{\lambda}(P_0^*) = \hat{\lambda}_6^*$.

The direct method of finding the least preferred distribution and the corresponding solution, as we can see, is extraordinarily difficult since it requires skill to solve the Bayes problem for a sufficiently broad class of partially assigned a priori distributions and to calculate for every solution the mean risk. However, in a number of cases it is possible to formulate the rule for finding the least preferred distribution by generalizing the rule, which follows from the the Walde theorem (Paragraph 6.9.2) for the case of complete a priori ignorance.

Let us assume that some statistical characteristics of $\lambda(t)$, allowing us to determine the class of assigned a priori distributions $P_0(\lambda)$, are assigned by conditions (6.9.22). Let us renumber components of function $\mathbf{f}(\lambda)$ and of the given quantity $\boldsymbol{\mu}$ so that

$$\mathbf{f}(\lambda) = \{f_1(\lambda), \ldots, f_k(\lambda), \ldots\}, \quad \boldsymbol{\mu} = \{\mu_1, \ldots, \mu_k, \ldots\}, \; f_j(\lambda)$$

are functions which depend on all or certain values of $\lambda_i (i = 1, \ldots, n)$, and index j runs through as many values as conditions assigned.

Let us assume that $\hat{\lambda}(\mathbf{y}, P_0)$ is the Bayes estimate with respect to distribution $P_0 \in \mathbf{P}_0$, which is the solution of equation (6.9.25). We use the Lagrange method for

finding the least preferred distribution of $P_0^*(\lambda)$. Here, function $P_0^*(\lambda)$ is determined from the condition of the maximum of functional

$$I = R(\hat{\lambda}(\mathbf{y}, P_0), P_0) - \varkappa_0 \int P_0(\lambda)\, d\lambda - \sum_j \varkappa_j \int f_j(\lambda) P_0(\lambda)\, d\lambda =$$

$$= \int \left[\int w(\lambda - \hat{\lambda}) P(\mathbf{y}|\lambda) d\mathbf{y} - \varkappa_0 - \sum_j \varkappa_j f_j(\lambda)\right] P_0(\lambda) d\lambda, \qquad (6.9.26)$$

where $\varkappa$, $\varkappa_j$ – indefinite factors, found from conditions

$$\int P_0(\lambda) d\lambda = 1, \quad \int f_j(\lambda) P_0(\lambda)\, d\lambda = \mu_j \quad (j = 1, 2, \ldots). \qquad (6.9.27)$$

Calculating variation of functional $I = I(P_0)$ with respect to function P_0, we obtain

$$\delta_{P_0} I = \int \left[\int w(\lambda - \hat{\lambda}) P(\mathbf{y}|\lambda)\, d\mathbf{y} - \varkappa_0 - \sum_j \varkappa_j f_j(\lambda)\right] \delta P_0(\lambda)\, d\lambda -$$

$$- \sum_{i=1}^{n} \int \left[\int \frac{\partial w(\lambda - \hat{\lambda})}{\partial \hat{\lambda}_i} P(\mathbf{y}|\lambda) P_0(\lambda)\, d\lambda\right] \delta_{P_0} \hat{\lambda}_i d\mathbf{y}, \qquad (6.9.28)$$

where $\delta_{P_0} \hat{\lambda}_i$ – variation of the vector component of the estimate with respect to distribution P_0.

The second component in (6.9.28) turns into zero by virtue of equality $\frac{\partial w}{\partial \hat{\lambda}_i} = -\frac{\partial w}{\partial \lambda_i}$, the evident independence of $\hat{\lambda}_i$ and, consequently, of $\delta_{P_0} \hat{\lambda}_i$ from λ and equations (6.5.12'). Therefore, from the requirement of equality to zero of variation $\delta_{P_0} I$ for an arbitrary δP_0 there follows the equation for conditional risk

$$r(\hat{\lambda}, \lambda) = \int w(\lambda - \hat{\lambda}) P(\mathbf{y}|\lambda)\, d\mathbf{y} = \varkappa_0 + \sum_j \varkappa_j f_j(\lambda), \qquad (6.9.29)$$

which permits us to formulate the following rule for finding $P^*_0(\lambda)$.

So that the a priori distribution with density $P^*_0(\lambda)$ is the least preferred in the class of distributions $\mathbf{P}_0$, satisfying assigned conditions (6.9.22), the Bayes solution built with this distribution should give conditional risk $r(\hat{\lambda}, \lambda)$, the functional dependence of which on λ is determined by expression (6.9.29).

This rule generalizes the requirement of independence of conditional risk from λ, valid for total absence of a priori information. The concrete form of $P_0(\lambda)$ besides the assigned a priori information depends on the method of coding $\lambda(t)$ and $y(t)$ and statistics of $y(t)$, i.e., on the form of likelihood function $P(\mathbf{y}|\lambda)$.

An important particular case is assignment of the correlation matrix $\mathbf{R}$ of vector λ. Here $\mathbf{f}(\lambda) = \{\lambda_i \lambda_j\}$ and the equation for $r(\hat{\lambda}, \lambda)$ has the form

$$r(\hat{\lambda}, \lambda) = \int w(\lambda - \hat{\lambda}) P(\mathbf{y}|\lambda)\, d\mathbf{y} = \varkappa_0 + \sum_{i,j=1}^{n} \varkappa_{ij} \lambda_i \lambda_j, \qquad (6.9.30)$$

where factors $\varkappa_0$ and $\varkappa_{ij}$ are determined from conditions

$$\int P^*{}_e(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} = 1, \quad \int \lambda_i \lambda_j P^*{}_e(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} = P_{ij}, \tag{6.9.31}$$

and $\hat{\boldsymbol{\lambda}}(\mathbf{y})$ — the Bayes estimate with respect to distribution $P^*(\boldsymbol{\lambda})$.

Thus, estimate $\hat{\lambda}(\mathbf{y})$, corresponding to assignment of only the correlation matrix of the measured parameters, is optimum when it is the Bayes estimate with respect to the a priori distribution of probabilities for which the correlation matrix is equal to the assigned one, and conditional risk for this estimate is the sum of a constant and the quadratic form of $\boldsymbol{\lambda}$ of form

$$\boldsymbol{\lambda}^+ \varkappa \boldsymbol{\lambda} = \sum_{i,j} \varkappa_{ij} \lambda_i \lambda_j. \tag{6.9.32}$$

We shall show that under certain conditions the least preferred distribution for the given case is a Gaussian distribution. Let us assume that the loss function is quadratic (6.5.8), and likelihood function $P(\mathbf{y}|\boldsymbol{\lambda})$ is presented in the form of (6.9.19), where function $F(\boldsymbol{\lambda} - \boldsymbol{\lambda}_0)$ is approximated by a Gaussian curve

$$F(\boldsymbol{\lambda} - \boldsymbol{\lambda}_0) = \exp\left\{-\frac{1}{2}(\boldsymbol{\lambda} - \boldsymbol{\lambda}_0)^+ \mathbf{A}(\boldsymbol{\lambda} - \boldsymbol{\lambda}_0)\right\}, \tag{6.9.33}$$

where $\mathbf{A} = \|A_{ij}\|$ — matrix of order $(n \times n)$, similar in structure to matrix $\mathbf{R}$.

Let us note that function $F(\boldsymbol{\lambda} - \boldsymbol{\lambda}_0)$ in (6.9.19) with the corresponding normalization is the density of the probability distribution for vector $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}_0(\mathbf{y})$ which is the maximum likelihood estimate for $\boldsymbol{\lambda}$. Let us consider the Gaussian a priori distribution of $\boldsymbol{\lambda}(t)$ with correlation matrix $\mathbf{R}$ and find the estimate of conditional mathematical expectation corresponding to the quadratic loss function. Then

$$\hat{\boldsymbol{\lambda}}(\mathbf{y}) = \frac{\int \boldsymbol{\lambda} \exp\left\{-\frac{1}{2}(\boldsymbol{\lambda} - \boldsymbol{\lambda}_0)^+ \mathbf{A}(\boldsymbol{\lambda} - \boldsymbol{\lambda}_0) - \frac{1}{2}\boldsymbol{\lambda}^+ \mathbf{V}\boldsymbol{\lambda}\right\} d\boldsymbol{\lambda}}{\int \exp\left\{-\frac{1}{2}(\boldsymbol{\lambda} - \boldsymbol{\lambda}_0)^+ \mathbf{A}(\boldsymbol{\lambda} - \boldsymbol{\lambda}_0) - \frac{1}{2}\boldsymbol{\lambda}^+ \mathbf{V}\boldsymbol{\lambda}\right\} d\boldsymbol{\lambda}} = \mathbf{C}\boldsymbol{\lambda}_0(y), \tag{6.9.34}$$

where matrix $\mathbf{C} = \|C_{ij}\|$ is determined from equation

$$\mathbf{C}[\mathbf{R} + \mathbf{A}^{-1}] = \mathbf{R}, \tag{6.9.35}$$

i.e., estimate $\hat{\boldsymbol{\lambda}}(\mathbf{y})$ is expressed linearly in terms of vector $\boldsymbol{\lambda}_0(\mathbf{y})$. Substituting (6.9.34) in the expression for conditional risk, we obtain

$$\begin{aligned} r(\hat{\boldsymbol{\lambda}}, \boldsymbol{\lambda}) &= \int w(\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}}) P(\mathbf{y}|\boldsymbol{\lambda})\, d\mathbf{y} = \\ &= \int (\boldsymbol{\lambda} - \mathbf{C}\boldsymbol{\lambda}_0)^+ \mathbf{B}(\boldsymbol{\lambda} - \mathbf{C}\boldsymbol{\lambda}_0) F(\boldsymbol{\lambda} - \boldsymbol{\lambda}_0)\, d\boldsymbol{\lambda}_0 = \\ &= \int [\boldsymbol{\lambda} - \mathbf{C}\boldsymbol{\lambda} + \mathbf{C}\mathbf{x}]^+ \mathbf{B}[\boldsymbol{\lambda} - \mathbf{C}\boldsymbol{\lambda} + \mathbf{C}\mathbf{x}] F(\mathbf{x})\, d\mathbf{x} = \end{aligned} \tag{6.9.36}$$

$$=\lambda^{+}(\mathbf{I}-\mathbf{C})^{+}\mathbf{B}(\mathbf{I}-\mathbf{C})\lambda\int F(\mathbf{x})\,d\mathbf{x}+\int\mathbf{x}^{+}\mathbf{C}^{+}\mathbf{B}\mathbf{C}\mathbf{x}F(\mathbf{x})\,d\mathbf{x}=$$
$$=\lambda^{+}\varkappa\lambda+\varkappa_{0}. \qquad (6.9.36) \text{ cont'd}$$

As we see, conditional risk has the required form (6.9.30), i.e., the Gaussian distribution indeed is the least preferred, and the optimum estimate itself has the form of (6.9.34).

Representation (6.9.19) is strictly satisfied for an additive mixture y(t) = = λ(t) + n(t); in this case $\lambda_0 = \mathbf{y}$ and $\hat{\lambda} = \mathbf{Cy}$. This result permits us to comprehend the character of optimality of solutions of the linear theory of filtration. It turns out that if measured parameter λ(t) is additively mixed with Gaussian interference, and we are given the correlation function of λ(t), there is no better (from the point of view of a quadratic loss function) operator of filtration than the linear operator, determined by the Wiener theory of filtration, and only more detailed statistical evidence can lead to change of the structure of the optimum operator.

Representation (6.9.19) is approximately realized in a rather large number of cases, of which we already spoke in Paragraph 6.5.1, and which occur in problems of radar measurement. This circumstance emphasizes the importance of Gaussian a priori distribution, which turns out to be interesting not only for itself, but as the least preferred distribution from the class of distributions with an assigned correlation function. In this connection we paid the greatest attention above to Gaussian a priori distribution.

One more interesting example of a problem with partially assigned statistical characteristics is the case when we are assigned the value of only the matrix of second moments of the derivative of λ(t) at any moment of time. To this case we can approximately reduce the case when there are assigned a priori limits of variation of the derivative at any moment of time. In the given example

$$\mathbf{f}(\lambda)=\{(\lambda_i-\lambda_{i-1})^2\},\ \mu=\{\overline{(\lambda_i-\lambda_{i-1})^2}\}=\{M_i\}$$

and equation (6.9.29) for conditional risk has the form

$$r(\hat{\lambda},\lambda)=\varkappa_0+\sum_{i=2}^{n}(\lambda_i-\lambda_{i-1})^2\varkappa_i. \qquad (6.9.37)$$

If there is also assigned the second moment for the value of λ(t) at the initial moment of time $\sigma_1^2 = \overline{\lambda_0^2}$, to the right side of (6.9.37) one should add component $\lambda_1^2\varkappa_1$. One can prove that under certain conditions, similar to those considered above, the least preferred distribution for the given case is the distribution for a Markovian

Gaussian process with transition probability

$$w(\lambda_i | \lambda_{i-1}) = \frac{1}{\sqrt{2\pi M_i}} \exp\left\{-\frac{(\lambda_i - \lambda_{i-1})^2}{2M_i}\right\} \tag{6.9.38}$$

and with uniform distribution for λ_1 when σ_1^2 is not assigned, or with a Gaussian distribution for λ_1 with variance σ_1^2 for an assigned σ_1^2.

Analogous results are obtained also when along with statistical characteristics of first derivatives there are assigned statistical characteristics of the second derivatives, so that the following moments are determined:

$$M_i^{(11)} = \overline{(\lambda_i - \lambda_{i-1})^2}, \quad M_i^{(12)} = \overline{(\lambda_i - \lambda_{i-1})(\lambda_i - 2\lambda_{i-1} + \lambda_{i-2})},$$

$$M_i^{(21)} = \overline{(\lambda_i - 2\lambda_{i-1} + \lambda_{i-2})(\lambda_i - \lambda_{i-1})},$$
$$M_i^{(22)} = \overline{(\lambda_i - 2\lambda_{i-1} + \lambda_{i-2})^2}. \tag{6.9.39}$$

In this case the least preferred distribution turns out to correspond to a Markovian Gaussian process of the second order with Gaussian transition probability, depending only on the quadratic form of variables λ_i, λ_{i-1}, λ_{i-2}, completely determined by assigned $M_i^{(11)}$, $M_i^{(12)}$, $M_i^{(21)}$, $M_i^{(22)}$.

Therefore, the optimum operator of estimation obtained for such a distribution is simultaneously optimum in the case of assignment of only the matrix of second moments for the first and second derivatives of measured parameters $\lambda(t)$. These statements show that the examples of measurement of Markovian and simultaneously Gaussian parameters given in Paragraph 6.8.2 have more universal meaning than supposed earlier.

§ 6.10. Conclusion

Basic results of the theory of radar meters developed in Chapter VI reduce to the following.

1. In the analysis of tracking meters it is convenient to divide their circuits into discriminators and smoothing circuits. Discriminators are completely described by two functions of mismatch — discrimination and fluctuating characteristics. The minimum parameters of these functions necessary for analysis of accuracy of meters are equivalent spectral density $S_{\text{экв}}$, the gain factor of the discriminator $K_{\text{д}}$, and in a number of cases spectral density of parametric fluctuations $S_{\text{пар}}$. If we know these characteristics and statistical properties of the measured quantity, error of

measurement usually can be calculated easily. Formulas for calculating errors for cases of rapid and slow fluctuations of the signal are given in § 6.2.

2. Breakoff of tracking can be characterized by two characteristics — average time to breakoff and magnitude of variance of errors of measurement in steady-state operating conditions, which in the presence of breakoffs of tracking increases sharply. In any case it turns out that breakoff is a threshold phenomenon, the probability of which increases with the ratio of mean square error (in a linearized system) to the width of the discriminator curve, exceeding a certain number which depends only on the type of smoothing circuits. This gives us the possibility of finding critical intensities of noises and interferences, which lead to breakoff of tracking.

3. In the analysis of nontracking meters it is convenient to divide them into estimator units and smoothing circuits. In §6.3 we obtained formulas for calculation of accuracy of measurement of nontracking circuits on the simplified assumptions of rapid fluctuations and a slowly changing measured quantity.

4. The most adequate means of synthesis of optimum meters at present one should consider the theory of nonlinear filtration. By application of this theory we solved the problem of synthesis of an optimum meter of a slowly changing parameter with Gaussian approximation of the likelihood function.

For a parameter with Gaussian distribution we found three variants of construction of optimum meter: single-loop and double-loop circuits of a tracking meter and a nontracking meter. All of them are identical in performance in identical conditions of work, but the tracking variants are usually more convenient technically.

Optimum tracking meters as basic elements contain nonlinear units (a discriminator and an accuracy unit) and smoothing circuits which for Gaussian statistics of the parameter are linear. An optimum discriminator, just as analogous devices known in practice, issues the measure of mismatch between current and measured values of the parameter, and the accuracy unit issues the measure of current accuracy of separate measurements, which, after smoothing, form the final result of measurement.

The type of discriminator and accuracy unit is determined by statistics of the input signal and the method of encoding the measured parameter in it. Therefore, without concretization of the form of this parameter we could only find certain general rules peculiar to discriminators, and we discussed possible types of them. Final synthesis of optimum discriminators, just as analysis of different closely

related circuits, will be conducted in subsequent chapters, in which we concretize the measured parameter (distance, speed, angles).

The form of smoothing circuits is determined by statistics of the measured parameter and a certain generalized characteristic of accuracy of the discriminator. Therefore, synthesis of smoothing circuits, and also analysis of accuracy of the optimum meter as a whole could be conducted in Chapter VI in sufficiently general form.

5. It should be noted that an optimum discriminator constructed with application of the developed theory coincides in certain assumptions with the discriminator found by the theory of estimation. Thus, we defined the meaning of optimality of devices synthesized by the theory of estimation.

6. Characteristics of optimum smoothing filters with a Gaussian law of distribution of measured parameters obey essentially the same equations as follow from filtration theory, which indicates the meaning of optimality determined by this wide-spread theory.

7. The developed method of synthesis is very universal. Thus, it remains in force for parameters with unknown statistics, if we are only given the correlation function of the parameter. With small changes it can be transfered to the case of a Markovian parameter. Here, the discriminator and accuracy unit remain constant, and only the smoothing circuits are subjected to certain complication.

Although the above findings concerning general rules of radar measurements pertain to a very broad class of radar systems, they nevertheless require further development. Besides problems of simultaneous measurement of several parameters and resolution of signals, to which we devote Chapters XII and XIII, it is possible to outline the following directions for further investigations.

During the analysis of meters above we considered fluctuating errors (due to noises, interferences and fluctuations of the signal) and dynamic errors. Meanwhile, in precision radar systems of importance, too, may be errors of different origin, for instance instrument errors. It would be desirable subsequently to expand the class of errors considered during analysis and synthesis.

If we are concerned with nonlinear phenomena in meters (phenomena of breakoff), results given in § 6.3 indicate a promising mathematical technique of investigation and give practical results for cases of systems with small astaticism. Solution of the problem of breakoff for more complicated forms of smoothing circuits, including

those with variable parameters, runs into greater mathematical difficulties, inasmuch as methods of solution of multi-dimensional diffusion equations have not been developed. This then poses a task for mathematicians, important in its application. Here fully permissible are formulations of problems, differing from these considered in § 6.3, inasmuch as actual definition of the concept of breakoff, obviously, depends on the purpose of the meter.

It is also important to consider that meters can be equipped with failure indicators, for instance, in the form of threshold circuits and "insuring" devices with "mateguard" channels, detuned relative to the measured value of the parameter. Questions of analysis of joint work of meters and such circuits remain practically unconsidered.

Further development of the theory of Bayes synthesis in the direction of non-Gaussian parameters, and also synthesis on the basis of the minimax method with limited statistics and the game-theory method is desirable. As initial propositions it is possible to consider in this case the material of § 6.9.

During concretization of the form of discriminators it is practically useful to consider non-Gaussian signals and interference of a fairly broad class, and also the case of slow fluctuations of the signal. Although the method of coupling such discriminators in optimum meters is known, their circuit construction at present is not completely clear. Analogous questions of concrete realization of circuit construction for non-Gaussian parameters are important also for smoothing circuits.

Although the direction of synthesis of smoothing circuits for quasi-regular parameters are outlined, it is necessary to generalize them for the real case when laws of variation of parameters are unknown and we are assigned only complex differential equations, when properties of errors are complicated, and from smoothing devices there is required maximum accuracy in combination with technical convenience.

Results of analysis and synthesis of radar meters, unboubtedly, may be useful also in those regions of technology where there are used information-bearing signals: in optics, infrared engineering, ultrasonic engineering, communications, broadcasting and television. In those places where the idealizations made above are not fully suitable the method of analysis and synthesis developed for radar systems remains in force.

CHAPTER VII

RANGE FINDING WITH A COHERENT SIGNAL

§ 7.1. Introduction

One of the most important problems of contemporary radar systems of various assignment is measurement of the distance to a target. The need to measure range appears both in survey radars and all the more so in acquisition and guidance radars, and also in navigation systems. Range finding can be based on different physical principles; in particular, methods based on integration of velocity and the triangulational method have been used; however, in extent of application and importance of greatest significance is the direct radar method, based on encoding distance to target in delay of the received signal and measurement of this delay [1, 26, 27].

We shall consider questions connected with range finding by this last method. The first two methods are completely based on measurement of velocity and angles, and all their consideration reduces only to application of results of the corresponding chapters of this book. Furthermore, of all systems finding range by delay of a reflected signal we shall be interested basically in the most widely used automatic tracking meters, carrying out range tracking of a selected target, and only to a smaller extent in other types of range finders.

Statistical analysis and synthesis of systems of range finding are the subject of a comparatively small number of published works. Most works known to the authors pertain actually only to analysis of the influence of random signals, noises, and certain interferences for the most widespread (until recently) pulse incoherent automatic range finder with various limiting assumptions [28-30]. To a considerably lesser extent they have touched on questions of analysis of range finders using other forms of signals [1, 31], and also questions of synthesis of an optimum system

of range finding. Certain particular solutions pertaining to the last question were obtained in [12,32].

Almost simultaneously with works on analysis of the fluctuating accuracy of range-only radars of a very specific type under the influence of noises and interferences there were undertaken attempts at investigation of the potential accuracy of range finding by radar. Apparently, the first work on this question is [4], where for very limiting assumptions there is found the variance of the efficient estimate of delay of a reflected signal for the case of a motionless target. The same problem with less limiting assumptions relative to the received signal was solved in [33]. Further advance on the given question was attained with application of the general results presented in Chapter VI. These results, in particular, give the possibility of obtaining a sufficiently correct solution of the problem of synthesis of an optimum system of range finding for a very broad and practically important class of cases, when the rate of change of the measured distance is small as compared to the rate of fluctuations of the received signal.

As already shown in Chapter VI, the optimum meter is realized here in the form of a closed-loop tracking system, including a nonlinear discriminator and smoothing circuits, controlling tuning of the discriminator (Fig. 7.1). Solution of the problem of synthesis permits us to establish the algorithm for finding operations of the discriminator and characteristics of smoothing circuits for an arbitrary form of modulation of the received signal and a broad class of cases of statistics of the change of distance, and also to conduct detailed investigation of potential accuracy of range finding without the limiting assumption of immobility of the target.

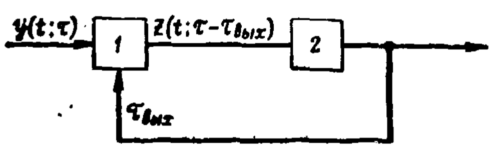

Fig. 7.1. Block diagram of a tracking range finder: 1 — discriminator; 2 — smoothing and control circuits.

Development of the theory of statistical solutions permits us to present statistical questions related to range finding by the deductive method. In accordance with this we shall first consider questions of the synthesis of optimum range finders and of the potential accuracy of range finding for an arbitrary case, and then also in general investigate basic methods of approximate realization of optimum operations and, finally, analyze more specifically concrete circuits of range finders for the most wide-spread and characteristic forms of signal molulation. Inasmuch as all results have great mathematical completeness for the case of a coherent signal, we shall start namely with it, and in the next chapter we shall

consider the case of an incoherent fluctuating signal.

In this and subsequent chapters both from the point of view of analysis, and also from the point of view of synthesis, we shall be interested basically in discriminators of range finding systems. This is explained by the fact that synthesis of smoothing circuits is conducted identically for any measured parameter of the signal. The structure of smoothing circuits is determined here only by the statistics of change of the parameter and does not depend on its physical nature. Such a problem of synthesis is considered in detail in Chapter VI, and here we can now use the available results. If we have determined basic characteristics of the discriminator (discrimination and fluctuation), further analysis of the measuring system, and in particular, determination of errors of measurement also no longer depend on the physical nature of the measured parameter of the signal, in which some target position datum is coded. This gives us the possibility of calculating errors of measuring of coordinates by a single method using previously found characteristics of the discriminator.

For instance, in the case of linearized consideration of the tracking system with smoothing circuits with constant parameters the calculation of the variance of error of measurement reduces to multiplication of the equivalent spectral density of the discriminator by the effective passband of the closed-loop tracking system. In other cases these methods are more complicated, but in general they have already been presented in Chapter VI, and for final calculations of accuracy of range finding systems we can use the available results. In contrast to this, questions pertaining to the construction of discriminators and their analysis have a specific character, characteristic to range finding systems, and require, of course, detailed consideration.

As shown in Chapter VI, the most complete characteristics of a discriminator are the discrimination and fluctuation characteristics. Such characteristics in the case when the measured distance changes slowly as compared to all random components of the signal completely determine, together with characteristics of smoothing circuits, the accuracy of range finding.

The discrimination characteristic is the dependence of the mathematical expectation of output voltage of the discriminator on the magnitude of mismatch between the true and measured value of delay; the fluctuation characteristic is the dependence of the spectral density of the output voltage of the discriminator at zero frequency on the same mismatch. In general these characteristics are arbitrary nonlinear functions, possessing definite symmetry, and in accordance with

this, the range finder is a nonlinear system with feedback. Only with small mismatch, when with high probability errors of measurement are sufficiently small, can this system be linearized (see Chapter VI).

For analysis of the linearized system it is sufficient to have simpler characteristics of the discriminator — null shift Δ_0, slope $K_Д$ of the discrimination characteristic and equivalent spectral density $S_{экв}$, i.e., the ratio of spectral density of the output voltage of the discriminator at zero frequency with zero mismatch to the square of the gain factor. Calculation of parametric fluctuations in the measuring system requires additional calculation of their spectral density, which is determined by the corresponding formulas of Chapter VI.

Subsequently in most cases we shall limit ourselves to linearized consideration of a range finder without taking into account parametric fluctuations, in accordance with which analysis of the discriminator and comparison of various discriminators lead to determination of magnitudes Δ_0, $K_Д$, $S_{экв}$ and their comparison for various discriminators. Such factors, as the signal-to-noise ratio, width of the spectrum of fluctuations of the received signal, the law of modulation and the method of processing the signal affect accuracy of range finding only through these three quantities.

In this chapter we first shall consider operation of a discriminator for a fluctuating arbitrarily modulated coherent signal in noises and investigate its characteristics, determining potential accuracy of range finding. Then we shall generally investigate basic methods of approximate realization of these operations and, more specifically, final circuits of discriminators for certain of the most important forms of modulation. Then, for various smoothing circuits, corresponding to the various forms of statistics of change of the measured range, we shall analyze the range finder as a whole. And, finally, at the end of the chapter we shall consider questions connected with the influence of organized interferences on coherent range finders.

§ 7.2. Optimum Discriminator

7.2.1. Operations of an Optimum Discriminator

In accordance with the definition given in Chapter I, the received coherent signal is a nonstationary normal random process with a correlation function which, according to (1.4.3), is equal to

$$R(t_1, t_2; \tau) = P_c \operatorname{Re} u(t_1 - \tau) u^*(t_2 - \tau) \times \\ \times \rho(t_1 - t_2) e^{i\omega(t_1 - t_2)} + N_0 \delta(t_1 - t_2), \qquad (7.2.1)$$

where $u(t) = u_a(t)e^{i\psi(t)}$ – complex notation of the law of modulation, in general describing the amplitude ($u_a(t)$) and phase ($\psi(t)$) modulation of the sounding signal;

$\tau = 2d/c$ – delay of the reflected signal;

P_c – mean signal power;

$\rho(t)$ – correlation coefficient of fluctuations of the signal reflected from the target;

N_0 – spectral density of noise, and function $u(t)$ is normalized so that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |u(t)|^2\,dt = 1. \tag{7.2.2}$$

With periodic modulation in (7.2.2) it is possible to be limited to averaging over one period.

All statistical properties of the received signal $y(t)$ are characterized by the functional of the density of the distribution of probabilities of process $y(t)$, which on the basis of (4.3.7) in the case interesting is presented in the following form:

$$P(y\,|\,\tau) = C\exp\left\{-\frac{1}{2N_0}\int_0^T y^2(t)\,dt + \frac{1}{N_0}\int_0^T |Q(t,\tau)|^2 dt\right\}, \tag{7.2.3}$$

where C does not depend on $y(t)$, and $|Q(t, \tau)|^2$ is defined as

$$|Q(t, \tau)|^2 = \left|\int_{-\infty}^{t} h(t-s)\,u(s-\tau)\,y(s)\,e^{i\omega_0 s}\,ds\right|^2. \tag{7.2.4}$$

The pulse response of the filter $h(t)$ is given by its own Fourier transform $H_0(i\omega)$, the square of the modulus of which is equal to

$$|H_0(i\omega)|^2 = \frac{hS_0(\omega)}{1+hS_0(\omega)}, \tag{7.2.5}$$

where $h = P_c/2\Delta f_c N_0$ – signal-to-noise ratio, repeatedly used in the preceding chapters;

$S_0(\omega)$ – normalized spectral density of fluctuations of the signal ($S_0(0) = 1$);

Δf_c – effective width of the spectrum of signal fluctuations.

Function $h(t)$ describes the pulse response of an optimum filter of a coherent receiver and appears in all problems of optimum detection and measurement in noises. Its properties were already discussed in sufficient detail in Chapter IV.

Assignment of functional $P(y|\tau)$ permits us to find operations which should be

performed on the received signal y(t) by an optimum discriminator. As shown in Chapter VI, these operations consist of formation of the derivative with respect to τ of the integrand in the exponent of $P(y|\tau)$, i.e., the output signal of an optimum discriminator z(t) is defined as

$$z(t) = \frac{1}{N_0}\frac{\partial |Q(t,\tau)|^2}{\partial\tau} =$$

$$= \frac{1}{N_0}\frac{\partial}{\partial\tau}\left|\int\limits_{\infty}^{t} h(t-s)\,u(s-\tau)\,y(s)\,e^{i\omega_0 t}ds\right|^2. \tag{7.2.6}$$

Derivative $\frac{\partial |Q(t,\tau)|^2}{\partial\tau}$ is taken for a value of τ, equal to the measured value of delay obtained at the output of the range finder.

The latter is inserted in the discriminator by feedback and controls its tuning. Quantity z(t), describing the output voltage of the optimum discriminator, turns out here to be a function of the mismatch $\tau_0 - \tau$ between the true value of delay of the signal τ_0 and its estimated (measured) value τ.

Differentiating in (7.2.6), we obtain the following final expression, determining the operation of an optimum discriminator:

$$z(t;\tau_0-\tau) = -\frac{2}{N_0}\,\mathrm{Re}\int\limits_{-\infty}^{t}\int\limits_{-\infty}^{t} h(t-s_1)\,h(t-s_2)\times$$

$$\times\, u(s_1-\tau)\,u^{*\prime}(s_2-\tau)\,e^{i\omega_0(s_1-s_2)}y(s_1)\,y(s_2)\,ds_1ds_2. \tag{7.2.7}$$

These operations can be realized by the block diagram of Fig. 7.2. The received signal y(t), possibly, after preamplification in amplifiers, broad-band as compared to the width of the spectrum of modulation, proceeds to two mixers. As reference signals of the mixers we use the expected signal and its derivative with respect to delay. Generators of reference signals (modulated local oscillators, strobing-pulse generators, and so forth) are controlled by the output value of measured delay. Converted signals from the output of the mixers enter optimum filters with pulse response $h(t)\cos\omega_{\text{пр}}(t)$, and then a phase detector, whose

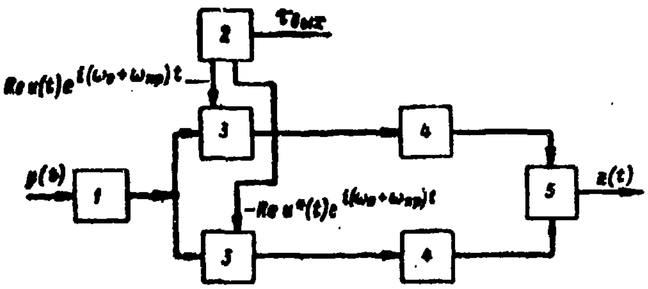

Fig. 7.2. Block diagram of an optimum discriminator: 1 – preamplifier; 2 – controlled coherent local oscillator; 3 – mixer; 4 – narrow-band filter; 5 – phase detector (multiplier).

output is the discriminator output. Processing of the signal in the first channel of the discriminator, obviously, coincides with predetection processing of a rapidly fluctuating coherent signal during detection (see Chapter IV), and processing in the second channel corresponds to calculation of correlation between the received signal and the derivative with respect to delay of the expected signal. By direct analysis of the block diagram of Fig. 7.2 it is simple to prove that, if only the magnitude of the intermediate frequency satisfies the usual requirement of elimination of image frequencies, output voltage of the phase detector indeed is described by formula (7.2.7), and, consequently, this circuit exactly executes optimum operations.

7.2.2. Discriminator Characteristics

Let us find now basic characteristics of an optimum discriminator, necessary for consideration of the measuring system as a whole. Averaging (7.2.7), replacing $s - \tau_0$ by s, introducing designation $\Delta = \tau_0 - \tau$ and using the slowness of change of function h(t) in comparison with u(t), we obtain

$$\overline{z(t,\ \Delta)} = -\frac{2}{N_0} \operatorname{Re} \int_{-\infty}^{t} \int_{-\infty}^{t} h(t - s_1) h(t - s_2) \times$$
$$\times u'(s_1 + \Delta) u^{*\prime}(s_2 + \Delta) e^{i\omega_0(s_1 - s_2)} [\operatorname{Re} P_c u(s_1) \times$$
$$\times u^*(s_2) e^{i\omega_0(s_1 - s_2)} \rho(s_1 - s_2) + N_0 \delta(s_1 - s_2)]\, ds_1 ds_2 =$$
$$= -\operatorname{Re} \left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T u^*(s) u(s + \Delta)\, ds \lim_{T\to\infty} \frac{1}{T} \int_0^T u(s) \times \right.$$
$$\times u^{*\prime}(s + \Delta) ds \frac{P_c}{N_0} \int_{-\infty}^{t} \int_{-\infty}^{t} h(t - s_1) h(t - s_2) \rho(s_1 - s_2)\, ds_1 ds_2 +$$
$$\left. + 2 \lim_{T\to\infty} \frac{1}{T} \int_0^T u(s) u^{*\prime}(s)\, ds \int_{-\infty}^{t} h^2(t - s_1)\, ds_1 \right\} =$$
$$= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{hS_0(\omega)}{1 + hS_0(\omega)} \operatorname{Re} \{C^{*\prime}(0) + hC(\Delta) C^{*\prime}(\Delta) S_0(\omega)\}\, d\omega, \qquad (7.2.8)$$

where $C(\Delta)$ – autocorrelation function of the law of modulation, determined by relationship

$$C(\Delta) = \lim_{T\to\infty} \frac{1}{T} \int_0^T u(s + \Delta) u^*(s)\, ds, \qquad (7.2.9)$$

where in accordance with the adopted normalization (7.2.2) $C(0) = 1$. Function $C(\Delta)$ from (7.2.9) is equal , obviously, to function $C(\Delta, 0)$, determined by formula (1.2.2).

For convenience we shall subsequently use the shorter notation. With a periodic modulation function, $C(\Delta)$ is periodic. However, in examining range finders mismatches Δ exceeding the period of repetition of the signal are not of practical interest. In accordance with this, subsequently, in examining all concrete examples it is sufficient to consider only one period of function $C(\Delta)$, corresponding to mismatches close to zero.

Function $C(\Delta)$ is the single characteristic of modulation of the sounding radar signal. Through it we find a description of the dependence of accuracy of range finding on the mode and parameters of modulation of the signal both in optimal, and also (we will prove this later) in nonoptimal systems. Furthermore, function $C(\Delta)$ is the characteristic of the potential resolution capability with respect to distance for a given signal [4]. Let us note certain important properties of function $C(\Delta)$. Separating in (7.2.9) the real and imaginary part

$$C(\Delta)=\lim_{T\to\infty}\frac{1}{T}\int_0^T u_a(s)\,u_a(s+\Delta)\cos[\psi(s+\Delta)-\psi(s)]\,ds+$$
$$+i\lim_{T\to\infty}\frac{1}{T}\int_0^T u_a(s)\,u_a(s+\Delta)\sin[\psi(s+\Delta)-\psi(s)]\,ds,$$

it is simple to prove that Re $C(\Delta)$ is an even function of Δ, and Im $C(\Delta)$ is an odd function. Therefore, $C'(0)$ is a purely imaginary quantity,* and the final expression for the discrimination characteristic $\overline{z(t,\Delta)}$ on the basis of (7.2.8) takes the form

$$\overline{z(t,\ \Delta)}=-\operatorname{Re}C(\Delta)\,C^{*\prime}(\Delta)\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{h^2S_0^2(\omega)\,d\omega}{1+hS_0(\omega)}. \qquad (7.2.10)$$

Furthermore, thanks to the fact that $C'(0)$ is purely imaginary, $\overline{z(t,\ 0)}=0$, i.e., there is no null shift of the discrimination characteristic in an optimum circuit.

Important parameters of function $C(\Delta)$ are values of its first and second derivatives at zero. Determining them by (7.2.9), we obtain

$$C'(0)=+ia=+i\lim_{T\to\infty}\frac{1}{T}\int_0^T u_a^2(t)\,\psi'(t)\,dt=$$
$$=+\frac{i}{2\pi}\int_{-\infty}^{\infty}\omega\lim_{T\to\infty}\frac{|U_T(i\omega)|^2}{T}\,d\omega, \qquad (7.2.11)$$

*Here and henceforth $C'(x)$ – derivative of function $C(x)$.

$$C''(0) = -b = -\lim_{T\to\infty}\frac{1}{T}\int_0^T\{[u'_a(t)]^2 + u_a^2(t)[\psi'(t)]^2\}\,dt =$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2\lim_{T\to\infty}\frac{|U_T(i\omega)|^2}{T}\,d\omega, \qquad (7.2.12)$$

where

$$U_T(i\omega) = \int_0^T u(t)\,e^{-i\omega t}\,dt \qquad (7.2.13)$$

is the spectrum of a finite segment of complex signal $u(t) = u_a(t)e^{i\psi(t)}$.

With periodic modulation, which occurs in most practical cases, it is sufficient to produce averaging in (7.2.11) and (7.2.12) in one period of modulation T_r. Here $U_T(i\omega)$ is the spectrum of one period of modulation. With stationary ergodic random modulation function $\lim_{T\to\infty}\frac{|U_T(i\omega)|^2}{T}$ coincides with the spectral density of modulation.

Positive values of a and b in (7.2.11) and (7.2.12) have the meaning of the mean frequency and the mean square of the frequency of the spectrum of modulation [4], and quantity $\sqrt{b - a^2}$, which, as we shall see presently, determines the accuracy of range finding, constitutes the mean square width of the spectrum of modulation, characterizing the rate of change of the modulating function. Quantity a differs from zero only with an asymmetric spectrum of modulation. In most real cases, for instance with only amplitude or only phase modulation, with simultaneous modulation in amplitude and phase by symmetric modulating functions, and in certain other cases, a = 0. Here, accuracy of range finding is determined only by the parameter of the law of modulation -- quantity b.

For analysis of a linearized measuring system without taking into account parametric fluctuations (see Chapter VI) the sufficient characteristic of the discriminator is the equivalent spectral density with zero mismatch, determined by relationship (6.2.8):

$$S_{\text{экв}} = \frac{S_{\text{экз}}}{K_A^2} = \int_{-\infty}^{\infty}\left[\overline{z(t,\,0)\,z(t+\tau,\,0)} - \right.$$

$$\left. - \overline{z(t,\,0)}\;\overline{z(t+\tau,\,0)}\right]d\tau \Big/ \left[\overline{\frac{\partial z(t,\,\Delta)}{\partial\Delta}}\bigg|_{\Delta=0}\right]^2.$$

In Chapter VI we show that for an optimum discriminator the magnitude of the equivalent spectral density $S_{\text{опт}}$ is

$$S_{\text{опт}} = \frac{1}{K_{\text{опт}}} = \frac{1}{\left.\frac{\partial \overline{z(t, \Delta)}}{\partial \Delta}\right|_{\Delta=0}}, \tag{7.2.14}$$

where $K_{\text{опт}}$ – slope of the discrimination characteristic of an optimum discriminator.

Substituting in (7.2.14) $\overline{z(t, \Delta)}$ from (7.2.10) and using (7.2.11) and (7.2.12), we obtain

$$S_{\text{опт}} = \left[(b - a^2)\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{h^2 S_0^2(\omega)\,d\omega}{1 + hS_0(\omega)}\right]^{-1} = \frac{F(h)}{2\Delta f_c(b - a^2)}, \tag{7.2.15}$$

where F(h) is a dimensionless function of the signal-to-noise ratio h, depending only on the form of spectral density of fluctuations of the received signal ($h = P_c/2\Delta f_c N_0$).

Thus, the dependence of $S_{\text{опт}}$ and potential errors of range finding on the parameters of the law of modulation and the signal-to-noise ratio has a fairly simple character. Quantity $S_{\text{опт}}$, and, consequently, variance of fluctuating error of measurement with linear smoothing circuits are proportional to $S_{\text{экв}}$, are inversely proportional to the mean square width of the spectrum of the modulating signal. This signifies, in particular, that from the point of view of accuracy of range finding all other characteristics of the modulating signal besides spectrum width do not play a role, and, applying various modulating signals with identical spectrum width, we, in principle, should obtain identical accuracy of range finding. Therefore, being interested only in accuracy of range finding, it is impossible to give preference to one or the other form of sounding signal, of course, with identical spectrum width, and final selection of the method of modulation of the sounding signal requires us to turn to other considerations, such as uniqueness, resolution capability, possibility of simple measurement of Doppler frequency, technical realizability, and so forth.

The dependence of $S_{\text{опт}}$ on the signal-to-noise ratio is described by function F(h), which has the following general properties. It is obvious that for large values of h

$$F(h) = \frac{1}{\frac{1}{2\pi\Delta f_c}\int_{-\infty}^{\infty}\frac{h^2 S_0^2(\omega)d\omega}{1 + hS_0(\omega)}} \xrightarrow[h\to\infty]{} \frac{1}{\frac{1}{2\pi\Delta f_c}\int_{-\infty}^{\infty} hS_0(\omega)d\omega} = \frac{1}{h}, \tag{7.2.16}$$

since, thanks to normalization of $S_0(\omega)$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_0(\omega)\,d\omega = \Delta f_c,$$

and for small h

$$F(h)\underset{h\to 0}{\to}\frac{1}{\frac{1}{2\pi\Delta f_c}\int_{-\infty}^{\infty} h^2 S_0^2(\omega)\,d\omega}=\frac{1}{\alpha h^2}, \qquad (7.2.17)$$

where α — numerical coefficient of order (α = 0.5-1.0).

The exact form of function F(h) depends on the form of spectrum $S_0(\omega)$; however, this dependence is not very essential, especially with such magnitudes of h, which can be considered working magnitudes in measuring systems, i.e., h > 5-10. For the two extreme cases (of the spectral density $S_0(\omega) = \left[1 + \left(\frac{\omega}{2\Delta f_c}\right)^2\right]^{-1}$ dropping most slowly to $\pm\infty$, corresponding to exponential correlation function, and a rectangular spectrum of fluctuations with the same effective width Δf_c) in Fig. 7.3 there are given dependences which have the form:

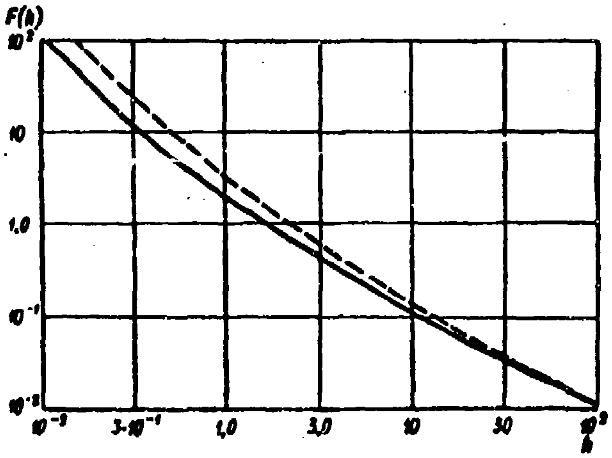

Fig. 7.3. Dependence of function F(h) on the signal-to-noise ratio. —— square spectrum of fluctuations; --- exponential correlation function of fluctuations.

for the first case

$$F(h)=\frac{\sqrt{1+h}\,(1+\sqrt{1+h})}{h^2}, \qquad (7.2.18)$$

for second case

$$F(h)=\frac{1+h}{h^2}. \qquad (7.2.19)$$

These curves simultaneously show the dependence of fluctuating error of range finding by a system with an optimum discriminator on the signal-to-noise ratio.

We shall also discuss the influence of the width of the spectrum of fluctuations of the signal on accuracy of measurement. It is obvious that for a sufficiently large h, when $F(h) \approx 1/h$, ratio $\frac{F(h)}{2\Delta f_c} \approx \frac{N_0}{P_c}$ does not depend on Δf_c.

However, for not very large h expansion of the spectrum of fluctuations worsens accuracy of measurement, since components of equivalent spectral density, caused by beats of noises, and having an order of $1/h^2$, increase with growth of Δf_c.

Let us now find the fluctuation characteristic of an optimum discriminator. Here, we recalculate fluctuations of output voltage of the discriminator into fluctuations of the measured parameter (delay) by dividing the output spectral density by the square of the slope. Then, in accordance with (6.2.8)

$$S_{\text{опт}}(\Delta)=\frac{1}{K^2_{\text{опт}}}\int\limits_{-\infty}^{\infty}\left[\overline{z(t,\ \Delta)\,z(t+\tau,\ \Delta)}-\right.$$
$$\left.-\overline{z(t,\ \Delta)}\ \overline{z(t+\tau,\ \Delta)}\right]d\tau.$$

Substituting (7.2.7) in this expression and averaging analogously to (7.2.8), we obtain

$$S_{\text{опт}}(\Delta)=\frac{1}{K^2_{\text{опт}}}\,\mathrm{Re}\left\{\left[|C(\Delta)|^2|C'(\Delta)|^2+C^{*2}(\Delta)C'^2(\Delta)\right]\frac{1}{\pi}\times\right.$$
$$\times\int\limits_{-\infty}^{\infty}\frac{h^4S_0^4(\omega)\,d\omega}{[1+hS_0(\omega)]^2}+\left[b|C(\Delta)|^2+|C'(\Delta)|^2+\right.$$
$$\left.+2iaC^*(\Delta)C'(\Delta)\right]\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{h^3S_0^3(\omega)\,d\omega}{[1+hS_0(\omega)]^2}+$$
$$\left.+(b-a^2)\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{h^2S_0^2(\omega)\,d\omega}{[1+hS_0(\omega)]^2}\right\}. \qquad (7.2.20)$$

From this expression it follows that when $\Delta \neq 0$ quantity $S_{\text{опт}}(\Delta)$ no longer tends to zero as $h\to\infty$, i.e., even with zero noises the output voltage of the discriminator contains random components, caused by fluctuations of the reflected signal and depending on mismatch. For illustration in Fig. 7.4 there is constructed a family of fluctuation characteristics, corresponding to a rectangular spectrum of fluctuations of the signal and the autocorrelation function of the law of modulation

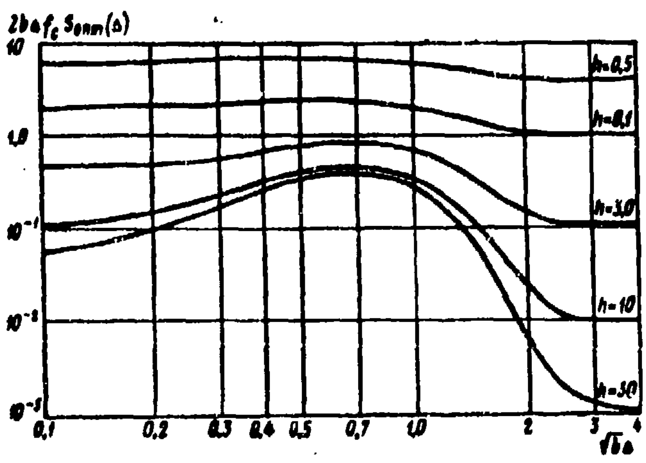

Fig. 7.4. Fluctuation characteristic of an optimum discriminator.

$C(\Delta) = \exp\left\{-\frac{1}{2}b\Delta^2\right\}$ for various values of h. From the figure it is clear that irregularities of the fluctuation characteristic start to appear when $h \sim 1$. A stationary level of equivalent spectral density is attained already when $\sqrt{b\Delta} =$ $= 1.5$-3, i.e., with mismatch exceeding the width of the basic lobe of autocorrelation function by a factor of one and a half to three.

Besides $S_{\text{экв}}$, of greatest practical interest for analysis of a range finder is the spectral density of equivalent parametric fluctuations $S_{\text{пар}}$, which determines parametric error of measurement (see Chapter IV) and is equal to the coefficient of Δ^2 in the expansion of $S_{\text{экв}}(\Delta)$ in powers of Δ. Then, as follows from (7.2.20), this quantity is equal to

$$S_{\text{пар}} = \frac{\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{h^4 S_0^4(\omega)\,d\omega}{[1+hS_0(\omega)]^2}}{\left[\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{h^2 S_0^2(\omega)\,d\omega}{1+hS_0(\omega)}\right]^2} \qquad (7.2.21)$$

and depends only on beats of signals in the two discriminator channels. The presence of noises does not increase parametric fluctuations in an optimum discriminator. The influence of parametric fluctuations on accuracy of the range finder as a whole and on the relationship between usual and parametric fluctuating errors we shall consider subsequently.

In conclusion let us discuss one more interpretation of quantity $S_{\text{опт}}$. In Chapter VI it is shown that if one were to assume constancy of the measured parameter throughout an interval of observation of duration T and construct some estimate of the parameter from values of the accepted realization of the signal, variance of this estimate, characterizing accuracy of measurement of the parameter, cannot be less than a certain quantity, called variance of the efficient estimate. Here, for instance, estimation of the parameter by the maximum likelihood method ([18], Chapter VI) ensures variance, practically equal to variance of the efficient estimate. In Chapter VI it is shown that variance of the efficient estimate is connected with quantity $S_{\text{опт}}$ in the following way:

$$\sigma^2_{\text{эф}} = \frac{S_{\text{опт}}}{T}. \qquad (7.2.22)$$

Consequently, maximum accuracy of measurement of delay of the signal received from the target (assuming its constancy during the time during which we produce this measurement) for the case when we make no assumptions with respect to statistical

characteristics of measured delay, but produce a simple estimate from the functional of likelihood, is determined by

$$\sigma^2_{э\phi} = \frac{S_{0э\tau}}{T} = \frac{F(h)}{2\Delta f_c T(b-a^2)}. \qquad (7.2.23)$$

This formula is valid when $\Delta f_c T \gg 1$. In the opposite case, when the time of observation is small as compared to the time of correlation of the signal($\Delta f_c T \ll 1$), which may be of interest for survey radars in which range finding is produced during every cycle of survey for a small time, we must return to the general formula for variance of the efficient estimate (6.7.33), from which it follows that

$$\sigma^2_{э\phi} = \left[\frac{1}{2}\int_0^T\int_0^T \frac{\partial R(t_1, t_2; \tau)}{\partial\tau}\,\frac{\partial W(t_1, t_2; \tau)}{\partial\tau}\,dt_1 dt_2\right]^{-1}, \qquad (7.2.24)$$

where $R(t_1, t_2; \tau)$ – function of correlation of the signal with noise (7.2.1);

$W(t_1, t_2; \tau)$ – its reciprocal in interval (0, T) (Chapters I, IV).

In accordance with (4.3.9) for $\Delta f_c T \ll 1$

$$W(t_1, t_2; \tau) = \frac{1}{N_0}\delta(t_1 - t_2) - \frac{P_c}{N_0^2(1+P_c T/2N_0)} \times$$
$$\times \operatorname{Re} u(t_1 - \tau) u^*(t_2 - \tau) e^{i\omega_0(t_1 - t_2)}.$$

Performing the calculation in (7.2.24), we obtain

$$\sigma^2_{э\phi} = \frac{1+\mu}{2\mu^2(b-a^2)}, \qquad (7.2.25)$$

where the signal-to-noise ratio $\mu = P_c T/2N_0$ is equal to the ratio of energy of the received signal during the time of observation T to the physical (one-way) spectral density of noise $2N_0$. Let us note that formula (7.2.25) formally coincides with the expression for $\sigma^2_{э\phi}$ from (7.2.23), if we consider the spectrum of fluctuations of the signal rectangular, with width $\Delta f_c = \frac{1}{T}$.

§ 7.3. Discriminator with Two Channels Detuned with Respect to Range

7.3.1. Block Diagram of the Discriminator

Exact technical fulfillment of an optimum discriminator circuit is hardly possibly. In practice, modulation of the reference signal utilized in the discriminator will somewhat differ from modulation of the sounding signal, the frequency response of the filter will differ from the optimum, etc. In this and subsequent paragraphs we shall consider the influence of different deviations from optimum

processing of the signal on accuracy of range finding.

One approximate method of realizing an optimum discriminator is directly indicated by formula (7.2.6) and consists of replacement of the derivative in the expression for z(t) by a finite difference. Since

$$\frac{\partial |Q(t, \tau)|^2}{\partial \tau} \approx \frac{1}{2\delta}\{|Q(t, \tau+\delta)|^2 - |Q(t, \tau-\delta)|^2\},$$

output voltage of the optimum discriminator with an accuracy of an immaterial constant coefficient is approximately determined by the following quantity:

$$\begin{aligned} z(t, \Delta) \approx |Q(t, \tau+\delta)|^2 - |Q(t, \tau-\delta)|^2 = \\ = \left|\int_{-\infty}^{t} h(t-s)\, u(s-\tau-\delta)\, y(s)\, e^{i\omega_0 s} ds\right|^2 - \\ - \left|\int_{-\infty}^{t} h(t-s)\, u(s-\tau+\delta)\, y(s)\, e^{i\omega_0 s} ds\right|^2. \end{aligned} \quad (7.3.1)$$

It is obvious that the degree of approximation in general is better, the less the magnitude of detuning δ; quantitative characteristics of the degree of approximation of operation (7.3.1) to (7.2.6) can be obtained in the course of analysis.

Operations (7.3.1) are realized by two channels, each of which is built just as an optimum receiver of a rapidly fluctuating signal in conditions of detection (Chapter IV). Reference signals utilized in the channels have a difference in delays of 2δ. The corresponding block diagram is shown in Fig. 7.5. The received signal y(t) is fed to two mixers, in which it is multiplied by the expected signal, shifted in frequency the magnitude of intermediate frequency $\omega_{\text{пр}}$ and in delay by quantities $\tau_0 - \Delta + \delta$ and $\tau_0 - \Delta - \delta$, respectively ($\Delta = \tau_0 - \tau$). In practice heterodyning of the signal and its multiplication by a modulating signal, for instance, a delayed strobe in pulse modulation, can be carried out separately, but for analysis this is not important. Output signals of mixers are filtered in narrow-band filters, tuned to the intermediate frequency, and, after detection by square-law detectors, they are subtracted. The physical

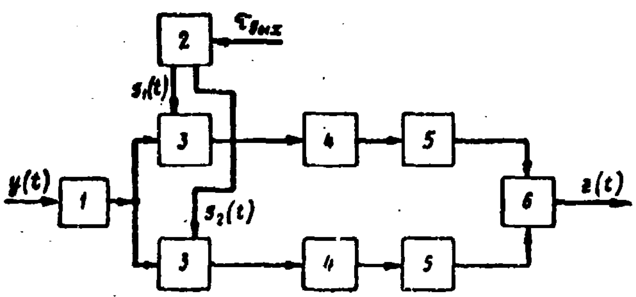

Fig. 7.5. Block diagram of a discriminator with two staggered channels: 1 — preamplifier; 2 — controlled modulated coherent local oscillator; 3 — mixer; 4 — narrow-band filter; 5 — square-law detector; 6 — subtractor.

$s_1(t) = \operatorname{Re} u_1(t-\tau-\delta)\, e^{i(\omega_0-\omega_{\text{пр}})t}$, $s_2(t) = \operatorname{Re} u_2(t-\tau+\delta)\, e^{i(\omega_0+\omega_{\text{пр}})t}$.

essence of all these operations, besides subtraction, obviously, is the same as during detection (Chapter IV), and needs no further discussion.

In practice, reference signals in such a circuit may differ somewhat from sounding signals (for instance, noncoincidence of the form of the strobe and the pulse), and also due to any kind of nonidentity may differ among themselves. Therefore, in general, in one of the channels there is used reference signal $u_1(t) \neq \neq u(t)$, and in other, $u_2(t) \neq u(t)$. Furthermore, the frequency response of the narrow-band filters may also differ from optimum (7.2.5). Taking into account these circumstances output voltage of the discriminator of Fig. 7.5, in general, is described by the following expression:

$$z(t,\ \Delta) = \left| \int_{-\infty}^{t} h_1(t-s)\, u_1(s-\tau_0+\Delta-\delta)\, y(s)\, e^{i\omega_0 s} ds \right|^2 - \\ - \left| \int_{-\infty}^{t} h_2(t-s)\, u_2(s-\tau_0+\Delta+\delta)\, y(s)\, e^{i\omega_0 s} ds \right|^2, \qquad (7.3.2)$$

where to pulse responses $h_1(t)$ and $h_2(t)$ there correspond frequency responses $H_1(i\omega)$ and $H_2(i\omega)$, and the true value of delay in the formula for the correlation function of $y(t)$ (7.1.1) is designated τ_0.

7.3.2. Identical Modulation of the Reference and Sounding Signals

Of all idealizations of real discriminators, probably, the most fundamental is the difference of the frequency responses of narrow-band filters from optimum, although quantitively this distinction is not necessarily the most important. The fact is that in an optimum system the frequency response of the filter depends on the magnitude of the signal-to-noise ratio and should, in principle, be corrected in the course of operation as the latter changes.

It is of considerable interest to consider a discriminator with filters not depending on the signal-to-noise ratio h, and to investigate its behavior during change of this ratio. In particular, it is interesting to consider that case when for a certain h we achieve coincidence of the frequency responses of optimum and real filters, but for all others they differ.

In order to investigate the influence of the difference of frequency responses of filters from optimum we shall first consider a somewhat idealized case, when $u_1(t) = u_2(t) = u(t)$, $h_1(t) = h_2(t) \neq h(t)$. Presenting in (7.3.2) the squares of moduli in the form of the product of complex conjugate integrals and averaging the the product $y(s_1)y(s_2)$, we obtain the following expression for the discrimination characteristic:

$$\overline{z(t,\ \Delta)}=\int_{-\infty}^{t}\int_{-\infty}^{t}h_1(t-s_1)\,h_1(t-s_2)\,[u(s_1-\tau_0+\Delta-\delta)\times$$
$$\times u^*(s_2-\tau_0+\Delta-\delta)-u(s_1-\tau_0+\Delta+\delta)\times$$
$$\times u^*(s_2-\tau_0+\Delta+\delta)]\,e^{i\omega_0(s_1-s_2)}[N_0\delta(s_1-s_2)+$$
$$+\operatorname{Re}P_c\,u(s_1-\tau_0)\,u^*(s_2-\tau_0)\,\rho(s_1-s_2)\,e^{i\omega_0(s_1-s_2)}]\,ds_1ds_2=$$
$$=\left\{\left|\lim_{T\to\infty}\frac{1}{T}\int_0^T u(s+\Delta-\delta)\,u^*(s)\,ds\right|^2-\right.$$
$$\left.-\left|\lim_{T\to\infty}\frac{1}{T}\int_0^T u(s+\Delta+\delta)\,u^*(s)\,ds\right|^2\right\}\times$$
$$\times\frac{P_c}{2}\int_{-\infty}^{t}\int_{-\infty}^{t}h_1(t-s_1)\,h_1(t-s_2)\,\rho(s_1-s_2)\,ds_1ds_2=$$
$$=\frac{P_c}{2\Delta f_c}\,[|C(\Delta-\delta)|^2-|C(\Delta+\delta)|^2]\,\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^2\,S_0(\omega)\,d\omega, \qquad (7.3.3)$$

from which it follows that null shift $\Delta_0 = 0$, and steepness of the discrimination characteristic

$$K_{\text{д}}=-\operatorname{Re}[C'(\delta)\,C^*(\delta)]\,\frac{P_c}{\pi\Delta f_c}\int_{-\infty}^{\infty}|H_1(i\omega)|^2\,S_0(\omega)\,d\omega. \qquad (7.3.4)$$

This equality is satisfied thanks to properties of the real and imaginary parts of $C(\Delta)$, which guarantees that $C(-\Delta) = C^*(\Delta)$ and $C'(-\Delta) = -C^{*\prime}(\Delta)$.

Likewise, we can obtain an expression for the correlation function, and through it an expression for the spectral density of the output voltage. We shall do this in some detail:

$$R_z(t_1,\ t_2)=\overline{z(t_1,\ \Delta)\,z(t_2,\ \Delta)}-\overline{z(t_1,\ \Delta)}\ \overline{z(t_2,\ \Delta)}=$$
$$=\int_{-\infty}^{t_1}\int_{-\infty}^{t_1}\int_{-\infty}^{t_2}\int_{-\infty}^{t_2}e^{i\omega_0(s_1-s_2+s_3-s_4)}h_1(t_1-s_1)\,h_1(t_1-s_2)\times$$
$$\times h_1(t_2-s_3)\,h_1(t_2-s_4)\,[u(s_1-\tau_0+\Delta-\delta)\,u^*(s_2+\tau_0+\Delta-\delta)-$$
$$-u(s_1-\tau_0+\Delta+\delta)\,u^*(s_2-\tau_0+\Delta+\delta)]\times$$
$$\times u(s_3-\tau_0+\Delta-\delta)\,u^*(s_4-\tau_0+\Delta-\delta)]-$$
$$-[u(s_3-\tau_0+\Delta+\delta)\,u^*(s_4-\tau_0+\Delta+\delta)]\times$$
$$\times[\overline{y(s_1)y(s_2)y(s_3)y(s_4)}-\overline{y(s_1)y(s_2)}\ \overline{y(s_3)y(s_4)}]\,ds_1ds_2ds_3ds_4. \qquad (7.3.5)$$

Further calculations can be made with the help of the known expression for the fourth mixed moment of a multi-dimensional normal distribution

$$\overline{y(s_1)y(s_2)y(s_3)y(s_4)}=\overline{y(s_1)y(s_2)}\ \overline{y(s_3)y(s_4)}+$$
$$+\overline{y(s_1)y(s_3)}\ \overline{y(s_2)y(s_4)}+\overline{y(s_1)y(s_4)}\ \overline{y(s_2)y(s_3)}=$$
$$=R(s_1,\ s_2)\,R(s_3,\ s_4)+R(s_1,\ s_3)\,R(s_2,\ s_4)+$$
$$+R(s_1,\ s_4)\,R(s_2,\ s_3). \qquad (7.3.6)$$

During calculation of statistical characteristics of quantity $z(t, \Delta)$ one should consider the following circumstance, appearing due to use of complex notation, and essentially simplifying the calculations. Output voltage of the discriminator $z(t, \Delta)$ in optimum and nonoptimal circuits is in the form of the sum of a product of form $\xi(t)\eta^*(t)$, where $\xi(t)$ and $\eta(t)$ are complex high-frequency normal random processes, whose total spectrum width is small as compared to the magnitude of the carrier frequency, identical for both processes.

A known property of such processes is equality to zero of the mathematical expectation of products $\overline{\xi(t_1)\xi(t_2)}$ and $\overline{\xi(t_1)\eta(t_2)}$ for any t_1 and t_2. Then from (7.3.6) it follows that

$$\overline{\xi(t_1)\eta^*(t_1)\xi(t_2)\eta^*(t_2)} = \overline{\xi(t_1)\eta^*(t_1)}\,\overline{\xi(t_2)\eta^*(t_2)} + \\ + \overline{\xi(t_1)\eta^*(t_2)}\,\overline{\xi(t_2)\eta^*(t_1)}.$$

This circumstance gives us the possibility during substitution of (7.3.6) in (7.3.5) to reject product $R(s_1, s_3)R(s_2, s_4)$, integration of which with factor $\exp\{i\omega_0(s_1 - s_2 + s_3 - s_4)\}$ all the same gives zero thanks to rapid oscillation of the integrand. Substituting the remaining terms in the integral, replacing $s - \tau_0$ by s, and rejecting the remaining high-frequency terms, we obtain

$$R_z(t_1, t_2) = \int_{-\infty}^{t_1}\int_{-\infty}^{t_2}\int_{-\infty}^{t_1}\int_{-\infty}^{t_2} h_1(t_1 - s_1)\,h_1(t_1 - s_3)\times \\ \times h_1(t_2 - s_2)\,h_1(t_2 - s_4)\,[u(s_1 + \Delta - \delta)\,u^*(s_2 + \Delta - \delta) - \\ - u(s_1 + \Delta + \delta)\,u^*(s_2 + \Delta + \delta)]\,[u(s_3 + \Delta - \delta)\,u^*(s_4 + \Delta - \delta) - \\ - u(s_3 + \Delta + \delta)\,u(s_4 + \Delta + \delta)]\times \\ \times\Big[\frac{1}{4}P_c^2\,u^*(s_1)\,u(s_2)\,u^*(s_3)\,u(s_4)\,\rho(s_1 - s_4)\,\rho(s_2 - s_3) + \\ + N_0^2\,\delta(s_1 - s_4)\,\delta(s_2 - s_3) + \frac{1}{2}P_cN_0u^*(s_1)\,u(s_4)\times \\ \times\rho(s_1 - s_4)\,\delta(s_2 - s_3) + \frac{1}{2}P_cN_0u(s_2)\,u^*(s_3)\,\rho(s_2 - s_3)\times \\ \times\delta(s_1 - s_4)\Big]\,ds_1ds_2ds_3ds_4.$$

Averaging in this expression modulating functions $u(t)$ over the time under the sign of the integral, integrating the expression for $R_z(t_1, t_2)$ over $t_2 - t_1$, and dividing the obtained expression for spectral density at zero by the square of steepness, for the fluctuation characteristic of the discriminator we have

$$S_{\text{экв}}(\Delta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^4\Big\{\frac{1}{2}[|C(\Delta+\delta)|^2-$$
$$-|C(\Delta-\delta)|^2]h^2S_0^2(\omega)+[|C(\Delta+\delta)|^2+|C(\Delta-\delta)|^2-$$
$$-2\operatorname{Re}C^*(2\delta)C^*(\Delta-\delta)C(\Delta+\delta)]hS_0(\omega)+[1-|C(2\delta)|^2]\Big\}d\omega:$$
$$:8h^2\left[\operatorname{Re}C'(\delta)C^*(\delta)\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^2S_0(\omega)\,d\omega\right]^2. \qquad (7.3.7)$$

It is not difficult to prove that with small detunings δ and optimum frequency response this expression completely coincides with (7.2.20). From (7.3.7) for equivalent spectral density $S_{\text{экв}} = S_{\text{экв}}(0)$ for zero mismatch we can obtain the following expression:

$$S_{\text{экв}}=\frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^4\{1-|C(2\delta)|^2+2hS_0(\omega)[|C(\delta)|^2-\operatorname{Re}C^*(2\delta)C^2(\delta)]\}d\omega}{8h^2\left[\operatorname{Re}C'(\delta)C^*(\delta)\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^2S_0(\omega)\,d\omega\right]^2}. \qquad (7.3.8)$$

Investigating the dependence of this expression on δ, it is simple to prove (this is easily done, considering small δ, when $C(\delta) \approx 1 + ia\delta - \frac{1}{2}b\delta^2$), that $S_{\text{экв}}$ monotonically decreases with decrease of δ, passing to the limit as $\delta \to 0$, which for $|H_1(i\omega)|^2 = |H_0(i\omega)|^2$ from (7.2.5) is equal to $S_{\text{опт}}$ from (7.2.15). This fact leads to the requirement of realization of small detuning between channels of the discriminator. However, with decrease of δ proportionally to $(b - a^2)\delta$ steepness of the discrimination characteristic decreases. This means that to preserve the required high speed operation of the closed measuring system it is necessary proportionally to decrease of δ to increase gain in the open circuit of the range finder. This circumstance forces us to take in real systems, built on the considered scheme, such δ for which steepness $K_{\text{д}}$ is not very small or is even maximum. Fortunately, as we shall subsequently prove in concrete examples, the value of $S_{\text{экв}}$ with increase of δ up to values of δ, turning into a maximum the steepness $K_{\text{д}}$, differs little from the value of $S_{\text{экв}}$ with zero detuning δ. This gives us the possibility, without specifying the form of modulation, to investigate the dependence of $S_{\text{экв}}$ on characteristics of filters.

For δ, close to zero, using the expansion for $C(\delta)$, from formula (7.3.8) we obtain:

$$S_{\text{экв}} = \frac{1}{2h^2(b-a^2)} \frac{\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} |H_1(i\omega)|^4[1+hS_0(\omega)]d\omega}{\left[\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} |H_1(i\omega)|^2 S_0(\omega)d\omega\right]^2}. \tag{7.3.9}$$

Let us calculate the magnitude of $S_{\text{ЭКВ}}$ for certain of the most commonly used forms of spectral density of fluctuations and frequency response of a filter. If spectral density $S_0(\omega)$ is uniform in band $\Delta f_{\text{С}}$, and the filter has a rectangular frequency response with band $\Delta f_{\text{Ф}}$, then

$$S_{\text{экв}} = \begin{cases} \dfrac{1+h}{2(b-a^2)\Delta f_{\text{ф}} h^2}, & \Delta f_{\text{ф}} < \Delta f_{\text{с}}, \\ \dfrac{\frac{\Delta f_{\text{ф}}}{\Delta f_{\text{с}}} + h}{2(b-a^2)\Delta f_{\text{с}} h^2}, & \Delta f_{\text{ф}} > \Delta f_{\text{с}}. \end{cases} \tag{7.3.10}$$

If the correlation function of the signal is approximated by an exponential function, i.e.,

$$S_0(\omega) = \frac{1}{1+\left(\frac{\omega}{2\Delta f_{\text{с}}}\right)^2}, \tag{7.3.11}$$

and narrow-band filtration is carried out by an LRC-filter with the response of a low-frequency equivalent

$$|H_1(i\omega)|^2 = \frac{1}{1+\left(\frac{\omega}{2\Delta f_{\text{ф}}}\right)^2}, \tag{7.3.12}$$

where $\Delta f_{\text{С}}$ – effective width of the spectrum of fluctuations;

$\Delta f_{\text{Ф}}$ – effective pass band of the filter, then

$$S_{\text{экв}} = \frac{\left(1+\frac{\Delta f_{\text{ф}}}{\Delta f_{\text{с}}}\right)^2 + h\left(1+2\frac{\Delta f_{\text{ф}}}{\Delta f_{\text{с}}}\right)}{4(b-a^2)h^2\Delta f_{\text{ф}}}. \tag{7.3.13}$$

If spectral density $S_0(\omega)$ has the form (7.3.11), and filtration is carried out by a filter with a rectangular frequency response, then

$$S_{\text{экв}} = \frac{\frac{\pi\Delta f_{\text{ф}}}{2\Delta f_{\text{с}}} + h\,\text{arctg}\,\frac{\pi\Delta f_{\text{ф}}}{2\Delta f_{\text{с}}}}{\frac{4h^2}{\pi}\Delta f_{\text{с}}(b-a^2)\left(\text{arctg}\,\frac{\pi\Delta f_{\text{ф}}}{2\Delta f_{\text{с}}}\right)^2}, \tag{7.3.14}$$

This expression practically coincides for all values of $\Delta f_\Phi / \Delta f_c$ with (7.3.10).

Let us further consider the case of a spectrum of fluctuations, approaching $\pm\infty$ somewhat more rapidly than (7.3.11), when

$$S_\varphi(\omega) = \frac{1}{\left[1+\left(\frac{\omega}{4\Delta f_c}\right)^2\right]^2}, \tag{7.3.15}$$

and that of the corresponding filter

$$|H_2(i\omega)|^2 = \frac{1}{\left[1+\left(\frac{\omega}{4\Delta f_\Phi}\right)^2\right]^2}. \tag{7.3.16}$$

In this case

$$S_{экв} = \frac{5(1+x)^4 + h[5+20x+29x^2+16x^3]}{16(b-a^2)\Delta f_\Phi x(1+2x)^3}, \tag{7.3.17}$$

where $x = \frac{\Delta f_\Phi}{\Delta f_c}$.

Comparing these expressions with formulas for the case of an optimum filter (7.2.15) for a uniform spectrum of fluctuations in band Δf_c and a rectangular frequency response, we obtain

$$\frac{S_{экв}}{S_{опт}} = \begin{cases} \frac{1}{x}, & x<1, \\ \frac{h+x}{1+h}, & x>1. \end{cases} \tag{7.3.18}$$

For spectrum (7.3.11) and filter (7.3.12) this ratio is equal to

$$\frac{S_{экв}}{S_{опт}} = \frac{(1+x^2)+h(1+2x)}{2x\sqrt{1+h}\,(1+\sqrt{1+h})}. \tag{7.3.19}$$

Dependences (7.3.18) and (7.3.19) are shown in Fig. 7.6. Analysis of them shows that, as one should have expected, ratio $S_{экв}/S_{опт}$ for a certain $x = \frac{\Delta f_\Phi}{\Delta f_c}$ attains a minimum, equal to one. For a rectangular spectrum this minimum is reached at $x = 1$, and for the spectrum of (7.3.11) it is when $x = \sqrt{1+h}$. Expansion of the passband of the filter, which is most interesting from the technical point of view, leads to especially unpleasant consequences for a small signal-to-noise ratio. When $h \ll 1$ ratio $S_{экв}/S_{опт}$ has the order of x, i.e., fluctuation error grows proportionally to the passband of the filter. For larger values of h expansion of the band plays a considerably smaller role. Ratio $S_{экв}/S_{опт}$ here has

the order 1 + (0.5 to 1)$\frac{x}{h}$, and with growth of h it approaches one. At working signal-to-noise ratios (h ~ 3 to 10) and expansion of the band to x ~ 10 increase of $S_{\text{экв}}$ does not exceed 50% for the spectrum of (7.3.11) and 3% for a rectangular spectrum.

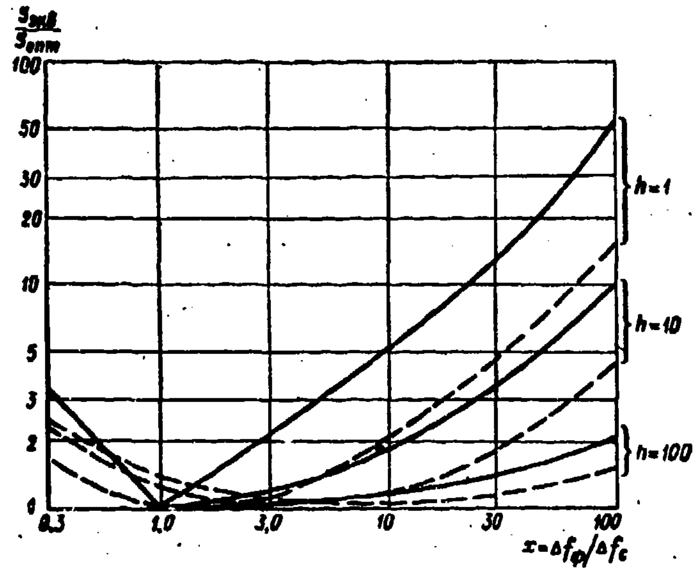

Fig. 7.6. Influence of the passband of the filter on accuracy of range finding: —— rectangular spectrum of fluctuations and rectangular frequency response of the filter; --- exponential correlation function of fluctuations and exponential pulse response of the filter.

Let us give another expression for the spectral density of parametric fluctuations. Expanding $S_{\text{экв}}(\Delta)$ from (7.3.7) in the vicinity of $\Delta = 0$, we obtain

$$S_{\text{пар}} = \frac{\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |H_1(i\omega)|^4 S_0^2(\omega)\, d\omega}{\left[\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |H_1(i\omega)|^2 S_0(\omega)\, d\omega\right]^2} + \frac{1}{4h}\,\frac{\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |H_1(i\omega)|^4 S_0(\omega)\, d\omega}{\left[\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |H_1(i\omega)|^2 S_0(\omega)\, d\omega\right]^2} \times$$

$$\times \frac{\mathrm{Re}\,\{|C'(\delta)|^2 + C''^{*}(\delta)\, C^{*}(2\delta) + C(\delta)\,[C^{*''}(\delta) - C^{*}(2\delta)\, C''(\delta)]\}}{[\mathrm{Re}\, C'(\delta)\, C^{*}(\delta)]^2}. \quad (7.3.20)$$

This expression shows that, in distinction from an optimum circuit, in the given case parametric fluctuations increase without limit with decrease of h;

however, since as $h \to 0$ normal fluctuations grow as $1/h^2$, they play a considerably large role. With high signal-to-noise ratios h, when the influence of parametric fluctuations is substantial, the term with $1/h$ in $S_{\text{пap}}$ is small and does not play a role. Furthermore, the coefficient of this term, depending on δ, for not very large values of δ is small. As $\delta \to 0$ it has the order of δ^2, increasing with increase of δ up to values, ensuring maximum steepness of the order of one. Let us note that the tendency of $S_{\text{пap}}$ for small δ to reach a finite limit, not depending on h, is one more basis for selection in real circuits of sufficiently small detunings.

It is interesting to note that the main component of parametric fluctuations, not depending on the signal-to-noise ratio in the two-channel circuit, depends neither on the magnitude of detuning nor the form of modulation in general. This component is equal to

$$\lim_{h\to\infty} S_{\text{пap}} = \frac{\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |H_1(i\omega)|^4 S_0^2(\omega)\,d\omega}{\left[\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |H_1(i\omega)|^2 S_0(\omega)\,d\omega\right]^2} = \frac{\alpha}{\Delta f_c}, \tag{7.3.21}$$

where coefficient α has the order of one, insignificantly changing with change of spectral density of the signal and of frequency response of the filter. In particular, for rectangular $S_0(\omega)$ and $|H_1(i\omega)|^2$ ($\Delta f_{\Phi} \geq \Delta f_c$) coefficient $\alpha = 1$, and for $S_0(\omega)$ and $|H_1(i\omega)|^2$ from (7.3.11) and (7.3.12)

$$\alpha = \frac{1+x}{x} + \frac{1}{1+x} \quad (x = \Delta f_{\Phi}/\Delta f_c).$$

7.3.3. Influence of the Finite Magnitude of Detuning

Without specifying yet the form of modulation, we can make certain important conclusions about the influence of the finite magnitude of detuning δ on characteristics of the discriminator. Turning to formula (7.3.8) for $S_{\text{экв}}$ with arbitrary detuning, it is easy to see that it reduces to formula (7.3.9) for equivalent spectral density with zero detuning if we introduce certain equivalent magnitudes of the signal-to-noise ratio $h_{\text{экв}}$ and mean square spectrum width $(b - a^2)_{\text{экв}}$ and takes the form

$$S_{экв} = \frac{1}{2h^2_{экв}(b-a^2)_{экв}} \times$$

$$\times \frac{\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |H_1(i\omega)|^4 [1 + h_{экв} S_0(\omega)]\, d\omega}{\left[\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |H_1(i\omega)|^2 S_0(\omega)\, d\omega\right]^2}, \tag{7.3.22}$$

where $h_{экв}$ and $(b - a^2)_{экв}$ depend on detuning and are equal to

$$h_{экв} = h\,\frac{2[|C(\delta)|^2 - \operatorname{Re} C^*(2\delta)\, C^2(\delta)]}{1 - |C(2\delta)|^2}, \tag{7.3.23}$$

$$(b - a^2)_{экв} = \frac{[\operatorname{Re} C'(\delta)\, C^*(\delta)]^2\,[1 - |C(2\delta)|^2]}{[|C(\delta)|^2 - \operatorname{Re} C^*(2\delta)\, C^2(\delta)]^2}. \tag{7.3.24}$$

Thereby, the nature of the dependence of accuracy of measurement on characteristics of narrow-band filters, on the ratio of bands of fluctuations and of the filter; studied at zero detuning, completely is preserved with arbitrary detuning, and quantitative calculations for the last case reduce to substitution in the formulas (7.3.9), (7.3.10), (7.3.13), (7.3.14) and (7.3.17) obtained above of quantities $h_{экв}$ and $(b - a^2)_{экв}$. Similarly we can also use graphs giving the dependence of $S_{экв}/S_{опт}$ and $S_{экв}$ on the ratio of bands $\Delta f_{ф}/\Delta f_{c}$ and the signal-to-noise ratio h.

Selection of a finite magnitude of detuning leads to a certain change of the frequency response of the filter, ensuring minimum equivalent spectral density. If from formula (7.3.9) for the case of zero detuning this characteristic is equal to the optimum (7.2.5), which it is possible to show by direct variation of expression (7.3.9), with finite detuning the minimum of $S_{экв}$, obviously, is at

$$|H_{1опт}(i\omega)|^2 = \frac{h_{экв} S_0(\omega)}{1 + h_{экв} S_0(\omega)}. \tag{7.3.25}$$

In particular, for a spectrum of form (7.3.11) this leads to change of the optimum band to

$$\Delta f_{согл} = \Delta f_c \sqrt{1 + h_{экв}}.$$

The nature of the dependence of $h_{экв}$ and $(b - a^2)_{экв}$ on the magnitude of detuning we shall study in examining concrete forms of modulation. However, it is possible to indicate that, in general, the magnitude of ratio $h_{экв}/h < 1$, and ratio

$(b - a^2)_{\text{экв}}/(b - a^2)$ may be larger or less than one, where $(b - a^2)_{\text{экв}}/(b - a^2) < 1$ only for such signals for which $C(\delta)$ does not have zeroes for finite δ and function $C(\delta)$ decreases to infinity no faster than its derivative $C'(\delta)$. In most real cases $(b - a^2)_{\text{экв}}/(b - a^2) > 1$, but so that the product $\frac{(b-a^2)_{\text{экв}}}{b-a^2}\cdot\frac{h_{\text{экв}}}{h}$, inversely proportional to which spectral density increases due to finite detuning for large h, and all the more so product $\frac{(b-a^2)_{\text{экв}}}{b-a^2}\cdot\frac{h^2_{\text{экв}}}{h^2}$, characterizing this increase for small h, is always smaller than one.

As we shall subsequently prove, in working conditions, i.e., for δ, smaller or of the order of $\delta_{\text{макс}}$, both $h_{\text{экв}}/h$, and also $(b - a^2)_{\text{экв}}/(b - a^2)$ in many cases differ so insignificantly from one that it is unnecessary to talk of any influence of detuning.

7.3.4. Noncoincidence of Modulations of Reference and Sounding Signals

Let us consider now the difference in the forms of modulating voltages of the sounding and reference signals. It is obvious that, assuming $u_1(t) \neq u_2(t)$, we cover also the most simple case of imbalance of gain in two channels, for which $u_1(t)$ and $u_2(t)$ coincide in form, but differ in amplitude. Returning to formula (7.3.2) for output voltage of the discriminator in general, considering again $h_1(t) = h_2(t)$ (the most important nonidentity of filters — the difference in gain factors — we already allowed for, considering $u_1(t) \neq u_2(t)$) and producing necessary operations of averaging and integration just as in the preceding paragraph, it is possible similarly to (7.3.3) to obtain the following expression for the discrimination characteristic:

$$\begin{aligned}\overline{z(t,\ \Delta)} = N_0\Delta f_\phi\,[C_{11}(0) - C_{22}(0)] + \\ + [|C_{10}(\Delta - \delta)|^2 - |C_{20}(\Delta + \delta)|^2] \times \\ \times \frac{P_0}{2\Delta f_0}\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^2 S_0(\omega)\,d\omega,\end{aligned} \tag{7.3.26}$$

where

$$C_{ik}(\delta) = \lim_{T\to\infty}\frac{1}{T}\int_0^T u_i(t+\delta)\,u^*_k(t)\,dt, \tag{7.3.27}$$

and $u_0(t) = u(t)$, i.e., subscript 0 pertains to the sounding signal. From (7.3.26) we can obtain the following formulas for steepness and systematic error:

$$K_{\text{д}} = \frac{P_c}{\Delta f_c}\frac{1}{2\pi}\int_{-\infty}^{\infty} |H_1(i\omega)|^2 S_0(\omega)\,d\omega \times$$

$$\times \operatorname{Re}[C'_{10}(-\delta)C^*_{10}(-\delta) - C'_{20}(\delta)C^*_{20}(\delta)], \tag{7.3.28}$$

$$\Delta_0 = \left\{\frac{[C_{11}(0) - C_{22}(0)]\,2\pi\Delta f_c}{h\int_{-\infty}^{\infty}|H_1(i\omega)|^2 S_0(\omega)\,d\omega} + |C_{10}(-\delta)|^2 - |C_{20}(\delta)|^2\right\}:$$

$$: \operatorname{Re}[C'_{10}(-\delta)C^*_{10}(-\delta) - C'_{20}(\delta)C^*_{20}(\delta)]. \tag{7.3.29}$$

Likewise, we can find the expression for equivalent spectral density (we shall limit ourselves to the formula for zero mismatch):

$$S_{\text{экв}} = \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} |H_1(i\omega)|^4 [|C_{11}(0)|^2 + |C_{22}(0)|^2 -\right.$$

$$- |C_{12}(2\delta)|^2 - |C_{21}(2\delta)|^2 + 2hS_0(\omega)(C_{11}(0)|C_{10}(-\delta)|^2 +$$

$$+ C_{22}(0)|C_{20}(\delta)|^2 - \operatorname{Re} C_{12}(-2\delta)C^*_{10}(-\delta)C^*_{20}(-\delta) -$$

$$- \operatorname{Re} C_{21}(2\delta)C^*_{20}(\delta)C^*_{10}(\delta)) + h^2 S_0^2(\omega)(|C_{10}(-\delta)|^2 -$$

$$\left.- |C_{20}(\delta)|^2)^2]\,d\omega\right\} : \left\{\frac{h}{\pi}\operatorname{Re}[C'_{10}(-\delta)C^*_{10}(-\delta) - C'_{20}(\delta)C^*_{20}(\delta)] \times\right.$$

$$\left.\times \int_{-\infty}^{\infty} |H_1(i\omega)|^2 S_0(\omega)\,d\omega\right\}^2. \tag{7.3.30}$$

These results cover a broad class of cases and permit us for all concrete examples to investigate the dependence of systematic and fluctuating errors on the form of the reference signals. Let us consider certain particular cases. We separate first of all conditions in which there is no systematic error. As follows from (7.3.29), $\Delta_0 = 0$ for all h, if

$$1)\ C_{11}(0) = C_{22}(0); \quad 2)\ |C_{10}(-\delta)|^2 = |C_{20}(\delta)|^2. \tag{7.3.31}$$

The first condition requires equality of powers of reference signals and includes the condition of equality of gain factors in the channels. The second condition requires symmetry of reference signals, expressed in equal correlation of "lagging" and "leading" reference signals with the expected signal. Both conditions are normal specifications, fulfillment of which should be sought during practical developments.

If (7.3.31) is satisfied, in the formula for $S_{\text{экв}}$ the component not depending on the signal-to-noise ratio turns into zero, and $S_{\text{экв}} \to 0$ as $h \to \infty$, as also for coinciding modulation.

The formula for $S_{\text{экв}}$ is considerably simplified if we assume that laws of

modulation of reference signals are identical. If the second of conditions (7.3.31) is simultaneously satisfied (the first is always satisfied when $u_1(t) \approx u_2(t)$), from (7.3.30) there follows:

$$S_{\text{экв}} = \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} |H_1(i\omega)|^4 [|C_{11}(0)|^2 - |C_{11}(2\delta)|^2 + \right.$$
$$+ 2hS_0(\omega)(C_{11}(0)|C_{10}(-\delta)|^2 -$$
$$\left. - \operatorname{Re} C_{11}(-2\delta) C^*_{10}(-\delta) C_{10}(\delta))] d\omega \right\} :$$
$$: 2h^2 \left\{ \operatorname{Re} [C'_{10}(-\delta) C^*_{10}(-\delta) - C'_{10}(\delta) C^*_{10}(\delta)] \times \right.$$
$$\left. \times \frac{1}{2\pi} \int_{-\infty}^{\infty} |H_1(i\omega)|^2 S_0(\omega) d\omega \right\}. \qquad (7.3.32)$$

If detuning of reference signals with respect to delay is so small that it is possible to be limited to first terms in the expansion of functions $C_{1k}(\delta)$, i.e.,

$$C_{1k}(\delta) = |C_{1k}(0)| \left(1 + ia_{1k}\delta - \frac{1}{2} b_{1k}\delta^2\right),$$

from (7.3.30) we can obtain a formula generalizing (7.3.9) for the case of noncoincident modulation:

$$S_{\text{экв}} = \frac{\frac{1}{2\pi} \int_{-\infty}^{\infty} |H_1(i\omega)|^4 [|C_{11}(0)|^2 (b_{11} - a_{11}^2) +}{2h^2 (\operatorname{Re} b_{10} - a_{10}^2)^2 |C_{10}(0)|^4 \times}$$
$$\frac{+ hS_0(\omega) C_{11}(0) |C_{10}(0)|^2 (b_{11} - 2a_{11}a_{10} - a_{10}^2)] d\omega}{\times \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} |H_1(i\omega)|^2 S_0(\omega) d\omega \right]^2}. \qquad (7.3.33)$$

Analysis of formulas (7.3.30), (7.3.32) and (7.3.33) shows that the presence of any nonidentities and noncoincidences leads to increase of the equivalent spectral density. In general there appears a component of $S_{\text{экв}}$, caused only by fluctuations of the signal and not vanishing with infinite increase of the signal-to-noise ratio.

If this component is absent, the presence of nonidentities changes the distribution between components of $S_{\text{экв}}$ caused by beats of the signal with noise and of noise with noise. In the simplest case, when $\sqrt{b}\delta \ll 1$ and $b_{11} \approx \operatorname{Re} b_{10} \approx b$, $a_{11} \approx$ $\approx a_{10} \approx a$, the presence of nonidentities reduces to equivalent decrease of the signal-to-noise ratio to

$$h_1 = h\frac{|C_{10}(0)|^2}{C_{11}(0)} \tag{7.3.34}$$

and (7.3.33) passes into (7.3.9) with replacement of h by h_1.

In the more general case we are able, just as in Paragraph 7.3.3, to reduce formulas (7.3.32) and (7.3.33) to a very simple formula for the case of coinciding modulations and zero detuning (7.3.9) by introducing the equivalent signal-to-noise ratio $h_{\text{экв}}$ and the mean square spectrum width $(b - a^2)_{\text{экв}}$. Then for arbitrary modulation of reference signals $u_1(t) = u_2(t) \neq u(t)$ and arbitrary detuning we again have formula (7.3.22), where

$$h_{\text{экв}} = h\,\frac{2[|C_{10}(-\delta)|^2 - \operatorname{Re} C_{11}(-2\delta)\,C^*_{10}(-\delta)\,C_{10}(\delta)]}{1 - |C_{11}(-2\delta)|^2}, \tag{7.3.35}$$

$$(b - a^2)_{\text{экв}} = \\ = \frac{\{\operatorname{Re}[C'_{10}(-\delta)\,C^*_{10}(-\delta) - C'_{10}(\delta)\,C^*_{10}(\delta)]\}^2\,[1 - |C_{11}(-2\delta)|^2]}{4[|C_{10}(-\delta)|^2 - \operatorname{Re} C_{11}(-2\delta)\,C^*_{10}(-\delta)\,C_{10}(\delta)]^2}. \tag{7.3.36}$$

Reference signal $u_1(t)$ for simplicity of recording in these formulas was normalized so that

$$C_{11}(0) = \lim_{T\to\infty}\frac{1}{T}\int_0^T |u_1(t)|^2\,dt = 1. \tag{7.3.37}$$

Thus, the already considerably more complex case of noncoincident modulations and finite detuning reduces to the very simple case of zero detuning with coinciding modulation. This gives us the possibility of using for calculation of final quantitative results the same few formulas and graphs of Paragraph 7.3.2, which together with calculation of quantities $(b - a^2)_{\text{экв}}$ and $h_{\text{экв}}$ permit us actually to answer all questions related to analysis of accuracy of a discriminator with detuned channels in linear approximation.

§ 7.4. Discriminator with Switching of Reference Signals

7.4.1. Block Diagram of the Discriminator

Solution of the problem of identification of channels in a two-channel circuit is connected with great technical difficulties. Therefore, preserving the principle of approximate replacement of the derivative of functional $|Q(t, \tau)|^2$ by a finite difference, it is possible to try, instead of simultaneous formation in the meter of quantities $|Q(t, \tau + \delta)|^2$, $|Q(t, \tau - \delta)|^2$ and their difference, to form the shown

quantities consecutively, periodically changing tuning of the discriminator relative to the measured value of τ. This is attained by application of the one channel of Fig. 7.5, to the mixer of which there is alternately fed "leading" and "lagging" reference signals, and the sign of output voltage of the detector correspondingly is changed by a phase switcher (Fig. 7.7).

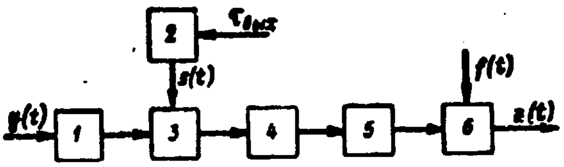

Fig. 7.7. Block diagram of a discriminator with switching of reference signals: 1 – preamplifier; 2 – controlled coherent local oscillator; 3 – mixer; 4 – narrow-band filter; 5 – square-law detector; 6 – phase commutator

$$s(t) = \mathrm{Re}\, u_1[t-\tau-\delta f(t)]\, e^{i(\omega_0+\omega_{пр})t}.$$

During analysis of the discriminator of Fig. 7.7 we naturally assume that reference signals are identical, although they may differ from the sounding signals, and that the time constant of the narrow-band filter is small as compared to the period of switching T_Π. Nonfulfillment of last condition would lead to averaging of the error signal in the filter and to sharp decrease of the gain factor of the discriminator. With these assumptions output voltage of the discriminator will be

$$z(t,\Delta)=\left|\int_{-\infty}^{t} h_1(t-s)\,u_1(s-\tau_0+\Delta-\delta)\,y(s)\,e^{i\omega_0 s}ds\right|^2 f(t)-$$
$$-\left|\int_{-\infty}^{t} h_1(t-s)\,u_1(s-\tau_0+\Delta+\delta)\,y(s)\,e^{i\omega_0 s}ds\right|^2 f\left(t-\frac{T_\Pi}{2}\right), \qquad (7.4.1)$$

where $f(t)$ is a sequence of meanders with period T_Π.

Voltage $z(t, \Delta)$ constitutes a nonstationary random process with periodic nonstationariness. In smoothing circuits of the servo system of a range finder, due to their inertia, there occurs averaging of this nonstationariness. In connection with this the magnitude of fluctuating error is determined by the spectral density corresponding to the time-averaged correlation function of voltage $z(t, \Delta)$. Then, in accordance with (6.2.4)-(6.2.5), the discrimination characteristic, the gain factor and the fluctuation characteristic are determined by the following expressions:

$$\left.\begin{aligned}
&\overline{\overline{z(t,\Delta)}}=\frac{1}{T_\Pi}\int_0^{T_\Pi}\overline{z(t,\Delta)}\,dt=\frac{1}{2}\overline{z(t,\Delta)},\\
&K_д=\left.\frac{\partial\overline{\overline{z(t,\Delta)}}}{\partial\Delta}\right|_{\Delta=0}=\frac{1}{2}\left.\frac{\partial\overline{z(t,\Delta)}}{\partial\Delta}\right|_{\Delta=0},\\
&S_{экв}(\Delta)=\frac{1}{K_д^2}\int_{-\infty}^{\infty}d\tau\,\frac{1}{T_\Pi}\int_0^{T_\Pi}[\overline{z(t+\tau,\Delta)\,z(t,\Delta)}-\\
&\qquad-\overline{z(t+\tau,\Delta)}\,\overline{z(t,\Delta)}]\,dt,
\end{aligned}\right\} \qquad (7.4.2)$$

where $\overline{z(t, \Delta)}$ corresponds to the case of a two-channel discriminator where $u_1(t) = u_2(t)$ and $h_1(t) = h_2(t)$ (7.3.15).

7.4.2. Characteristics of the Discriminator

Formulas (7.4.2) permit us to directly use for a discriminator with switching of channels the results obtained earlier for a two-channel discriminator. In particular, the discrimination characteristic and the gain factor differ, other things being equal, by only a factor of 1/2. Systematic error is absent when there is symmetry of the reference signal, where

$$|C_{10}(-\delta)|^2 = |C_{10}(\delta)|^2.$$

In other cases Δ_0 is determined by formula (7.3.18) with $C_{10}(\delta) = C_{20}(\delta)$.

Thus, as in preceding paragraphs, with the given assumptions with respect to the frequency of switching we can obtain an expression for the equivalent spectral density.

$$S_{экв} = \frac{\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |H_1(i\omega)|^4 \left[|C_{11}(0)|^2 + 2hS_0(\omega)\,|C_{10}(-\delta)|^2\right] d\omega + h^2 \Delta f_c A\left(\frac{1}{\Delta f_c T_п}\right)\left[|C_{10}(-\delta)|^4 + |C_{10}(\delta)|^4\right]}{h^2 \left\{\mathrm{Re}\left[C'_{10}(-\delta)\, C^*_{10}(-\delta) - C'_{10}(\delta)\, C^*_{10}(\delta)\right] \times \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |H_1(i\omega)|^2 S_0(\omega)\, d\omega\right\}^2}, \tag{7.4.3}$$

where

$$A\left(\frac{1}{\Delta f_c T_п}\right) = \frac{2}{\Delta f_c} \sum_{k=-\infty}^{\infty} \frac{B\left(2\pi \frac{2k+1}{T_п}\right)}{\pi^2 (2k+1)^2}; \tag{7.4.4}$$

$B(\omega)$ – a quantity, proportional to the spectral density of fluctuations at the detector output,

$$B(\omega) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |H_1(i\omega - is)|^2 S_0(\omega - s)\,|H_1(is)|^2 S_0(s)\, ds. \tag{7.4.5}$$

An essential peculiarity of the considered discriminator is the presence in $S_{экв}$ of a component not depending on the signal-to-noise ratio but proportional to $A\left(\frac{1}{T_п \Delta f_c}\right)$. This means that in a range finder with such a discriminator fluctuating error does not approach zero with unlimited decrease of noises. Physically this is explained by incomplete correlation of those components of output voltage of the discriminator in various half-periods of frequency switching which are

caused by the presence of fluctuations of the reflected signal, and the related impossibility of their compensation, as occurs in a two-channel system carrying out instantaneous comparison with "lagging" and "leading" reference signals. Such components of $S_{ЭКВ}$ occur in all measuring systems where the discrimination characteristic is created by consecutive measurement of correlation between the received and expected signals for various values of the measured parameter. A characteristic example, as we shall subsequently see, is angle measurement by scanning. The presence of such components is a serious deficiency of such systems.

With slow switching, when $\Delta f_C T_П \gg 1$,

$$A\left(\frac{1}{\Delta f_c T_П}\right)=\frac{B(0)}{2\Delta f_c}=\frac{1}{4\pi\Delta f_c}\int_{-\infty}^{\infty}|H_1(i\omega)|^4 S_0^2(\omega)\,d\omega. \qquad (7.4.6)$$

The case of switchings, rapid as compared to signal fluctuations, when $A\left(\frac{1}{\Delta f_C T_П}\right) \to 0$, can be presented only if the band of the filter exceeds by many times the width of the spectrum of fluctuations Δf_C. Here, the period of switching may be small as compared to the time constant of the filter, but large as compared to the time of correlation of fluctuations. For exponential correlation and an LRC-filter with band $\Delta f_Ф$ calculation of function $A\left(\frac{1}{\Delta f_C T_П}\right)$ can be conducted for an arbitrary switching frequency $F_П = \frac{1}{T_П}$. In the case interesting us $x = \frac{\Delta f_Ф}{\Delta f_C} \gg 1$:

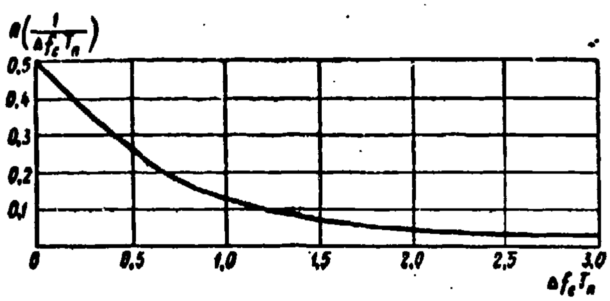

Fig. 7.8. Dependence of the relative magnitude of the residual equivalent spectral density on the product $\Delta f_C T_П$.

$$A\left(\frac{1}{\Delta f_c T_П}\right)=\frac{1}{2}\left(1-\frac{1}{\Delta f_c T_П}\,\mathrm{th}\,\Delta f_c T_П\right). \qquad (7.4.7)$$

With increase of $F_П$ function $A\left(\frac{1}{\Delta f_C T_П}\right)$ decreases rather slowly. Its graph is constructed in Fig. 7.8.

With slow switchings and fulfillment of the condition of the absence of systematic error the formula for $S_{ЭКВ}$ is simplified and takes the form

$$S_{\text{экв}} = \frac{\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |H_1(i\omega)|^4 [1 + h|C_{10}(\delta)|^2 S_0(\omega)]^2 \, d\omega}{\left\{ \operatorname{Re}[C'_{10}(-\delta) C^*_{10}(-\delta)] \cdot \frac{h}{\pi} \int\limits_{-\infty}^{\infty} |H_1(i\omega)|^2 S_0(\omega) \, d\omega \right\}^2}, \tag{7.4.8}$$

where we again normalized $u_1(t)$ so that $C_{11}(0) = 1$. With coinciding modulations function $C_{10}(\delta)$ is replaced by $C(\delta)$. Noncoincidence of modulations of the reference and sounding signals again can be described by introduction of the equivalent signal-to-noise ratio, depending on δ:

$$h_{\text{эkв}} = h \frac{|C_{10}(\delta)|^2}{|C(\delta)|^2}. \tag{7.4.9}$$

Then, if we introduce for the case of coinciding modulation the special designation

$$S^{(0)}_{\text{экв}}(h) = \frac{\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |H_1(i\omega)|^4 [1 + h|C(\delta)|^2 S_0(\omega)]^2 \, d\omega}{\left[\operatorname{Re} C'(\delta) C^*(\delta) \frac{h}{\pi} \int\limits_{-\infty}^{\infty} |H_1(i\omega)|^2 S_0(\omega) \, d\omega \right]^2}, \tag{7.4.10}$$

the equivalent spectral density with noncoincident modulations is

$$S_{\text{экв}} = M(\delta) S^{(0)}_{\text{экв}}(h_{\text{экв}}), \tag{7.4.11}$$

where

$$M(\delta) = \frac{|C_{10}(-\delta)|^4 [\operatorname{Re} C'(\delta) C^*(\delta)]^2}{|C(\delta)|^4 [\operatorname{Re} C'_{10}(-\delta) C^*_{10}(-\delta)]^2} \tag{7.4.12}$$

is the equivalent change of the square of the slope of the discrimination characteristic.

In formulas (7.4.8), (7.4.9), (7.4.11) and (7.4.12) and henceforth we assume that the condition of the absence of systematic error is satisfied.

We also give an expression for the fluctuation characteristic and spectral density of parametric fluctuations for the case

$$u_1(t)=u(t), \quad C_{10}(\delta)=C(\delta):$$

$$S_{\text{экв}}(\Delta)=\frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^4\{[1+h\,|C(\Delta+\delta)|^2 S_0(\omega)]^2+[1+h\,|C(\Delta-\delta)|^2 S_0(\omega)]^2\}\,d\omega}{8h^2\left[\operatorname{Re}C'(\delta)C^*(\delta)\times\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^2 S_0(\omega)\,d\omega\right]^2}, \tag{7.4.13}$$

$$S_{\text{пар}}=\left[1+\frac{\operatorname{Re}[C''(\delta)C^*(\delta)+C'(\delta)C^{*\prime}(\delta)]\,|C(\delta)|^2}{2[\operatorname{Re}C'(\delta)C^*(\delta)]^2}\right]\times$$

$$\times\frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^4 S_0^2(\omega)\,d\omega}{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^2 S_0(\omega)\,d\omega\right]^2}+$$

$$+\frac{\operatorname{Re}[C''(\delta)C^*(\delta)+C'(\delta)C^{*\prime}(\delta)]}{2h[\operatorname{Re}C'(\delta)C^*(\delta)]^2}\,\frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^4 S_0(\omega)\,d\omega}{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^2 S_0(\omega)\,d\omega\right]^2}. \tag{7.4.14}$$

Formulas (7.4.8) and (7.4.14) show that both nonparametric and parametric fluctuations essentially depend on the form of signal modulation and the magnitude of detuning δ. For any finite values of h and any $C_{10}(\delta)$ quantity $S_{\text{экв}}$ changes monotonically depending upon δ, turning for a certain δ into a minimum. The magnitude of this optimum value of detuning essentially depends on the form of the signal and the magnitude of the signal-to-noise ratio h. For small values of h the minimum of $S_{\text{экв}}$ is attained at value of δ, which turns the slope of the discrimination characteristic into a maximum; for larger h the optimum value of δ increases. We shall investigate the character of the dependence of $S_{\text{экв}}$ on δ in more detail when we examine concrete forms of modulation; however, in general it is possible to indicate that in circuits with switching there are all grounds on which to select detuning equal to that value of it which maximizes the gain factor, or somewhat larger.

In distinction from (7.3.14), in this case both components of $S_{\text{пар}}$ depend on the form of modulation and detuning δ. This dependence is such that, for small δ, $S_{\text{пар}}$ may be negative, i.e., the fluctuation characteristic may have its maximum in the vicinity of $\Delta = 0$.

Such a character of the dependence $S_{ЭКВ}(\Delta)$ means that in the considered case, if $S_{пар} < 0$, there is no sense in talking about parametric fluctuations. In examining linearized conditions we should consider fluctuations not to depend on mismatch. If, however, we try to allow for the dependence of $S_{ЭКВ}(\Delta)$ on mismatch, then, even being limited to small Δ, we have to solve a nonlinear problem, considering also the dependence of the discrimination characteristic on Δ. For large δ the dependence of $S_{ЭКВ}(\Delta)$ on Δ has a normal character, and analysis of the range finder as a whole can be conducted by the usual means. As follows from (7.4.10), for δ, turning the gain factor $K_Д$ into a maximum, the component of $S_{пар}$ of the order $1/h$ disappears, just as the addition to the basic term, and $S_{пар}$ turns out to coincide with (7.3.14), i.e., $S_{пар} = \frac{\alpha}{\Delta f_c}$. For a certain smaller value of δ depending on h, the magnitude of $S_{пар}$ turns into zero. The considered character of change of $S_{ЭКВ}$ and $S_{пар}$ from δ is rather unique and essentially distinguishes a circuit with switching from the optimum and two-channel circuits.

7.4.3. Optimum Filter for a Discriminator with Switching

The dependence of $S_{ЭКВ}$ on the form of frequency response of the filter has an interesting character. Let us consider this dependence for the case of low frequency switching. Composing the variation of $S_{ЭКВ}$ from (7.4.8) with respect to $|H_1(i\omega)|^2$ and equating it to zero, we obtain the following equation, determining the optimum frequency response:

$$|H_1(i\omega)|^2[1+h|C_{10}(\delta)|^2 S_0(\omega)] \int_{-\infty}^{\infty} |H_1(ix)|^2 S_0(x)\,dx - \\ - hS_0(\omega) \int_{-\infty}^{\infty} |H_1(ix)|^4 [1+h|C_{10}(\delta)|^2 S_0(x)]^2\,dx = 0, \tag{7.4.15}$$

which, obviously, is satisfied when

$$|H_1(i\omega)|^2 = |H_{опт}(i\omega)|^2 = \frac{h|C_{10}(\delta)|^2 S_0(\omega)}{[1+h|C_{10}(\delta)|^2 S_0(\omega)]^2}. \tag{7.4.16}$$

This result shows that an optimum filter for a discriminator with switching, ensuring a minimum equivalent spectral density, may substantially differ in its characteristics from the usual optimum filter (7.2.5). With fulfillment of (7.4.16)

$$S_{ЭКВ} = \frac{1}{4h^2[\operatorname{Re} C'_{10}(\delta) C^*_{10}(\delta)]^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{S_0^2(\omega)\,d\omega}{[1+h|C_{10}(\delta)|^2 S_0(\omega)]^2}}. \tag{7.4.17}$$

For any real spectrum of fluctuations [$S_0(\omega)$ is different from a rectangular function] $S_{\text{экв}} \to 0$ as $h \to \infty$. The order of decrease of $S_{\text{экв}}$ depends on the form of the spectral density and is always slower than $1/h$.

The difference between optimum filter (7.4.16) and a usual one from the theoretical point of view is explained by the fact that in this case during synthesis of the optimum discriminator we impose an additional technical condition. This condition is that at every moment of time we can form the likelihood functional [actually functional $Q(t, \tau)$] for only one value of τ. In principle in an optimum system values of this functional should be formed for all τ; however, as we proved in the example of a discriminator with staggered channels, with observance of certain conditions it is practically sufficient to form this functional simultaneously only for two differing values of parameter τ. The additional technical limitation, naturally, leads to change of the form of the optimum system and, as any limitation, leads to loss in characteristics of the synthesized system. This loss may be slight, but it always exists in principle, being the cost of the attained technical simplification.

Let us investigate in more detail the frequency response of an optimum filter. For small $h|C_{10}(\delta)|^2$ this characteristic coincides with the form of the spectrum of fluctuations and with the usual optimum frequency response for small h. With growth of $h|C_{10}(\delta)|^2$ the difference between (7.4.16) and (7.2.5) becomes essential. For frequency response curve (7.4.16) there is a dip at $\omega = 0$, the depth of which decreases as $1/h|C_{10}(\delta)|^2$, and two maxima, the separation of which increases with increase of $h|C_{10}(\delta)|^2$. In the particular case of an exponential correlation of fluctuations

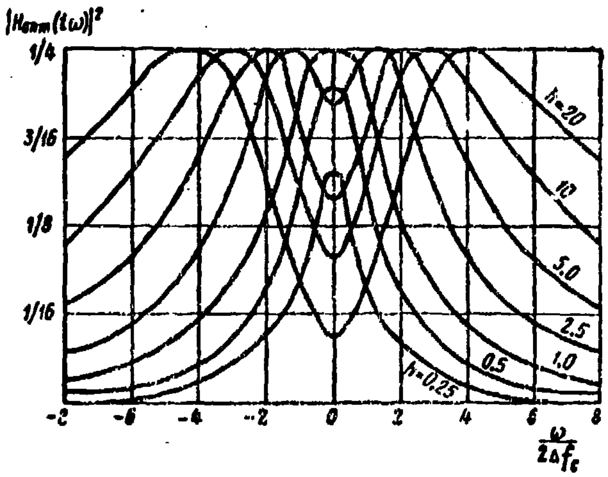

Fig. 7.9. Frequency response curve of an optimum filter.

$$|H_{\text{опт}}(i\omega)|^2 = \frac{h|C_{10}(\delta)|^2\left(1+\frac{\omega^2}{4\Delta f_c^2}\right)}{\left[1+h|C_{10}(\delta)|^2+\frac{\omega^2}{4\Delta f_c^2}\right]^2}. \quad (7.4.18)$$

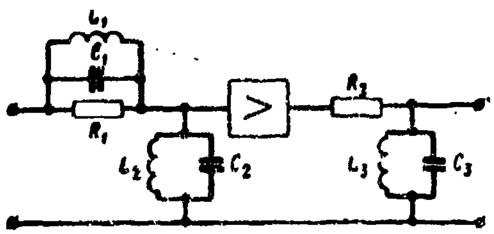

Fig. 7.10. Fundamental circuit of an optimum filter.

The graph of this function is constructed in Fig. 7.9. Frequency response (7.4.18), taking into account transfer to intermediate frequency

$\omega_{\text{пр}}$, can be formed by the circuit of Fig. 7.10 when

$$\frac{1}{\sqrt{L_1C_1}}=\frac{1}{\sqrt{L_2C_2}}=\frac{1}{\sqrt{L_3C_3}}=\omega_{\text{пр}},\quad \frac{1}{R_1C_1}=2\Delta f_c,$$
$$\frac{1}{R_2(C_1+C_2)}=\frac{1}{R_3C_3}=2\Delta f_c\sqrt{1+h|C_{10}(\delta)|^2}.$$

In a system with optimum filter (7.4.18) $S_{\text{экв}}$ is expressed by the formula

$$S_{\text{экв}}=\frac{1}{2\Delta f_c}\,\frac{[1+h|C_{10}(\delta)|^2]^{3/2}}{h^2[\operatorname{Re}C'_{10}(\delta)\,C^*_{10}(\delta)]^2}. \tag{7.4.19}$$

This expression, as also (7.3.8), for a certain δ turns into a minimum, but in distinction from the general case it approaches zero as $h\to\infty$ as $1/\sqrt{h}$. Thus, even with an optimum filter the circuit with switching for a large signal-to-noise ratio has a loss as compared to a two-channel circuit, growing as $\sqrt{h}$. Detailed comparison of circuits with detuning and with switching requires specification of the form of function $C_{10}(\delta)$, and therefore, we postpone it to subsequent paragraphs. Here we can only note that in general both for an optimum filter (7.4.16) and for an arbitrary one loss in accuracy during use of a circuit with switching with respect to an optimum and a two-channel one is greater, the greater h. For small h it may be more or less immaterial, but for large h, when fluctuating error of the range finder is caused basically by the randomness of the signal, it attains considerable magnitudes. For a normal filter loss grows proportionally to h; for an optimum filter it is somewhat slower, for instance, for filter (7.2.18), as $\sqrt{h}$. This peculiarity is an unpleasant property of a circuit with switching.

Let us estimate now what gain application of an optimum filter gives in a circuit with switching. Let us consider for this again the spectral density of a signal of form (7.3.11) and frequency response of form (7.3.12). Substituting them in the formula for (7.4.8), we obtain

$$S_{\text{экв}}=\frac{(1+x)^2+h|C_{10}(\delta)|^2(1+2x)+h^2|C_{10}(\delta)|^4\left(2x+\frac{1}{1+x}\right)}{2\Delta f_c\cdot 4h^2x\,(\operatorname{Re}C'_{10}(\delta)\,C^*_{10}(\delta))^2}, \tag{7.4.20}$$

where x, as before, is $\Delta f_{\Phi}/\Delta f_c$.

Then ratio γ of equivalent spectral densities during use of LRC-filter (7.3.12) and optimum filter (7.4.18) is

$$\gamma=\frac{(1+x)^2+h|C_{10}(\delta)|^2(1+2x)+h^2|C_{10}(\delta)|^4\left(2x+\frac{1}{1+x}\right)}{4x\,[1+h|C_{10}(\delta)|^2]^{3/2}}. \tag{7.4.21}$$

Dependence $\gamma(x)$ for various $h|C_{10}(\delta)|^2$ is shown in Fig. 7.11. For small $h|C_{10}(\delta)|^2$ this dependence has a clearly expressed minimum at $x \approx 1$, where the magnitude of the minimum is close to one, which is naturally explained by practical coincidence when $h|C_{10}(\delta)|^2 << 1$ of characteristics of an optimum and an LRC-filter when $x \approx 1$. With increase of $h|C_{10}(\delta)|^2$ the minimum is shifted in the direction of large x, and curve $\gamma(x)$ in the vicinity of the minimum is all the more shallow, so that there appears a zone within which change of the band of the LRC-filter practically does not affect $S_{\text{эв}}$. The loss as compared to an optimum circuit in this zone comprises approximately a factor of $|C_{10}(\delta)|\sqrt{h}/2$. Width of the zone grows rapidly with increase of $h|C_{10}(\delta)|^2$. If for $h|C_{10}(\delta)|^2 = 10$ error of measurement almost does not change with triple change of band, then for $h|C_{10}(\delta)|^2 = 100$ error remains practically constant with change of the band by a factor of 200.

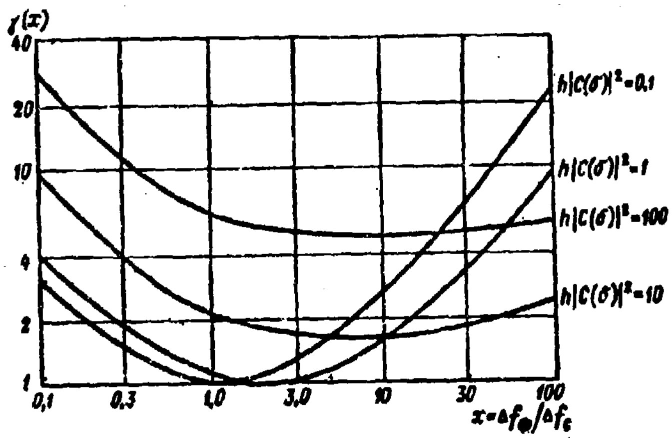

Fig. 7.11. Influence of nonoptimality of the filter on equivalent spectral density.

The gain during use of an optimum filter is explained by the fact that in a circuit with switching useful components of the spectrum are transferred to the frequency of switching, and therefore, suppression of signal fluctuations concentrated near zero frequency and playing, at low noises, the basic role in formation of fluctuating error, which is attained by use of a frequency response curve with a dip near zero, turns out to be useful. The magnitude and width of the dip by formula (7.4.16) is automatically selected in such a way as to ensure minimum equivalent spectral density.

§ 7.5. Discriminator with Differentiation of the Reference Signal

7.5.1. Block Diagram of the Discriminator

In § 7.2 we already saw that an optimum discriminator can be built by the circuit of Fig. 7.2 and consist of two channels, to one of which there proceeds the reference signal, and to the other, its derivative with respect to τ. The advantage of the circuit with differentiation of the reference signal as compared to a circuit with staggered channels is that operations performed by it can be made identical to optimum operations without decrease of the slope of the discrimination characteristic.

One of the basic difficulties connected with realization of such a circuit consists in forming reference voltage for the channel in which the received signal should be mixed with the derivative of the sounding signal. However, in certain cases this difficulty is surmountable. For instance, with frequency modulation reference signals of the local oscillator are equal, respectively, to

$$\operatorname{Re} u(t-\tau)\, e^{i(\omega_0+\omega_{\text{пр}})t} = \cos\left[(\omega_0+\omega_{\text{пр}})t + \psi(t-\tau)\right],$$
$$\frac{\partial}{\partial\tau} \operatorname{Re} u(t-\tau)\, e^{i(\omega_0+\omega_{\text{пр}})t} =$$
$$= \Delta\omega(t-\tau) \sin\left[(\omega_0+\omega_{\text{пр}})t + \psi(t-\tau)\right].$$

It is possible to form these signals with the help of the circuit of Fig. 7.12. The signal of the master oscillator, delayed τ, enters the frequency-modulated local oscillator at whose output there is obtained the first reference signal. The second reference signal is obtained from the first as a result of a shift of $\pi/2$ and amplitude modulation by the law $\Delta\omega(t - \tau)$. In the case of sinusoidal frequency modulation ($\Delta\omega(t) = \omega_m \cos \Omega t$) the second reference signal is obtained by simple shift of the first in frequency by $\pm\Omega$ and by phase shift.

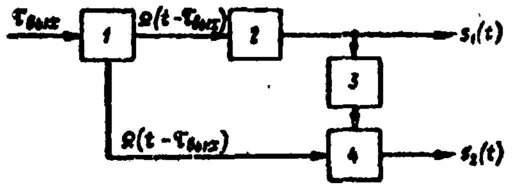

Fig. 7.12. Block diagram of forming of reference signals with frequency modulation: 1 — controlled master oscillator; 2 — frequency-modulated local oscillator; 3 — $\pi/2$ phase shifter; 4 — amplitude modulator.

During technical realization of the block diagram of Fig. 7.2 there are possible all sorts of deviations from optimum operations, the influence of which we shall now estimate. Denoting, as in § 7.3, by $u_1(t)$ and $u_2(t)$ the laws of modulation of the reference signals in the first and second channels and considering the filters identical, we obtain the following expression for discriminator output, generalizing (7.2.7):

$$z(t, \Delta) = \operatorname{Re} \int_{-\infty}^{t} \int_{-\infty}^{t} h_1(t-s_1)\, h_1(t-s_2)\, u_1(s_1-\tau_0-\Delta) \times$$
$$\times u^*_2(s_2-\tau_0-\Delta)\, y(s_1)\, y(s_2)\, e^{i\omega_0(s_1-s_2)} ds_1 ds_2. \qquad (7.5.1)$$

7.5.2. Characteristics of the Discriminator

Averaging expression (7.5.1) as usual, we obtain the following formula for the discrimination characteristic:

$$\overline{z(t,\Delta)}=\frac{N_0}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^2\operatorname{Re}[C_{12}(0)+$$
$$+hS_0(\omega)C_{10}(-\Delta)C^*_{20}(-\Delta)]\,d\omega, \tag{7.5.2}$$

where $C_{ik}(\Delta)$ are determined by formula (7.3.16), from which it follows that

$$\left.\begin{aligned}
&K_{д}=-\frac{N_0 h}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^2 S_0(\omega)\,d\omega\times\\
&\times\operatorname{Re}[C'_{10}(0)C^*_{20}(0)+C^*_{10}(0)C'_{20}(0)],\\
&\Delta_0=\left\{-\operatorname{Re}C_{10}(0)C^*_{20}(0)+\operatorname{Re}C_{12}(0)\frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^2 d\omega}{\frac{h}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^2 S_0(\omega)\,d\omega}\right\}:\\
&:\operatorname{Re}[C'_{10}(0)C^*_{20}(0)+C^*_{10}(0)C'_{20}(0)].
\end{aligned}\right\} \tag{7.5.3}$$

From the last expression it follows that systematic error is absent for any h, if

$$\operatorname{Re}C_{12}(0)=\operatorname{Re}C_{10}(0)C^*_{20}(0)=0. \tag{7.5.4}$$

These conditions, obviously, lead to the requirement that when there is a symmetric sounding signal one reference signal be symmetric, and the other, antisymmetric.

Determining the correlation function of voltage $z(t, \Delta)$ and subjecting it to the required transformations, it is possible to obtain an expression for equivalent spectral density

$$\begin{aligned}
S_{экв}=&\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^2[C_{11}(0)C_{22}(0)+\operatorname{Re}C_{12}^2(0)+\right.\\
&+hS_0(\omega)(C_{11}(0)|C_{02}(0)|^2+C_{22}(0)|C_{10}(0)|^2+\\
&+2\operatorname{Re}C_{12}(0)C_{02}(0)C^*_{01}(0))+h^2S_0^2(\omega)(|C_{01}(0)C_{02}(0)|^2+\\
&\left.+\operatorname{Re}C_{01}^2(0)C_{02}^2(0))\,d\omega\right\}:2h^2\left\{\operatorname{Re}[C'_{10}(0)C^*_{02}(0)+\right.\\
&\left.+C^*_{10}(0)C'_{02}(0)]\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^2S_0(\omega)\,d\omega\right\}^2.
\end{aligned} \tag{7.5.5}$$

From this expression it again follows that the presence of nonidealnesses leads to the appearance of a component of $S_{экв}$, not depending on h. As also in the circuit of Fig. 7.5, this component disappears together with disappearance of systematic error, i.e., with fulfillment of conditions (7.5.4). In general, formula (7.5.5) with an accuracy of coefficients coincides with the corresponding formula (7.3.19)

for a discriminator with staggered channels. If, in particular, differentiation of the reference signal is sufficiently accurate, which in certain cases is fully practicable, i.e., $u_2(t) = u'_1(t)$, and the main reference signal $u_1(t)$ differs from the sounding signal $u(t)$ due to the difference of modulation characteristics of the local oscillator and the transmitter, with fulfillment of (7.4.4) the spectral density of $S_{экв}$ is determined by a formula which coincides with (7.3.33) for a discriminator with staggered channels with small detuning. Obviously, expression (7.3.9), satisfied for $u_1(t) = u(t)$, also remains valid.

All these circumstances indicate that the dependence of equivalent spectral density for the considered discriminator on the signal-to-noise ratio h and the band of the filter actually has already been investigated earlier in § 7.3, and it remains for us only to repeat the conclusions presented there.

It is essential that in the considered circuit nonidentity of gain factors in channels does not lead to the appearance of stationary error, and that density of phase responses plays the basic role. If $u_1(t)$ and $u_2(t)$ coincide with $u(t)$ and $u'(t)$ with an accuracy of constant phase shift, for instance:

$$u_1(t) = u(t)\, e^{i\varphi}, \quad u_2(t) = u'(t),$$

where φ — phase delay in one channel relative to the other, then, as it is easy to prove,

$$C_{10}(0)\, C^*_{20}(0) = C_{12}(0) = ia e^{i\varphi},$$

$$\Delta_0 = \frac{a \sin\varphi}{b - a^2} \cdot \frac{\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |H_1(i\omega)|^2 [1 + hS_0(\omega)]\, d\omega}{h\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |H_1(i\omega)|^2 S_0(\omega)\, d\omega}. \qquad (7.5.6)$$

The condition of the absence of systematic error consists, thus, of the equality to zero of quantity a, i.e., the requirement of symmetry of the power spectrum of modulation. Phase shift leads also to increase of equivalent spectral density. When a = 0 quantity $S_{экв}$ increases as compared to (7.3.9) by a factor of $1/\cos^2\varphi$.

In conclusion let us give the expression for the fluctuation characteristic of a discriminator with differentiation of the reference signal. Limiting ourselves to the case $u_1(t) = u(t)$, $u_2(t) = u'(t)$, we obtain, completely analogously to (7.2.20)

$$
\begin{aligned}
S_{\text{экв}}(\Delta) = \operatorname{Re}\Big\{\frac{1}{2\pi}\int_{-\infty}^{\infty} |H_1(i\omega)|^4\,[h^2 S_0^2(\omega)(|C(\Delta)|^2|C'(\Delta)|^2 + \\
+ C^2(\Delta)\,C^{*\prime 2}(\Delta)) + hS_0(\omega)(b\,|C(\Delta)|^2 + \\
+ |C'(\Delta)|^2 - iaC(\Delta)\,C^{*\prime}(\Delta)) + \\
+ b - a^2]\,d\omega\Big\} : 2h^2(b-a^2)^2\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_1(i\omega)|^2 S_0(\omega)\,d\omega\right]^2.
\end{aligned}
\tag{7.5.7}
$$

This expression, obviously, differs from (7.2.20) only by replacement of $hS_0(\omega)|[1 + hS_0(\omega)]$ by an arbitrary frequency response $|H_1(i\omega)|^2$. Calculating the second derivative of $S_{\text{экв}}(\Delta)$ from (7.5.7) when $\Delta = 0$, it is simple to prove that spectral density of parametric fluctuations of $S_{\text{пар}}$ is determined by the very simple formula (7.3.21), occurring for a circuit with staggered channels as $h \to \infty$.

With this we complete our general consideration of time discriminators. In the following paragraphs we shall turn to concrete forms of modulation of the sounding signal. Inasmuch as the dependence of characteristics of discriminators on the signal-to-noise ratio and characteristics of filters was considered in general form, subsequently we shall be interested basically in the dependence of $S_{\text{экв}}$ and $K_{\text{Д}}$ on detuning of channels with respect to delay and on nonidentity of the reference and sounding signals. Characteristics of filters we consider optimum. Corrections for noncoincidence of bands and frequency responses of filters with the optimum can be introduced by the above results. Furthermore, we subsequently consider that spectral density of the fluctuating signal is determined by formula (7.3.11).

§ 7.6. Pulse Radiation

Pulse radiation, thanks to a series of well-known technical advantages, is widely used in contemporary radar. With pulse radiation we can use, obviously, all the schemes for construction of a discriminator considered earlier. Furthermore, in this case there is one more possibility — multiplication of the received signal by the expected can be replaced by transmission of it through a filter physically realizable due to finite pulse duration, with frequency response conjugate with the spectrum of the signal, and with subsequent gating by a very short delta-shaped pulse. Equivalence of these methods of processing was proven in [4] and at present is well-known. An advantage of replacement of multiplication by filtration with subsequent gating is that one filter can be used both in the channel of detection and in tracking channels for measurement of coordinates of one or several targets. In more detail questions of realizing an optimum coherent receiver on the basis of the principle of filtration were discussed previously in Chapter IV in examining

the problem of detection.

The circuit of a coherent time discriminator with two detuned channels, using filtration, has the form shown in Fig. 7.13. After the mixer, the signal enters a filter with frequency response $\frac{1}{2}\{U^*(i\omega - i\omega_{\text{пр}}) + U(-i\omega - i\omega_{\text{пр}})\}$ and is gated by two gate pulses, detuned $\pm\delta$ from the measured value of delay. After gating, the signals are filtered by narrow-band filters, detected and subtracted. Analogously, we can convert a circuit with differentiation of the reference signal. Gate pulses in this case coincide, and the filter corresponding to the channel with the derivative has frequency response

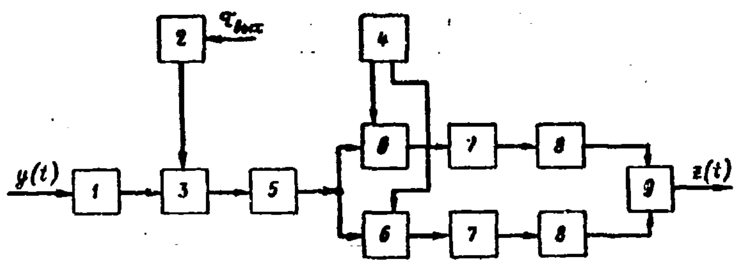

Fig. 7. 13. Block diagram of an optimum discriminator with a matched intermediate-frequency amplifier (UPCh): 1 — preamplifier; 2 — coherent local oscillator; 3 — mixer; 4 — controlled generator of gate pulses; 5 — filter (UPCh) with a frequency response matched with the spectrum of modulation; 6 — gated amplifier; 7 — narrow-band filter; 8 — square-law detector; 9 — subtractor.

$$\frac{1}{2}\{i(\omega-\omega_{\text{пр}})U^*(i\omega-i\omega_{\text{пр}})-i(\omega+\omega_{\text{пр}})U(-i\omega-i\omega_{\text{пр}})\}.$$

Likewise we realize a circuit with switching. In the case of pulse signals without additional intrapulse modulation it is sufficient that the band of the filter coincide with the band of the signal.

From equivalence of circuits with filtration and with multiplication by the expected signal (such a realization of an optimum circuit for processing is often called a correlation receiver), it follows that analysis of their accuracy can be performed by the same formulas. In this paragraph we investigate accuracy of rangefinding during pulse radiation without additional modulation for different forms of the pulse and gate pulse (or, which is the same, for different signal spectra and frequency response of the filter in circuits using the principle of optimum filtration). In the absence of intrapulse modulation and with arbitrary forms of pulses and gate pulses all functions $C_{1k}(\delta)$ are real, and general formulas of the preceding paragraphs are somewhat simplified.

Most practicable for pulse modulation are circuits with detuned channels and with switching; therefore, we shall basically consider discriminators of these forms, but at the end of the paragraph we shall briefly discuss the application of a discriminator with differentiation of the reference signal.

7.6.1. Square Pulse

Let us consider first modulation of the signal by square, periodica ly repeated pulses of duration $\tau_{И}$. It is obvious that the autocorrelation function for such a signal is

$$C(\delta)=1-\frac{|\delta|}{\tau_{и}},\quad |\delta|<\tau_{и},$$
$$C(\delta)=0,\quad |\delta|>\tau_{и}. \tag{7.6.1}$$

A characteristic feature of such a signal is infinite root mean square spectrum width $\sqrt{F}$, which formally leads to infinite accuracy of range finding for any signal-to-noise ratio. Actually this does not take place, since the pulse edge always has finite duration τ_{Φ}. Thanks to this the vertex of function $C(\delta)$ does not have a break and the magnitude of b is finite. In examining an optimum circuit and its characteristics allowance for these circumstances is necessary; however, during the analysis of a circuit with detuned channels and a circuit with switching we with full right can use function $C(\delta)$ in form (7.6.1), calculated without taking into account the edge, if we always consider that detuning δ exceeds the edge duration, and equality of δ to zero is understood in the sense that $\delta \ll \tau_{И}$, but $\delta > \tau_{\Phi}$. If we consider the hypothetical case of a square pulse of strictly zero edge duration, then, in general, the approach utilized by us becomes inapplicable. In this case the logarithm of the likelihood functional does not have a finite second derivative, and operations, calculated on the basis of replacement of this functional by first terms of a Taylor expansion, do not have meaning. Such a nonanalytical case requires special consideration, which leads anew to finite accuracy of measurement. Such a consideration, however, does not have any essential practical interest.

Substituting (7.6.1) in the corresponding general formulas for $K_{Д}$ and $S_{ЭКВ}$, for the gain factor and equivalent spectral density we obtain the following expressions:

$$K_{Д}=D\frac{1}{\tau_{и}}\left(1-\frac{\delta}{\tau_{и}}\right), \tag{7.6.2}$$

where $D = 2P_C\sqrt{1 + h}/[1 + \sqrt{1 + h}]$ – designation, which we introduce to shorten notation;

for a two-channel circuit

$$S_{экв}=\frac{\delta}{\tau_{и}}\frac{(1+\sqrt{1+h})^2+h(1+2\sqrt{1+h})\left(1-\frac{\delta}{\tau_{и}}\right)}{4h^2\sqrt{1+h}\Delta f_c\left(1-\frac{\delta}{\tau_{и}}\right)}\tau_{и}^2; \tag{7.6.3}$$

for a circuit with switching

$$S_{экв}=\frac{\left[1+h\left(1-\frac{\delta}{\tau_и}\right)^2\right]^{3/2}}{2h^2\Delta f_c\left(1-\frac{\delta}{\tau_и}\right)^2}\tau_и^2. \qquad (7.6.4)$$

Let us remember that characteristics of filters are considered optimum. In order to pass in expressions pertaining to the two-channel circuit to the case of an LRC-filter with an arbitrary band $\Delta f_Ф = x\Delta f_c$ here and subsequently one should replace in the formulas $\sqrt{1+h}$ by x.

The dependence of $K_Д$ on δ is very simple; the gain factor decreases proportionally to δ. Dependences (7.6.3) and (7.6.4) are constructed in Figures 7.14 and 7.15, respectively. For δ close to $\tau_И$, spectral density in both cases increases without limit; as $\delta \to 0$ for a two- hannel circuit it approaches zero (actually a value, determined by the duration of the edge); and for a circuit with switching it approaches $(1+h)^{3/2}\tau_И^2/2\Delta f_c\ h^2$. For the last circuit $S_{экв}(\delta)$ has a minimum, the existance of which one should use when selecting parameters of the discriminator. This question already was discussed in § 7.4. For the considered case the minimum is reached when

$$\delta_0=\tau_и\left(1-\sqrt{\frac{2}{h}}\right). \qquad (7.6.5)$$

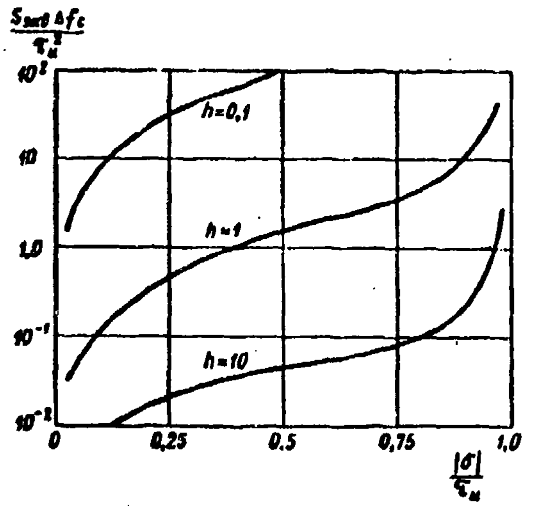

Fig. 7.14. Dependence of equivalent spectral density on detuning for a two-channel discriminator.

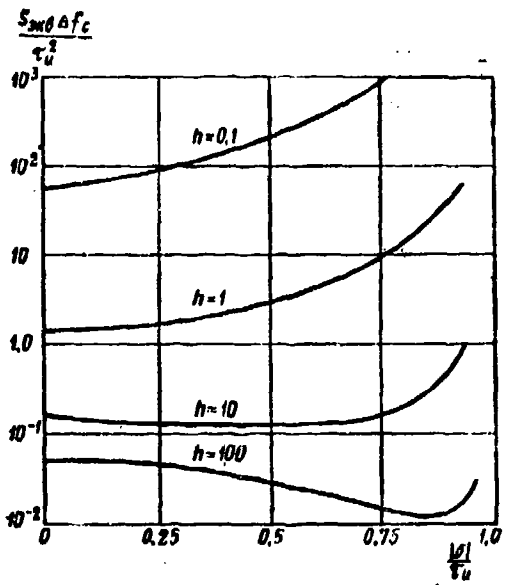

Fig. 7.15. Dependence of equivalent spectral density on detuning for a discriminator with switching of reference signals.

When h < 2 quantity $\delta_0 < 0$. This means that when h < 2 one should select detuning close to zero. When h > 2 the magnitude of optimum detuning is greater, the larger h. All this confirms the conclusions of Paragraph 7.4.2 about the character of the dependence of optimum δ on the signal-to-noise ratio. In normal conditions (when h = 10 to 30) detuning should comprise (0.45 to 0.8) τ_y. If $\delta = \delta_0$,

$$S_{экв} = \frac{1{,}3\kappa_и^2}{\Delta f_c h}. \tag{7.6.6}$$

Now let the duration of gate pulses τ_c differ from pulse duration. We shall consider for definitiveness $\tau_c > \tau_и$ (the opposite case if not of interest for practice). Designating $\frac{\delta}{\tau_и} = z$, and $\frac{\tau_c}{\tau_и} = \mu$, in accordance with (7.2.16) we obtain

$$C_{11}(\delta) = 1 - \frac{|\delta|}{\tau_c} = 1 - \frac{|z|}{\mu}, \quad C_{10}(\delta) = \sqrt{\frac{1}{\mu}}(1 - |z|) = \\ = \sqrt{\frac{1}{\mu}}\, C_{00}(\delta), \tag{7.6.7}$$

Substituting these expressions in formulas (7.3.17) and (7.3.21) for the two-channel circuit, we prove that the gain factor decreases as compared to (7.6.2) by a factor of μ, and

$$S_{экв} = \mu z \frac{(1+\sqrt{1+h})^2\left(1-\frac{z}{\mu}\right) + h(1+2\sqrt{1+h})(1-z)^2}{4h^2 \Delta f_c \sqrt{1+h}\,(1-z)^2}\tau_и^2. \tag{7.6.8}$$

The dependence of ratio $S_{экв}(\mu, z)/S_{экв}(1, z)$ on μ for various relative detunings z is constructed in Fig. 7.16. This ratio varies little with change of h in a sufficiently wide range, especially if detuning is small. From the graph it is possible to establish the expansion of the gate pulse and under what conditions it can be considered permissible. With sufficiently good approximation the loss in magnitude of spectral density $S_{экв}$ can be considered equal to μ in all operating conditions.

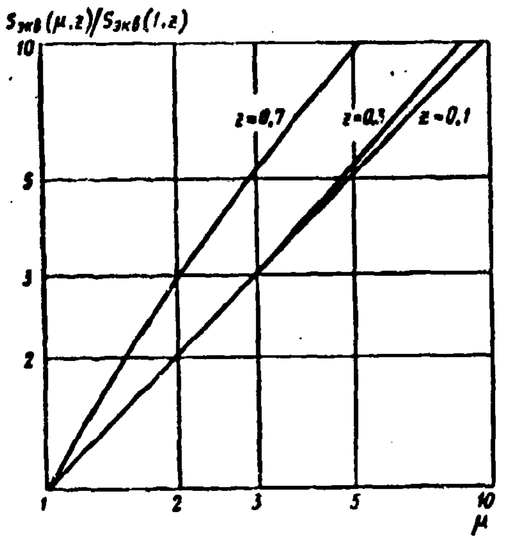

Fig. 7.16. Influence of duration of the gate pulse on $S_{экв}$ for a square pulse.

Results for the circuit with switching are still simpler. Since coefficient $M(\delta)$ from (7.4.12) for $C_{10}(\delta)$ from (7.6.7) is equal to one for all δ, and $h_{1экв}$ from (7.4.9) is h/μ, everything reduces to equivalent decrease of the signal-to-noise ratio by a factor of μ. Here, the magnitude of optimum detuning decreases to

$$\delta_1 = \tau_и\left(1 - \sqrt{\frac{2\mu}{h}}\right).$$

and equivalent spectral density for any μ, as before, is characterized by formula (7.6.4) and the graph of (7.15) with replacement of $h_{1\,\text{экв}} = h/\mu$ for h.

7.6.2. Gaussian Pulse

Very often the shape of the pulse essentially differs from rectangular. Here a convenient approximation, corresponding well to reality in many cases, is a Gaussian curve of form

$$u(t)=\sqrt{\frac{T_r}{\tau_и}}\,e^{-\frac{\pi t^2}{2\tau_и^2}}, \tag{7.6.9}$$

where $\tau_И$ – duration of the equivalent square pulse of the same amplitude $u_0 = \sqrt{\frac{T_r}{\tau_и}}$, introduced generally by the following relationship:

$$\tau_и=\int_{-\infty}^{\infty}\frac{u^2(t)}{u^2_{\text{макс}}}\,dt=\int_{-\infty}^{\infty}\frac{u^2(t)}{u^2(0)}\,dt. \tag{7.6.10}$$

Quantity $\tau_И$ is connected with the duration of a pulse at level 0.5 by the obvious relationship

$$\tau_{0,5}=\sqrt{\frac{5,6}{\pi}}\,\tau_и=1,33\tau_и. \tag{7.6.11}$$

We shall consider that gate pulses (reference signals) $u_1(t)$ also have Gaussian form and duration $\tau_C = \mu\tau_И$, i.e.,

$$u_1(t)=\sqrt{\frac{T_r}{\tau_c}}\,e^{-\frac{\pi t^2}{2\tau_c^2}}=\sqrt{\frac{T_r}{\mu\tau_и}}\,e^{-\frac{\pi}{2\mu^2}\frac{t^2}{\tau_и^2}}. \tag{7.6.12}$$

Then

$$C_{00}(\delta)=C(\delta)=e^{-\frac{\pi z^2}{4}},\quad C_{11}(\delta)=e^{-\frac{\pi z^2}{4\mu^2}};$$
$$C_{10}(\delta)=\sqrt{\frac{2\mu}{1+\mu^2}}\,e^{-\frac{\pi z^2}{2(1+\mu^2)}}, \tag{7.6.13}$$

where, as before, $z = \delta/\tau_И$.

The mean square spectrum width for a Gaussian pulse, obviously, is equal to

$$b=-C''(0)=\frac{\pi}{2\tau_и^2}. \tag{7.6.14}$$

The slope of the discrimination characteristic for a two-channel circuit according to (7.3.28) is determined by the following expression:

$$K_{д} = D\frac{1}{\tau_{и}}\sqrt{\frac{2\mu}{1+\mu^2}}\frac{\pi z}{1+\mu^2}e^{-\frac{\pi z^2}{1+\mu^2}}. \tag{7.6.15}$$

Its maximum is attained, obviously, when

$$z_{макс} = \sqrt{\frac{1+\mu^2}{2\pi}} = \sqrt{\frac{1+\mu^2}{4b\tau_{и}^2}}, \tag{7.6.16}$$

which increases with growth of μ. Substituting (7.6.13) in (7.3.35) and (7.3.36), we obtain expressions, determining $h_{экв}$ and $b_{экв}$ for a circuit with detuned channels

$$\frac{h_{экв}}{h} = \frac{4\mu}{1+\mu^2}\frac{e^{-\frac{\pi z^2}{1+\mu^2}}}{1+e^{-\frac{\pi z^2}{\mu_1}}} \xrightarrow[z\to 0]{} \frac{2\mu}{1+\mu^2}, \tag{7.6.17}$$

$$\frac{b_{экв}}{b} = \frac{2\pi z^2}{(1+\mu^2)^2}\frac{1+e^{-\frac{\pi z^2}{\mu^2}}}{1-e^{-\frac{\pi z^2}{\mu^2}}} \xrightarrow[z\to 0]{} \frac{4\mu^2}{(1+\mu^2)^2}. \tag{7.6.18}$$

The dependence of ratios $K_{д}(\delta)/K_{д\ макс}$, $h_{экв}(\delta)/h$, $b_{экв}(\delta)/b$ and $S_{экв}(\delta)/S_{опт}$ (for small and large h) on detuning is shown in Fig. 7.17 for the case of coinciding signal and gate pulses ($\mu = 1$). This figure graphically shows how weakly the finite quantity of detuning affects accuracy of range finding in this case. Actually, for $z = z_{макс}$ $b_{экв}/b = 1.09$, $h_{экв}/h = 0.88$, $bh/b_{экв}h_{экв} = 1.04$, $bh^2/b_{экв}h^2_{экв} = 1.18$, i.e., in the most unfavorable case the equivalent spectral density is increased by not more than 20%, and at working signal-to-noise ratios this increase does not exceed 5%.

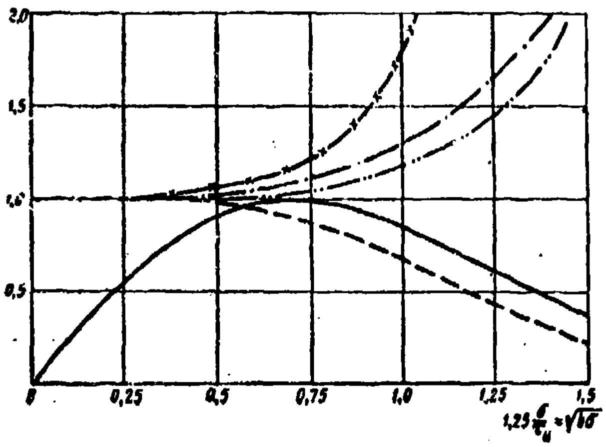

Fig. 7.17. Influence of detuning on characteristics of a two-channel discriminator with Gaussian rounding of the sounding pulse: ——— $K_{д}(\delta)/K_{д\ макс}$; – – – – – – $h_{экв}(\delta)/h$; —·—·— $b_{экв}(\delta)/b$; —··—··— $S_{экв}(\delta)/S_{опт}$ at $h \gg 1$; —×—×— $S_{экв}(\delta)/S_{опт}$ at $h \ll 1$.

For μ, differing from one (we are interested in practically realizable values of $\mu > 1$), the nature of the dependence of $b_{экв}/b$ and $h_{экв}/h$ on detuning does not change, and we are equally justified for values of $z \leq z_{макс}$ in using the formulas for zero detuning. Ratios $b_{экв}/b$ and $h_{экв}/h$ are given here by limiting forms of expressions (7.6.17) and (7.6.18), allowing us easily to analyze the influence of mismatch of durations of the pulse and gate pulse on accuracy of range finding. In particular, for small h ratio

$$\frac{S_{экв}(\mu)}{S_{экв}(1)} = \frac{(1+\mu^2)^4}{16\mu^4}, \tag{7.6.19}$$

for sufficiently large $h_{ЭКВ} = 2h\mu/(1 + \mu^2)$ the spectral density increases in accordance with

$$\frac{S_{экв}(\mu)}{S_{экв}(1)} = \frac{(1+\mu^2)^2}{8\mu^2}. \qquad (7.6.20)$$

Dependences (7.6.19) and (7.6.20) are shown in Fig. 7.18. Formulas (7.6.17)-(7.6.20) and Fig. 7.18 show that in this case expansion of the gate pulse leads to considerably more unpleasant consequences than in the case of a square pulse and gate pulse. For sufficiently large μ spectral density increases as $\mu^4/16$ for small h and as $\mu^2/8$ for large h, whereas for a square pulse we had an approximately linear dependence on μ. The physically stronger dependence $S_{ЭКВ}(\mu)/S_{ЭКВ}(1)$ for a Gaussian pulse and gate pulse is explained by the fact that along with increase of power of noise proceeding to the input of the narrow-band filter, which occurs with expansion of the gate pulse, there simultaneously occurs a sharp decrease of the gain factor, leading immediately to increase of equivalent spectral density. All this imposes more stringent requirements on duration of gate pulses in the case of a Gaussian pulse.

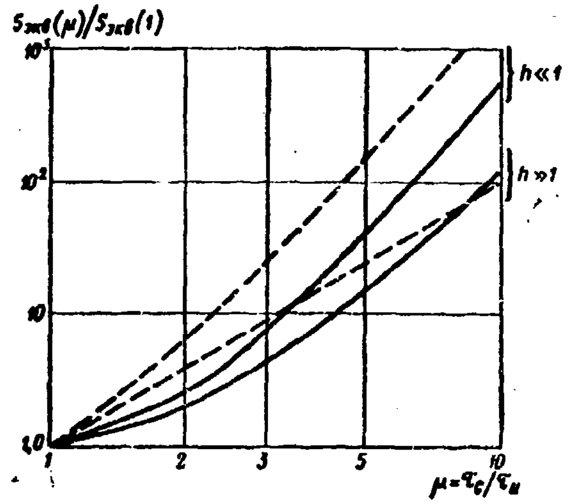

Fig. 7.18. Influence of duration of the gate pulse on $S_{ЭКВ}$ with a Gaussian pulse envelope: —— two-channel discriminator; --- discriminator with switching of reference signals.

Analogous dependences are simply obtained for a circuit with switching, and this consideration leads to analogous conclusions. Being limited to the case of the spectrum of (7.3.11) and of optimum filter (7.4.18) for a circuit with switching, we obtain the following expression for equivalent spectral density:

$$S_{экв} = \frac{\tau_u^2}{2\Delta_{f\,e}^{2}} \frac{\left[1 + \frac{2h\mu}{1+\mu^2} e^{-\frac{\pi z^2}{1+\mu^2}}\right]^{3/2} (1+\mu^2)^4}{4h^2\mu^2\pi^2 z^2 \exp\left(-\frac{2\pi z^2}{1+\mu^2}\right)}. \qquad (7.6.21)$$

The dependence of $S_{ЭКВ}/S_{ОПТ}$ on z for various values of h and μ is shown in Fig. 7.19. It turns out that the optimum value of z, minimizing $S_{ЭКВ}$, comparatively weakly depends on h and approximately coincides with the value of detuning which ensures the maximum gain factor (7.6.16). Noncoincidence of durations of the pulse

and gate pulses leads in a circuit with switching approximately to the same consequences as in a two-channel circuit.

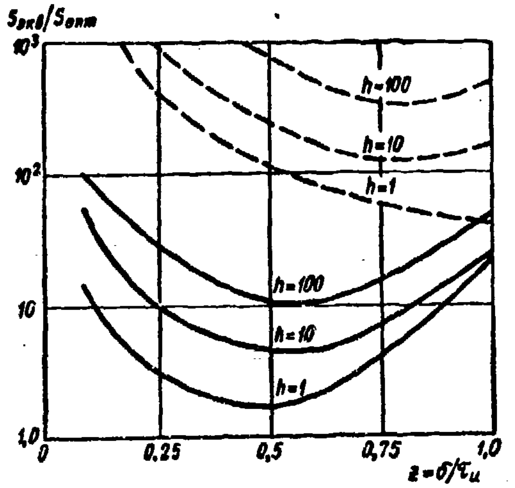

Fig. 7.19. Influence of parameters of a discriminator with switching of the reference signal on $S_{экв}$: —— $\mu = 1$; ---- $\mu = 3$.

In general, if only detuning $z = \frac{\delta}{\tau_и}$ is selected close to the optimum for any μ, spectral density increases as:

for small h

$$\frac{S_{экв}(\mu)}{S_{экв}(1)} = \frac{(1+\mu^2)^2}{4}; \qquad (7.6.22)$$

for large h

$$\frac{S_{экв}(\mu)}{S_{экв}(1)} = \mu^2. \qquad (7.6.23)$$

These dependences are shown in Fig. 7.18 by the dotted line.

7.6.3. Discriminator with Differentiation of the Reference Signal

Technical realization of such a discriminator during pulse radiation will cause, obviously, serious difficulties. If narrow-band filtration of the signal is carried out at intermediate frequency, multiplication by the differentiated reference signal u'(t), always having positive and negative values (Fig. 7.20a), can be carried out by means of multiplication of the signal from the output of the mixer by function |u'(t)| (Fig. 7.20b) and change of the phase of the amplified signal at the time of passage of function u'(t) through zero.

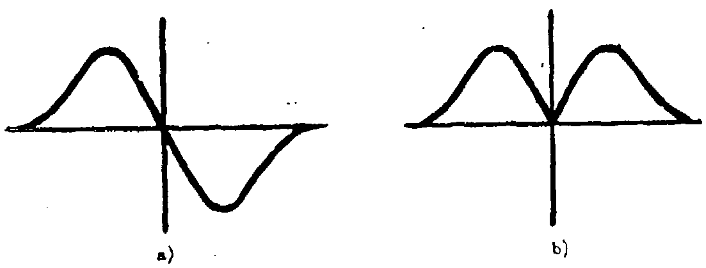

Fig. 7.20. Derivative of the modulating signal and its modulus.

Possibly simpler from the technical point of view will be the circuit of a discriminator with sine and cosine channels and filtration of the received signal at low frequency, analogous to the corresponding circuits of detectors (Chapter IV). This circuit is shown in Fig. 7.21, and in the operations executed by it is completely equivalent to the circuit of Fig. 7.2. Formation of correlation integrals (multiplication by reference signals and filtration) is carried out in it at low frequency. The necessity of using two channels with quadrature mixers was discussed in Chapter IV. In each of these channels a discrimination characteristic

can be created, of course, both with the help of two delay-detuned channels and with the help of switching of reference signals. The advantage of this kind of circuit for the case of coherent radiation as compared to the circuit of Fig. 7.2 and approximations to it can be possibility of measurement of the distance simultaneously of several targets using only one filter in each channel. Such a filter, carrying out accumulation of the signal immediately for all distances may be realized by a charge-storage tube.

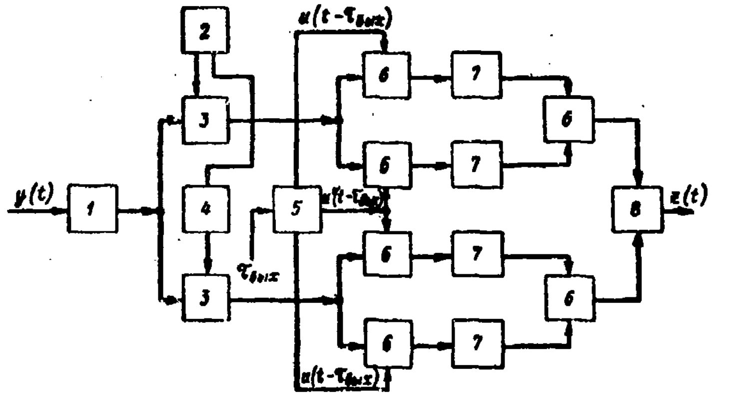

Fig. 7.21. Block diagram of an optimum discriminator with quadrature channels: 1 – preamplifier; 2 – coherent local oscillator; 3 – phase detector; 4 – $\pi/2$ phase shifter; 5 – controlled generator of reference signals; 6 – multiplier; 7 – narrow-band low-frequency filter; 8 – circuit of addition.

Since operations of the circuits of Figures 7.2 and 7.21 completely coincide, for the latter there are preserved all results obtained earlier. This pertains also to all possible methods of approximate realization of the circuit of Fig. 7.21 analogous to those considered earlier for the case of filtration in one channel at intermediate frequency. Let us consider for illustration of a circuit with differentiation the case when the signal pulse has cosine form

$$u(t)=\sqrt{\frac{T_r}{\tau_и}}\cos\frac{\pi t}{2\tau_и}, \qquad (7.6.24)$$

where $\tau_и$ – duration of the equivalent square pulse, connected with the duration of a pulse of zeroes τ_0 by relationship

$$\tau_и=\frac{1}{2}\tau_0. \qquad (7.6.25)$$

We assume that the reference signal of the first channel $u_1(t)$ coincides in form with the signal pulse $u_1(t) = u(t)$, and the reference signal of the second channel constitutes two adjoining identical square pulses of duration $\tau_0/2$, one of which has positive, and the other, negative polarity. In the variant of construction of a discriminator with filtration at intermediate frequency such a signal is generated in the form of a square pulse of duration τ_0, in the middle of which there is carried out change of phase of the filtered signal, or of heterodyne voltage, by π. If the reference pulses of the second channel have, not rectangular form,

but are two adjoining quarters of a sine wave, then, obviously, we have complete matching of the form of modulations of the reference and sounding signals.

In the considered case functions $C_{ik}(\delta)$ have the form

$$\left.\begin{aligned} C_{00}(\delta)=C(\delta)=\left(1-\frac{|\delta|}{\tau_и}\right)\cos\frac{\pi\delta}{2\tau_и}+\\ +\frac{1}{\pi}\cos\frac{\pi\delta}{2\tau_и}\sin\frac{|\delta|}{\tau_и}=C_{11}(\delta)=C_{10}(\delta),\\ C_{22}(\delta)=1-3\frac{|\delta|}{\tau_c}\quad C_{20}(\delta)=C_{21}(\delta)=\\ =\frac{8}{\pi}\sqrt{\frac{\tau_и}{\tau_c}}\sin^2\frac{\pi\tau_c}{8\tau_и}\sin\frac{\pi\delta}{2\tau_и}. \end{aligned}\right\} \tag{7.6.26}$$

The mean square spectrum width b for a cosinusoidal pulse

$$b=\frac{\pi^2}{4\tau_и^2},$$

and quantities necessary for substitution in the formula for spectral density are

$$C_{10}(0)=1,\quad C_{20}(0)=C_{12}(0)=0,\quad C_{22}(0)=1,\ C_{11}(0)=1.$$
$$C'_{10}(0)=0,\quad C'_{20}(0)=\frac{4}{\sqrt{\tau_и\tau_c}}\sin^2\frac{\pi\tau_c}{8\tau_и}.$$

As follows from (7.5.3), there is no systematic error in this case, and equivalent spectral density according to (7.5.5) is

$$S_{экв}=\frac{\tau_и\tau_c}{16\sin^4\frac{\pi\tau_c}{8\tau_и}}\,\frac{\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}|H_1(i\omega)|^2[1+hS_0(\omega)]\,d\omega}{2h^2\left[\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}|H_1(i\omega)|^2S_0(\omega)\,d\omega\right]^2}. \tag{7.6.27}$$

Expression (7.6.27) differs from $S_{экв}$ for the case of coinciding modulations only by factor

$$\frac{b}{|C'_{20}(0)|}=\frac{\pi^2\tau_c}{64\tau_и\sin^4\frac{\pi\tau_c}{8\tau_и}}, \tag{7.6.28}$$

characterizing the decrease of the slope of the discrimination characteristic and the related increase of equivalent spectral density. The dependence of $S_{экв}/S_{опт}$ on $\frac{\tau_c}{\tau_и}=\mu$ is shown in Fig. 7.22. With sufficiently good selection of duration of the gate pulse [τ_c = (2 to 5)$\tau_и$] the loss caused by noncoincidence of modulations is

practically immaterial, and with selection of the proper frequency response of the filter the considered circuit almost does not differ from the optimum.

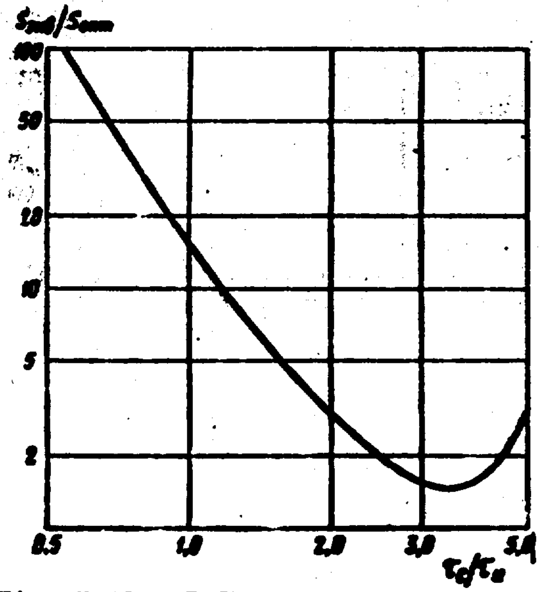

Fig. 7.22. Influence of mismatch of the reference signal with the sounding signal on $S_{\text{экв}}$.

§ 7.7. Continuous Radiation

In the case of a coherent signal of special interest is application of continuous radiation. This is caused by the fact that in a number of cases it may be the most profitable in terms of power, since it permits us for a fixed peak power of the transmitter to ensure maximum average power and, consequently, the greatest accuracy of range finding. Another advantage of continuous radiation is simplicity of realization of frequency selection, which is the most effective means of combatting passive interferences [34]. During continuous radiation we can use the most diverse forms of modulation: amplitude, frequency, phase-code manipulation, random amplitude, phase or frequency modulation [1, 12, 27, 35-37]. Below we shall consider some of the possible forms of modulation.

7.7.1. Frequency Modulation

Let us consider first a signal with periodic frequency modulation according to the law $\Delta\omega(t)$ (sinusoidal and triangular). We shall consider the amplitude of function $\Delta\omega(t)$ equal to ω_m, so that deviation (maximum change) of frequency is equal to $2\omega_m$. The corresponding laws of modulation are shown in Fig. 7.23. A discriminator with frequency modulation can be constructed on any of the previously considered circuits. It is possible that the most suitable from the technical point of view will be the circuit with differentiation of the reference signal.

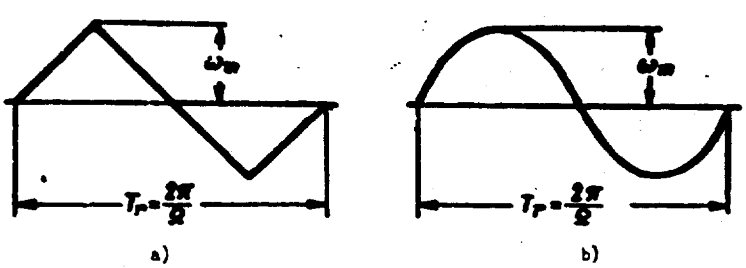

Fig. 7.23. Change of frequency during frequency modulation: a) triangular modulation; b) sinusoidal modulation.

We consider first the case when modulation of reference signals is not distorted. Autocorrelation functions $C(\delta)$ for FM signals are determined for values of δ which are small as compared to the period of modulation by the following relationships:

for sinusoidal modulation

$$C(\delta)=J_0(\omega_m\delta); \tag{7.7.1}$$

for triangular modulation

$$C(\delta)=\frac{\sin\omega_m\delta}{\omega_m\delta}, \tag{7.7.2}$$

where $J_0(x)$ — Bessel function.

Equivalent spectral density for a circuit with differentiation is determined by formula (7.3.9), where in both cases $a = 0$, and $b = \frac{\omega_m^2}{2}$ for sinusoidal modulation and $b = \frac{\omega_m^2}{3}$ for triangular modulation. Thus, frequency modulation by triangular law in principle by a factor of $\sqrt{1.5}$ gives greater fluctuation error than sinusoidal modulation with the same deviation of frequency. Remaining dependences, essential for a circuit with differentiation (dependences of $S_{экв}$ on the signal-to-noise ratio, bands and frequency response of filters) were already studied earlier and need no further discussion.

We shall investigate now how qualities of a discriminator change depending upon detuning δ when we use a two-channel circuit and a circuit with switching. The slope of the discrimination characteristic in both cases is proportional to $J_0(\omega_m\delta)J_1(\omega_m\delta)$ for sinusoidal FM and to $\frac{\sin^2\omega_m\delta}{\omega_m^2\delta}\left(\frac{1}{\omega_m\delta}-\operatorname{ctg}\omega_m\delta\right)$ for triangular FM. For a circuit with two detuned channels quantities $h_{экв}$ and $b_{экв}$, characterizing change of properties of the discriminator with finite detuning, in accordance with (7.3.23) and (7.3.24), are expressed in the following way:

with sinusoidal modulation

$$h_{экв}=k\frac{2J_0^2(\omega_m\delta)}{1+J_0(2\omega_m\delta)}, \tag{7.7.3}$$

$$b_{экв}=\omega_m^2\frac{J_1^2(\omega_m\delta)[1+J_0(2\omega_m\delta)]}{J_0^2(\omega_m\delta)[1-J_0(2\omega_m\delta)]}; \tag{7.7.4}$$

with triangular modulation

$$h_{экв} = h\,\frac{2\left(\frac{\sin\omega_m\delta}{\omega_m\delta}\right)^2}{1+\frac{\sin 2\omega_m\delta}{2\omega_m\delta}}, \tag{7.7.5}$$

$$b_{экв} = \omega_m^2\,\frac{\left[\frac{1}{\omega_m\delta}-\operatorname{ctg}\omega_m\delta\right]^2\left[1+\frac{\sin 2\omega_m\delta}{2\omega_m\delta}\right]}{1-\frac{\sin 2\omega_m\delta}{2\omega_m\delta}}. \tag{7.7.6}$$

The magnitude of equivalent spectral density is determined here by formula (7.3.22) with substitution in it of (7.7.3)-(7.7.6). In particular, for small h for sinusoidal FM

$$\frac{S_{экв}(\delta)}{S_{экв}(0)} = \frac{h^2 b}{h^2_{экв} b_{экв}} = \frac{1-J_0^2(2\omega_m\delta)}{8J_1^2(\omega_m\delta)\,J_0^2(\omega_m\delta)}; \tag{7.7.7}$$

for triangular FM

$$\frac{S_{экв}(\delta)}{S_{экв}(0)} = \frac{1-\left(\frac{\sin 2\omega_m\delta}{2\omega_m\delta}\right)^2}{12\left(\frac{\sin\omega_m\delta}{\omega_m\delta}\right)^4\left[\frac{1}{\omega_m\delta}-\operatorname{ctg}\omega_m\delta\right]^2}. \tag{7.7.8}$$

With sufficiently large $h_{экв}$ the corresponding formulas have the form

$$\frac{S_{экв}(\delta)}{S_{экв}(0)} = \frac{1-J_0^2(2\omega_m\delta)}{4J_1^2(\omega_m\delta)}, \tag{7.7.9}$$

$$\frac{S_{экв}(\delta)}{S_{экв}(0)} = \frac{1-\frac{\sin 2\omega_m\delta}{2\omega_m\delta}}{6\left(\frac{\sin\omega_m\delta}{\omega_m\delta}\right)^2\left[\frac{1}{\omega_m\delta}-\operatorname{ctg}\omega_m\delta\right]^2}. \tag{7.7.10}$$

The dependence of ratios $b_{экв}(\delta)/b$, $h_{экв}(\delta)/h$, $K_П(\delta)/K_{П\,макс}$ and $S_{экв}(\delta)/S_{опт}$ according to (7.7.7) and (7.7.9) on detuning δ with sinusoidal FM is shown in Fig. 7.24. Analogous dependences for triangular FM are shown in Fig. 7.25. From these graphs it is clear that the magnitude of detuning plays practically an identical role in both cases. Selection of detuning, just as in the previously considered case of modulation by Gaussian pulses, is best produced from condition of a maximum gain factor i.e., $\delta_0 \approx \frac{1}{\omega_m}$ for sinusoidal FM and $\delta_0 \approx \frac{1.2}{\omega_m}$ for triangular FM. The equivalent spectral density increases here to a still smaller degree than in Paragraph 7.6.2.

For large $h_{экв}$ in both cases $S_{экв}(\delta_0)/S_{опт}$ is practically equal to one, but for small h spectral density $S_{экв}$ with such detuning is increased in all by 4-6%. To

find the character of relationship $S_{экв}(\delta)/S_{опт}$ for intermediate values of h in Fig. 7.26 we show this ratio during sinusoidal FM for various h. The frequency response of filter $H_1(i\omega)$ is assumed to coincide with the optimum (7.2.5) for spectrum (7.3.11). It is obvious that here $S_{экв}(0) = S_{опт}$.

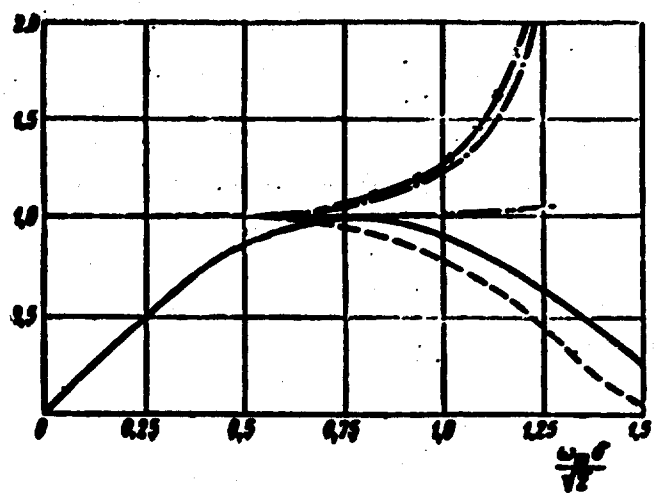

Fig. 7.24. Influence of detuning on the characteristic of a two-channel discriminator with sinusoidal FM: —— $K_д(\delta)/K_{д\,макс}$; – – – $b_{экв}(\delta)/b$; –·–·– $b_{экв}(\delta)/b$; –··–··– $S_{экв}(\delta)/S_{экв}$ at $h>1$; –×–×– $S_{экв}(\delta)/S_{экв}$ at $h<1$.

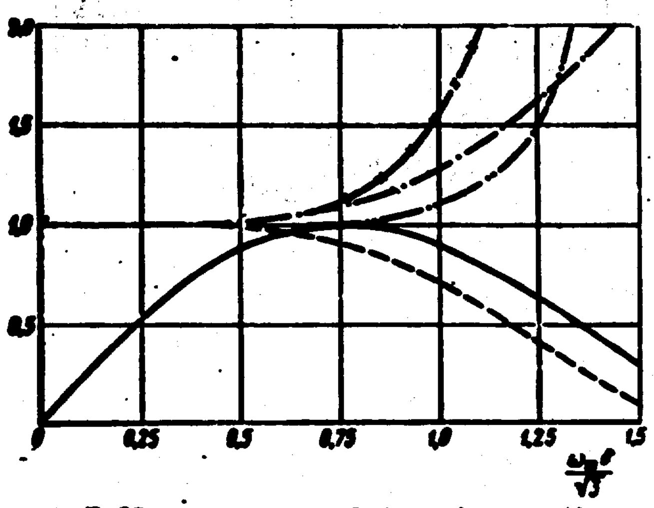

Fig. 7.25. Influence of detuning on the characteristic of a two-channel discriminator with triangular FM: —— $K_д(\delta)/K_{д\,макс}$; – – – $b_{экв}(\delta)/b$; –·–·– $b_{экв}(\delta)/b$; –··–··– $S_{экв}(\delta)/S_{экв}$ at $h>1$; –×–×– $S_{экв}(\delta)/S_{экв}$ at $h<1$.

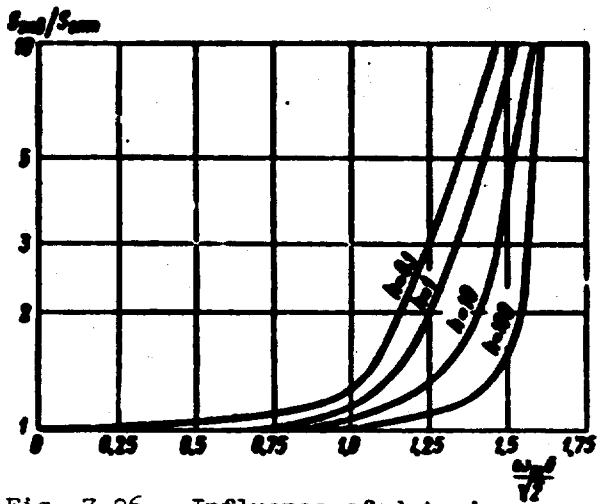

Fig. 7.26. Influence of detuning on $S_{экв}$ for a two-channel discriminator with sinusoidal FM.

Thus, the examples considered here once again confirm the fact that with any reasonable selection of detuning a two-channel circuit practically does not differ from the optimum when there are coinciding laws of modulation of the reference and sounding signals.

To find the character of $S_{экв}(\delta)$ in a circuit with switching we shall limit ourselves to the case of spectrum (7.3.11) and the corresponding optimum filter (7.4.18). Then, in accordance with (7.4.19), with sinusoidal modulation

$$S_{экв}(\delta) = \frac{[1 + hJ_0^2(\omega_m\delta)]^{3/2}}{2\Delta f_e h^2 \omega_m^2 J_1^2(\omega_m\delta) J_0^2(\nu_m\delta)}. \qquad (7.7.11)$$

Ratio $S_{экв}(\delta)/S_{опт}$ for the given circuit is shown for various values of h in Fig. 7.27. This dependence is more substantial than with modulation by Gaussian pulses and an analogous discriminator circuit. The minimum, occurring for a certain

value δ_0, is rather deep. The optimum value of δ, equal to δ_0, as follows from the graph, increases with increase of the signal-to-noise ratio h. For small h this value coincides with δ, for which the gain factor will be maximum; for larger h it increases, approaching a value, corresponding to the first root of function $J_0(x)$.

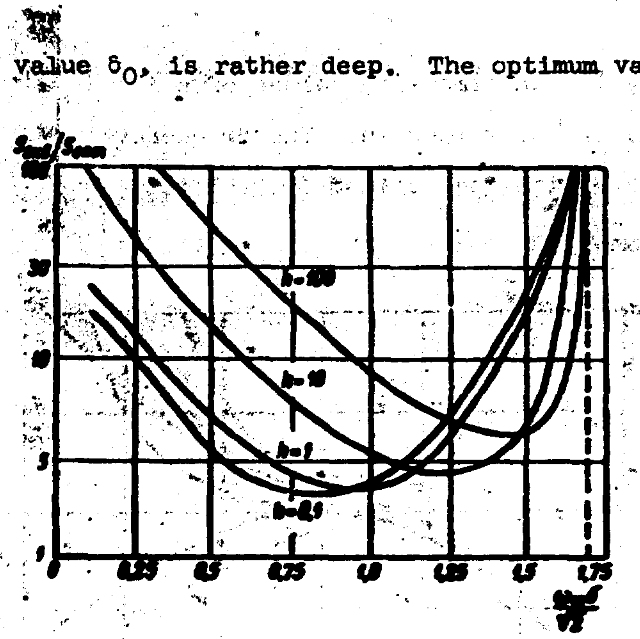

Fig. 7.27. Influence of detuning on $S_{экв}$ for a discriminator with switching of reference signals during sinusoidal FM.

Presence of a minimum for $S_{экв}(\delta)$ should be allowed for during practical construction of a system. Incorrect selection of detuning, as follows from Fig. 7.27, leads to impermissibly great losses of accuracy. Inasmuch as normally the same system should be used for different values of signal power (at different distances), one should select δ equal to δ_0, corresponding to such a value of h at which the most stringent requirements are made on accuracy of the system; this usually takes place at maximum ranges to the target, when the value of h has the order of 5-10. The optimum value of δ_0 here is about $1.5/\omega_m$, and fluctuation error is 1.5-2 times larger than the minimum. The dependence of $S_{экв}(\delta)/S_{опт}$ on h has a normal character. For small h the loss is comparatively small; with growth of h quantity $S_{экв}(\delta)/S_{опт}$ increases, which is caused by a different order of decrease of $S_{экв}$ for large h in an optimum circuit and in a circuit with switching.

Results for triangular FM turn out to be completely analogous; optimum values of δ, at which $S_{экв}(\delta)$ will be minimal, and the actual curves for $S_{экв}(\delta)/S_{опт}$ practically coincide with the curves for sinusoidal FM.

In real circuits with FM the reference and sounding signals are obtained by means of frequency modulation of separate generators. With such a construction due to noncoincidence of modulation characteristics of the transmitter and local oscillator the laws of modulation may not coincide. A very simple case of noncoincidence is difference in frequency deviations. The influence of this phenomenon on accuracy can be calculated by the formulas of the preceding paragraphs. We shall limit ourselves to circuits with two channels and with differentiation of the reference signal. In § 7.6 with the example of pulse radiation we already proved that the influence of nonidentity in the two-channel circuit practically does not

depend on detuning for not very large values of δ. Then, considering sufficiently small detunings δ, with the help of formulas (7.3.33) and (7.5.5) we can simply prove that all results for both circuits are identical.

If the difference of frequency deviations of the sounding and reference signals is equal to $\Delta\omega_m$, with sinusoidal FM

$$C_{11}(0)=1,\ |C_{10}(0)|=|J_0(\mu)|,\ a_{10}=a_{11}=0,\quad b_{11}=\frac{\omega_m^2}{2},\ b_{10}=\frac{J_1(\mu)}{\mu|J_0(\mu)|}\omega_m^2, \tag{7.7.12}$$

where $\mu = \frac{\Delta\omega_m}{\omega_r} = \frac{\Delta\omega_m T_r}{2\pi}$ – ratio of the difference of deviations to the frequency of repetition of modulation.

With triangular FM

$$C_{11}(0)=1,\ C_{10}=\int_0^1\cos\frac{\pi\mu}{4}(1-z^2)\,dz,\ a_{11}=a_{10}=0,$$

$$b_{11}=\frac{\omega_m^2}{3},\ b_{10}=\omega_m^2\cos\frac{\pi\mu}{4}\frac{\int_0^1 z^2\cos\frac{\pi\mu}{4}(1-z^2)\,dz}{\int_0^1\cos\frac{\pi\mu}{4}(1-z^2)\,dz}. \tag{7.7.13}$$

Substituting (7.7.12) and (7.7.13) in (7.3.33), we can obtain the dependence of equivalent spectral density on μ. This relationship is shown in Fig. 7.28 on the assumption that the frequency response of the filters is optimum. As follows from the graphs, when μ > 1 there occurs rapid increase of spectral density. When μ ≈ 2.4 for sinusoidal FM and μ = 2 for triangular, the reference and sounding signals become orthogonal and $S_{экв}$ turns into infinity. Deviations $\Delta\omega_m$ > (1 to 1.5)ω_r, apparently, already are impermissible. Noncoincidence of deviations is somewhat more evident for triangular modulation and, as in the general case of any noncoincidences, leads to greater

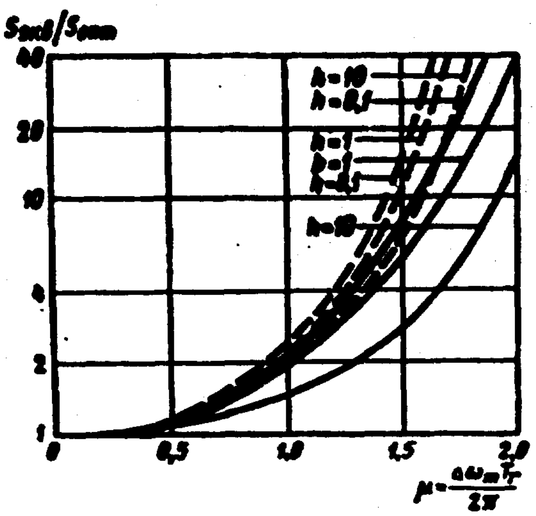

Fig. 7.28. Influence of the difference of frequency deviations of the sounding and reference signals on $S_{экв}$ in a two-channel discriminator: —— sinusoidal FM, --- triangular FM.

impairments, the less the signal-to-noise ratio.

7.7.2. Phase-Code Manipulation (FKM)

Characteristics of different forms of FKM are considered in Chapter I. In the same place it is shown that with any reasonable selection of the code, ensuring a sufficiently low level of side lobes, the autocorrelation function of a phase-code-manipulated signal within the main lobe with a properly selected code is described by the following approximate expression:

$$C(\delta)\approx 1-\frac{|\delta|}{\tau_K},\ |\delta|<\tau_K, \tag{7.7.14}$$

which becomes more exact, the larger the numbers of code intervals n in the period of repetition of the signal. Here τ_K — duration of the code interval. For certain codes (e.g., for binary code with values of phase 0 and $\pi - \arccos\frac{n-1}{n+1}$) this expression is exact.

Function (7.7.14) for values of δ from $-\tau_K$ to τ_K, which are only of interest for the work of an automatic range finder, coincides with the autocorrelation function for square pulse modulation. Since all dependences of characteristics of the discriminator on the form and parameters of modulation are completely determined by function $C(\delta)$, it is obvious that it remains for us only to repeat everything said in Paragraph 7.6.1 with respect to range finding with modulation by square pulses with duration $\tau_H = \tau_K$ and with coincidence of forms of the sounding and reference signals. With FKM there can be used either a two-channel discriminator, or a discriminator with switching, and in both cases all results of Paragraph 7.6.1 are simply repeated.

We consider that duration τ_Π of switching of phase of an FKM signal from one value to the other is finite. This is necessary when the magnitude of detuning is less than the duration of switching. The coefficients of the expansion of function $C(\delta)$ in a series of powers of δ are equal to

$$a=0,\ b=\frac{1}{T_r}\int_0^{T_r}|\Psi'(t)|^2\,dt=\frac{l}{T_r}\int_0^{\tau_\Pi}|\Psi'(t)|^2\,dt,$$

where l — number of changes of phase, and the integral in the final formula is taken over the edge of a code pulse.

If phase changes by linear law, $[\Psi'(t)]^2=\frac{\pi^2}{\tau_\Pi^2}$, and taking into account $l\approx\frac{n}{2}$

$$b = \frac{\pi^2}{2}\frac{\tau_u}{\tau_n}\frac{1}{\tau_n^2}. \tag{7.7.15}$$

It is interesting to note that the maximum value of b is reached when the phase changes at the end of every interval, i.e., $l = n$. Quantity b here is increased by a factor of 2; however, simultaneously, in function $C(\delta)$ there appear large side lobes. Thus, increase of fluctuation error is the price we pay for the high resolution capability provided by FKM.

7.7.3. Noise Phase Modulation

Let us consider a signal whose phase changes proportionally to the instantaneous value of a normal stationary random process

$$\varphi(t) = a\xi(t).$$

If process $\xi(t)$ is ergodic, time-averaging, produced during calculation of function $C(\delta)$, is equivalent to ensemble-averaging and

$$C(\delta) = \overline{e^{ia[\xi(t-\delta)-\xi(t)]}} = e^{-a^2\sigma_m^2[1-\rho_m(\delta)]}, \tag{7.7.16}$$

where σ_m^2 and $\rho_m(\delta)$ – variance and correlation coefficient of modulating process $\xi(t)$.

The mean square spectrum width of modulation in accordance with (7.2.11) and (7.2.12) is equal to

$$b = -C''(0) = -a^2\sigma_m^2\rho''(0)\ (a = 0), \tag{7.7.17}$$

i.e., is $a^2\sigma_m^2$ times greater than the mean square spectrum width of the modulating process $\xi(t)$. In particular, if the spectral density of $\xi(t)$ is uniform in frequency band $(-\omega_m, \omega_m)$,

$$\left.\begin{aligned} C(\delta) &= e^{-a^2\sigma_m^2\left(1-\frac{\sin\omega_m\delta}{\omega_m\delta}\right)},\\ b &= a^2\sigma_m^2\frac{\omega_m^2}{3}. \end{aligned}\right\} \tag{7.7.18}$$

A characteristic feature of the considered form of modulation is the aperiodicity of function $C(\delta)$ and the related absence of ambiguity, and also the fact that $C(\delta)$ does not decrease to zero as $\delta \to \infty$, but has a finite limit $C(\infty) = e^{-a^2\sigma_m^2}$. The magnitude of this limit, determining the possibility of resolution of targets, sufficiently removed in distance, depends on the magnitude of product $a\sigma_m$. For instance,

for $C(\delta)$ from (7.7.18) the first spurious maximum has magnitude $e^{-0.86(\alpha\sigma_m)^2}$. This means that for a fixed magnitude of b it is more expedient to use narrow-band modulating voltage with a larger magnitude of $\alpha\sigma_m$ than broad-band modulating voltage with smaller $\alpha\sigma_m$.

In the considered case the slope of the discrimination characteristic changes proportionally to

$$\frac{\sin\omega_m\delta}{\omega_m\delta}\left(\frac{1}{\omega_m\delta}-\operatorname{ctg}\omega_m\delta\right)e^{-2(\alpha\sigma_m)^2\left(1-\frac{\sin\omega_m\delta}{\omega_m\delta}\right)},$$

and all formulas necessary for calculations ($h_{экв}$, $b_{экв}$, $S_{экв}(\delta)/S_{экв}(0)$) are obtained, for instance, by replacement in the formulas for sinusoidal FM of $J_0(\omega_m\delta)$ by (7.7.18), and of $\omega_m J_1(\omega_m\delta)$ by

$$\omega_m\frac{\sin\omega_m\delta}{\omega_m\delta}\left(\frac{1}{\omega_m\delta}-\operatorname{ctg}\omega_m\delta\right)(\alpha\sigma_m)^2\times$$
$$\times\exp\left\{-(\alpha\sigma_m)^2\left[1-\frac{\sin\omega_m\delta}{\omega_m\delta}\right]\right\}$$

or by the general formulas of §§ 7.3-7.5.

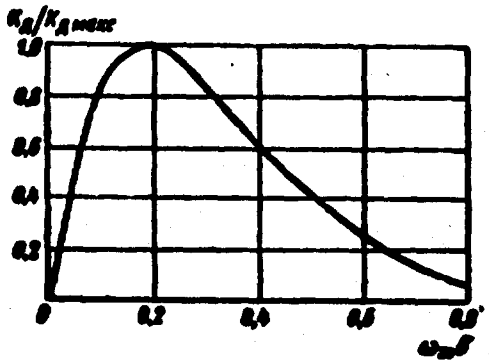

Fig. 7.29. Dependence of the gain factor on detuning with noise phase modulation.

Dependences of $K_д/K_{д\,макс}$ and of $S_{экв}(\delta)/S_{экв}(0)$ on detuning for various values of h for the two-channel circuit and the circuit with switching are shown in Figures 7.29-7.31, respectively. Quantity $\alpha\sigma_m$ in the graphs is taken equal to five. In accordance with general affirmations of Paragraph 7.3.3 the influence of detuning for a two-channel circuit in this case is more substantial than in those considered earlier. When $\delta = \delta_{макс} \approx \frac{0.2}{\omega_m}$ the equivalent spectral density is increased by a factor of 1.3 when h = 100 and a factor of 2 when h = 0.1. In the circuit with switching the dependence of $S_{экв}(\delta)/S_{экв}$ on detuning has a rather complicated character; minimum values of this ratio are rather large, and increase of it for δ, differing from δ_0, occurs comparatively rapidly. The optimum value δ_0 essentially depends on the signal-to-noise ratio and for large h greatly differs from $\delta_{макс}$.

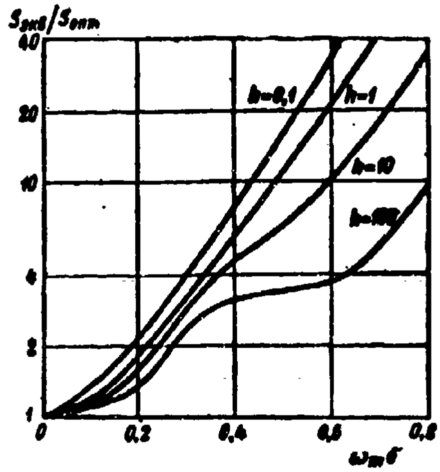

Fig. 7.30. Influence of detuning on $S_{\text{ЭКВ}}$ in a two-channel discriminator with noise phase modulation.

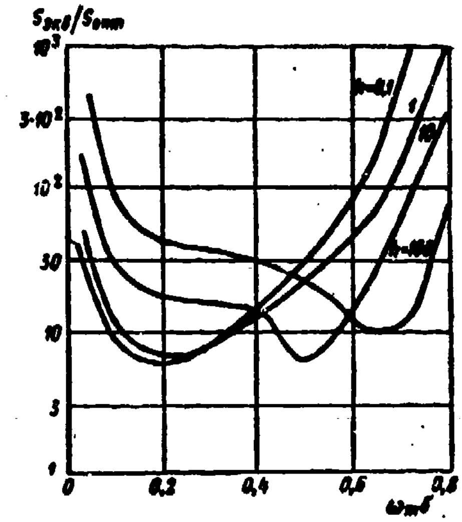

Fig. 7.31. Influence of detuning on $S_{\text{ЭКВ}}$ in a discriminator with switching of reference signals with noise phase modulation.

7.7.4. Noise Signal

We next consider range finding with purely noise radiation. Such a signal can be formed by transmission of broad-band noise through a bandpass filter with subsequent amplification, transfer to the frequency of radiation and amplification at this frequency. Modulating voltage u(t) here is a normal stationary random process, and the autocorrelation function C(δ) coincides with the correlation function of this process, so that

$$C(\delta)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S_m(\omega)\,e^{i\omega\delta}\,d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\delta}\frac{|H_{\phi}(i\omega)|^2}{\Delta f_{\text{эф}}}\,d\omega, \qquad (7.7.19)$$

where $S_m(\omega)$ – spectral density of process u(t)

$$\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} S_m(\omega)\,d\omega=1\right);$$

$H_{\phi}(i\omega)$ – frequency response of the low-frequency equivalent of the shaping filter;

$\Delta f_{\text{эф}}$ – its passband.

In spite of the fact that technical methods of delaying such a signal, just as the signal with phase noise modulation considered in Paragraph 7.7.3, at present are far from clear, it is of definite interest both from the point of view of simplicity of generation, and also from the point of view of absence of ambiguity.

In the optimum system and in a circuit with differentiation accuracy of range finding is determined by the mean square spectrum width

$$b=\frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2 S_m(\omega)\,d\omega,$$

and the influence of finite detuning in the two-channel circuit and in the circuit with switching is determined by the form of function $C(\delta)$. If the spectral density of modulation $S_m(\omega)$ is described by a Gaussian curve or has rectangular form, function $C(\delta)$ has the form of (7.6.13) or (7.7.2), and all results of Paragraphs 7.4.2 and 7.7.3, pertaining to pulse modulation by Gaussian pulses and frequency modulation by triangular law, remain in force. We next consider the case, when the shaping filter is a sequency of two decoupled LRC-filters, i.e., frequency response $H_{\Phi}(i\omega)$ has the form of (7.3.16). Substituting (7.3.16) in (7.7.19), we obtain

$$\begin{gathered} C(\delta)=(1+4\Delta f_{э\phi}|\delta|)\,e^{-4\Delta f_{э\phi}|\delta|}, \\ a=0,\ b=-C''(0)=16\Delta f_{э\phi}^{2}. \end{gathered} \tag{7.7.20}$$

Dependences of $K_Д(\delta)/K_{Д\ макс}$, $b_{экв}(\delta)/b$, $h_{экв}(\delta)/h$ and $S_{экв}(\delta)/S_{экв}(0)$ on detuning, calculated for a two-channel circuit by formulas of § 7.3, pertaining to the case of coinciding modulations of autocorrelation determined by formula (7.7.20), are constructed in Fig. 7.32. The influence of detuning in this case is somewhat more substantial than for other forms of $C(\delta)$. When $\delta = \delta_{макс}$, $S_{экв}(\delta_{макс})/S_{экв}(0)$ increases already by 1.7 times for large h and by 2.5 times for small h. This difference is explained by the fact that due to the slowness of decrease of $C(\delta)$ ratio $\frac{b_{экв}}{b} \approx 1$ for all δ.

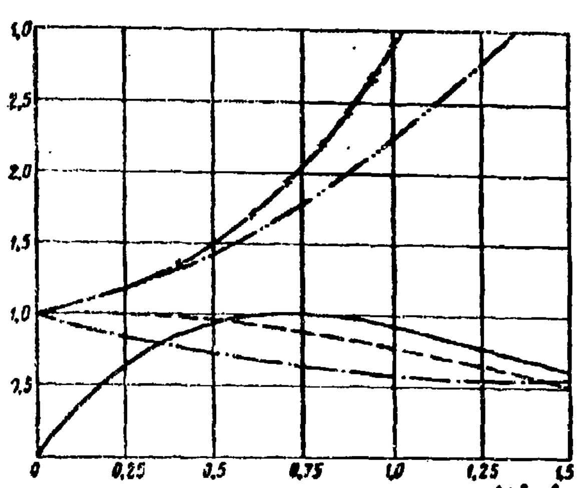

Fig. 7.32. Influence of detuning on characteristics of a two-channel discriminator with a noise signal:
——— $K_Д(\delta)/K_{Д\ макс}$; — — — — $h_{экв}(\delta)/h$; —·—·— $b_{экв}(\delta)/b$; —··—··— $S_{экв}(\delta)/S_{опт}$ at $h\gg 1$; —×—×— $S_{экв}(\delta)/S_{опт}$ at $h\ll 1$.

7.7.5. Amplitude Modulation

Amplitude modulation during continuous radiation has the deficiency that for a fixed peak power of the sounding signal it provides, other things being equal, lower mean

power and, consequently, lower accuracy of measurement than frequency or phase modulation. However, in certain cases its application, possibly, is sensible, and we shall briefly discuss it, limiting ourselves to the very simple case of sinusoidal modulation by one frequency. The modulating signal u(t) here has the form

$$u(t)=\frac{1+m\cos\omega_m t}{\sqrt{1+\frac{m^2}{2}}}, \tag{7.7.21}$$

and its autocorrelation function and mean square spectrum width are

$$C(\delta)=\frac{1+\frac{m^2}{2}\cos\omega_m\delta}{1+\frac{m^2}{2}}, \tag{7.7.22}$$

$$b=\frac{m^2\omega_m^2}{2\left(1+\frac{m^2}{2}\right)},\ a=0. \tag{7.7.23}$$

The most reasonable scheme of construction of a discriminator in this case is, in all probability, a circuit with differentiation of the reference signal. Multiplication by reference signals reduces to heterodyning the received signal with the signal amplitude-modulated by law (7.7.21) in the first channel and with signal $\sin[(\omega_0+\omega_{\text{пр}}+\omega_m)t-\omega_m\tau]-\sin[(\omega_0+\omega_{\text{пр}}-\omega_m)t+\omega_m\tau]$ in the second channel. The required coincidence of laws of modulation in this case can be ensured without great difficulties. The magnitude of the equivalent spectral density is determined here by expression (7.3.9).

The form of formulas (7.7.22) and (7.7.23) already indicates the essential deficiences of such a signal. It is characterized by low resolution capability and great ambiguity. The period of ambiguity and accuracy of range finding are determined by the same magnitude — the frequency of modulation ω_m, where $1/\sqrt{b}$, having the meaning of a certain equivalent pulse duration, for small m is even larger than the period of ambiguity. Accuracy of range finding with sinusoidal AM essentially depends on the modulation percentage. For small m fluctuation error is proportional to 1/m, and with increase of m to infinity it approaches a finite limit, determined only by the frequency of modulation. Physically the signal with $m > 1$ signifies radiation at three frequencies ω_0, $\omega_0-\omega_m$ and $\omega_0+\omega_m$, where with increase of m the relative intensity of the center frequency decreases, while that of the sidebands increases. The case $m=\infty$ signifies radiation at two frequencies, $2\omega_m$ apart and

having identical intensity. As $m \to \infty$ autocorrelation function $C(\delta) = \cos \omega_m \delta$ and $b = \omega_m^2$.

§ 7.8. Pulse Radiation with Additional Modulation

The effort to ensure high power potential in pulse radars leads to the requirement of decreasing the off-duty factor. At the same time in certain cases the only method of eliminating ambiguity remaining is selection of a sufficiently low frequency of repetition. These two requirements, in turn, lead to the necessity of increase of pulse duration to such limits that in a number of cases it is impossible to ensure the needed accuracy of range finding. With selection of a high frequency of repetition with ambiguity with respect to range pulse duration can be so great that the requirements on accuracy of measurement and resolution capability with respect to range are not satisfied. Therefore, it is of interest to consider intrapluse modulation. Use of pulse signals with additional modulation permits us to immediately obtain high power potential, great accuracy of range finding, and good resolving power [38-39].

As additional modulation we can use all those forms of modulation which were considered for continuous radiation. We shall limit ourselves to only two examples — linear frequency modulation and phase-code manipulation within limits of a pulse. Signals with such forms of modulation possess, apparently, the greatest advantages. As discriminators here we can use all the circuits considered earlier, both discriminators of correlation type of all three forms, and also discriminators with optimum filtration and pulse shortening (§ 7.5).

7.8.1. Linear Frequency Modulation

In the given case additional modulation consists of linear change of frequency within the limits of a pulse. Without loss of generality one may assume that this

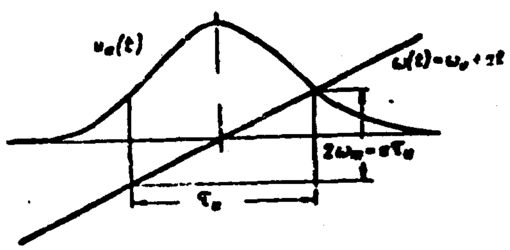

Fig. 7.33. Pulse signal with linear frequency modulation.

change occurs symmetrically with respect to the center of the pulse with speed $a = \frac{d\omega}{dt}$ (Fig. 7.33). Then signal u(t) has the form

$$u(t) = u_a(t) e^{\frac{iat^2}{2}}, \qquad (7.8.1)$$

where $u_a(t)$ is a function, describing the pulse shape, where, as before,

$$\frac{1}{T_r}\int_{-\frac{T_r}{2}}^{\frac{T_r}{2}} u_a^2(t)\,dt = 1, \quad u_a(0) = \sqrt{\frac{T_r}{\tau_u}}, \qquad (7.8.2)$$

$\tau_{И}$ – duration of an equivalent square pulse of the same energy.

Subsequently we assume that $u_a(t)$ is a function symmetric with respect to zero. Furthermore, we consider that the width of the spectrum of the modulated pulse is much greater than the width of the spectrum of the unmodulated pulse, i.e., $a\tau_И^2 \gg 1$. This condition means that in practice $u_a(t - \frac{1}{a\tau_И}) = u_a(t)$. Then the autocorrelation function of the signal is

$$C(\delta) = \frac{1}{T_r}\int_{-T_r/2}^{T_r/2} u_a(t)\,u_a(t+\delta)\,e^{ia\delta t + \frac{ia\delta^2}{2}}\,dt \approx$$
$$\approx \frac{1}{T_r}\int_{-T_r/2}^{T_r/2} u_a^2(t)\cos a\delta t\,dt, \qquad (7.8.3)$$

or, introducing frequency deviation for pulse duration $2\omega_m = a\tau_И$ and considering finiteness of pulse $u_a(t)$, we have

$$C(\delta) = \frac{1}{T_r}\int_{-\infty}^{\infty} u_a^2(t)\,e^{ia\delta t}\,dt = \frac{1}{T_r}\int_{-\infty}^{\infty} u_a^2(t)\,e^{i\frac{2\omega_m\delta}{\tau_И}t}\,dt. \qquad (7.8.4)$$

Let us consider several examples for modulation by pulses of different shape. If, for instance, $u_a(t)$ is a square pulse,

$$C(\delta) = \frac{\sin\frac{a\tau_И\delta}{2}}{\frac{a\tau_И\delta}{2}} = \frac{\sin\omega_m\delta}{\omega_m\delta}, \qquad (7.8.5)$$

which coincides for identical deviations with (7.72) for triangular FM during continuous radiation.

During modulation by Gaussian pulses (7.6.9)

$$C(\delta) = e^{-\frac{a^2\tau_И^2\delta^2}{4\pi}} = e^{-\frac{\omega_m^2\delta^2}{\pi}}, \quad b = \frac{2\omega_m^2}{\pi}, \qquad (7.8.6)$$

which coincides with the case of absence of intrapulse modulation, if pulse duration has equivalent magnitude

$$\tau_{экв} = \frac{\pi}{2\omega_m} = \frac{\pi}{a\tau_И}. \qquad (7.8.7)$$

In this case it is also possible to easily obtain an exact expression for $C(\delta)$ without the above-indicated assumption about the relationship of spectrum width of modulated and unmodulated pulses. Expressions for $C(\delta)$, b and $\tau_{экв}$ here have the form

$$C(\delta)=\exp\left\{-\frac{\omega_m^2\delta^2}{\pi}\left(1+\frac{\pi^2}{4\omega_m^2\tau_и^2}\right)\right\}, \qquad (7.8.8)$$

$$b=\frac{2\omega_m^2}{\pi}\left(1+\frac{\pi^2}{4\omega_m^2\tau_и^2}\right), \quad \tau_{экв}=\frac{\pi}{2\omega_m}\frac{1}{\sqrt{1+\frac{\pi^2}{4\omega_m^2\tau_и^2}}}.$$

Thus, from the given example it is clear that corrections to results (7.8.6) and (7.6.7) have the order $\frac{\pi^2}{4\omega_m^2\tau_и^2}=\frac{\pi^2}{a^2\tau_и^4}$. If we require that duration of the shorter pulse $\tau_{экв}$ be less than the duration of the basic pulse by at least a factor of 5-10, the correction for allowance for finite duration of the basic pulse will have the order $(1-5)10^{-2}$ and can be considered immaterial.

From comparison of (7.8.5) and (7.8.6) it follows that the shape of the pulse envelope comparatively weakly affects accuracy of range finding. With a given rate of change of frequency a and effective pulse duration $\tau_и$ or with assigned frequency deviation for a pulse duration $2\omega_m = a\tau_и$ coefficient b is approximately identical both for the Gaussian and for the square pulse. However, the shape of the pulse envelope essentially affects behavior of function $C(\delta)$ with large δ, i.e., the resolving power with respect to distance. With a square envelope we obtain comparatively large side lobes of $C(\delta)$, and with a Gaussian envelope $C(\delta)$ monotonically decreases. For pulses of sufficiently great duration to recreate a Gaussian envelope which ensures high resolving power may be difficult. Therefore, it is useful to consider a more realistic envelope shape, which at the same time would ensure good resolving power. If, for instance, the envelope has the shape of a cosine wave (7.6.24),

$$C(\delta)=\frac{\sin 2\omega_m\delta}{2\omega_m\delta}\,\frac{1}{1-\frac{4}{\pi}\omega_m^2\delta^2} \quad (2\omega_m=a\tau_и),$$

$$b=\frac{4(\pi^2-b)}{3\pi^2}\omega_m^2\approx 0{,}52\omega_m^2.$$

This function decreases in the vicinity of zero somewhat faster than $\frac{\sin\omega_m\delta}{\omega_m\delta}$, and more slowly than Gaussian (7.8.7). Its first zero coincides with the first zero of $\sin\omega_m\delta/\omega_m\delta$ when $\delta=\pi/\omega_m$, and further zeroes occur with twice the frequency. Already the first spurious maximum of this function comprises in absolute value only 2.7% of $C(0)$ in distinction from the case of a square envelope, where this maximum is 22%. Thus, modulation by cosinusoidal pulses ensures practically the same resolving power as with a Gaussian envelope.

It should be noted that relationship (7.8.4) permits us to find the shape of the pulse envelope necessary for producing a signal with any assigned autocorrelation function $C(\delta)$. Therefore, we have the possibility of assigning any shape of shortened pulse and here must only correctly select the shape of the envelope of the fundamental pulse. Producing in (7.8.4) reverse Fourier transformation, we obtain the following expression, establishing the relationship between the assigned function $C(\delta)$ and the required form of the pulse envelope $u_a(t)$

$$u_a(t)=\sqrt{\frac{T_r a}{2\pi}\int_{-\infty}^{\infty} C(\delta)\, e^{-ia\delta t}\, d\delta}. \tag{7.8.9}$$

It is necessary, of course, to assign $C(\delta)$ in such a manner that it indeed is an autocorrelation function and possesses the required properties for this (the Fourier transform of $C(\delta)$ should be positive everywhere). Furthermore, for an assigned power and effective pulse duration $u_a(t)$, i.e., with fulfillment of (7.8.2), function $C(\delta)$ should obey normalizing condition

$$\int_{-\infty}^{\infty} C(\delta)\, d\delta=\frac{2\pi}{a\tau_u}=\frac{\pi}{\omega_m}, \tag{7.8.10}$$

which for an assigned rate of change of frequency a fixes the effective duration of $C(\delta)$, and for an assigned duration of $C(\delta)$ assigns the required magnitude of a.

For instance, if we require that $C(\delta)$ be a triangular function of form (7.6.1), condition (7.8.10) gives

$$C(\delta)=1-\frac{\omega_m|\delta|}{\pi},\quad |\delta|<\frac{\pi}{\omega_m}, \tag{7.8.11}$$

and the shape of the pulse envelope has the form

$$u_a(t)=\sqrt{\frac{T_r}{\tau_u}}\left|\frac{\sin\frac{\pi t}{\tau_u}}{\frac{\pi t}{\tau_u}}\right|, \tag{7.8.12}$$

i.e., is described by a curve of form $(\sin x)/x$. Then a signal with such an envelope and linear change of frequency will be equivalent to a square pulse of duration $\frac{\pi}{\omega_m}$.

Functions $C(\delta)$, corresponding to (7.8.5), (7.8.6), (7.8.8) and (7.8.12), are shown in Fig. 7.34. Questions of range finding for a signal with linear change of frequency and a Gaussian envelope, and for a square signal and of a shape of form

(7.8.12) we considered earlier. A signal with a cosinusoidal envelope has, as follows from Fig. 7.34, an autocorrelation function $\rho(\theta)$ so close in form to the Gaussian that without essential differences for it we have all the results of Paragraph 7.5.2.

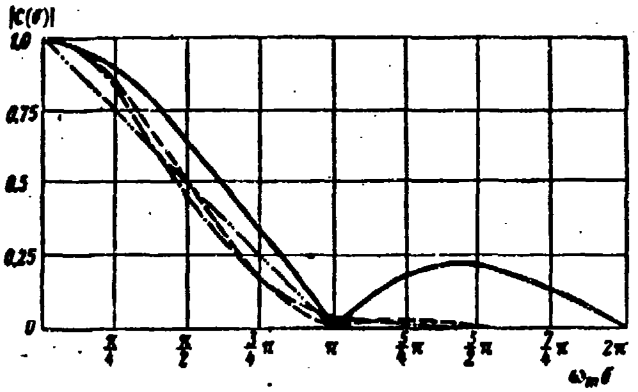

Fig. 7.34. Autocorrelation functions for a pulse signal with linear frequency modulation:

—— square pulse; ---- $u_n(t)=\sqrt{\frac{T_r}{\tau_n}}\cos\frac{\pi t}{2\tau_n}$;

-·-·- $u_n(t)=\sqrt{\frac{T_r}{\tau_n}}\exp\left(-\frac{\pi t^2}{2\tau_n^2}\right)$;

-··-··- $u_n(t)=\sqrt{\frac{T_r}{\tau_n}}\left|\frac{\sin \pi t/\tau_n}{\pi t/\tau_n}\right|$.

7.8.2. Pulse Radiation with PCM

Phase-code manipulation of a signal can also be used during pulse radiation. Here, on codes we make the same requirement of high resolving power as during continuous radiation. Different only is selection of the code, and this is caused by the fact that to extreme code intervals in a shifted signal, occurring beyond the limits of the initial pulse, there correspond in the initial signal empty places, and not the start of the next or the end of the preceding period, as in the continuous case.

With a sufficiently large number of code intervals in a pulse the autocorrelation function practically coincides with the autocorrelation function of a continuous signal (see Chapter I), and all results in the study of accuracy completely coincide with results of Paragraph 7.6.1.

§ 7.9. Multifrequency Radiation Without Modulation

As we know [1], range finding, in principle, is possible in the presence in the spectrum of the sounding signal of only two frequency components. Here, distance is measured by the difference of phase advances of these frequency components. A deficiency of such a method is low accuracy. It can be improved by increasing the number of radiated sine waves and rational selection of frequencies. In the process of measurement one should use here difference of phases of all sine waves, where the minimum difference is used for rough but also unambiguous range finding, and other differences larger in magnitude, are for more exact measurement. In order to ensure tuning of frequency away from passive interference the frequencies should be selected sufficiently far from each other, and for simple range finding one should use a difference of phases of the second order (the difference of differences of phases).

In the general case of m radiated oscillations of different frequencies with arbitrary amplitudes a_j and phases ψ_j sounding signal u(t) can be presented in the form

$$u(t)=\sum_{j=1}^{m} A_j e^{i\omega_j t}, \tag{7.9.1}$$

where $A_j = a_j e^{i\psi_j}$;

ω_j – difference between the j-th radiated frequency and an arbitrarily selected carrier frequency ω_0, and $\sum_{j=1}^{m} |A_j|^2 = 1$.

It is obvious that all results obtained earlier are applicable to a signal of such form, and for range finding using a multifrequency signal we can use all the above considered discriminator circuits. The mean square spectrum width of the signal, determining accuracy of range finding, is equal to

$$b-a^2=\sum_{j=1}^{m} \omega_j^2 |A_j|^2 - \left(\sum_{j=1}^{m} \omega_j |A_j|^2\right), \tag{7.9.2}$$

and function C(δ) is recorded in the form

$$C(\delta)=\sum_{j=1}^{m} |A_j|^2 e^{i\omega_j \delta}. \tag{7.9.3}$$

Resolving power, ensured with use of a set of a small number of sine waves, is very low. As an example we shall consider the case of three sinusoids of identical amplitude. Here,

$$|C(\delta)|^2=\frac{1}{3}\left\{1+\frac{2}{3}\left[\cos(\omega_2-\omega_1)\delta + \cos(\omega_3-\omega_1)\delta + \cos(\omega_3-\omega_2)\delta\right]\right\}. \tag{7.9.4}$$

The dependence of $|C(\delta)|^2$ on product $(\omega_2 - \omega_1)\delta$ for various values of $\alpha = \left|\frac{\omega_3-\omega_2}{\omega_2-\omega_1}\right|$ is shown in Fig. 7.35. As can be seen from the figure, the considered relationship has spurious maxima, attaining a magnitude of 0.6-0.8.

Thus, the advantage connected with the possibility of tuning frequency away from passive interference with simultaneous simple measurement of large distances (as compared to the case when tuning is achieved by increase of the frequency of repetition), is accompanied for a given signal by considerable impairment of resolving

power. With increase of the number of frequencies utilized and proper selection of them the resolving power can be increased, and here in its properties the signal, apparently, inevitably nears normally considered forms of modulation.

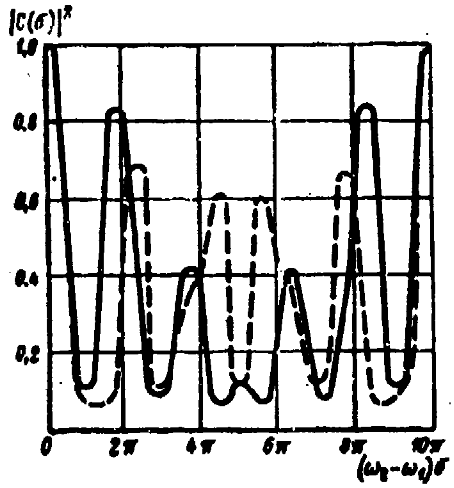

Fig. 7.35. Autocorrelation function of a three-frequency signal:
——α = 0.8; ---α = 1.2.

7.9.1. Case of Arbitrarily Correlated Frequency Components of the Reflected Signal

Till recently, in examining different forms of modulation of the sounding signal we implicitly assumed that the width of the spectrum of modulation is limited from above by the condition of complete correlatedness of the most distant frequency components of the spectrum. Physically, this means that the interval of resolution with respect to distance exceeds the distance to the irradiated target. Here, all frequency components of the reflected signal fluctuate harmoniously, and we have its usual presentation in the form of a normal nonstationary random process with correlation function (7.2.1).

For signals presented in the form of (7.9.1) it is possible to obtain a number of results pertaining to the case of nonharmoniously fluctuating frequency components. It should be noted that presentation (7.9.1) is possible for a very broad class of signals. In particular, these can be periodic signals or signals of any form, considered in a finite time interval and represented in this interval by a Fourier series. Therefore, results obtained for a signal of form (7.9.1), are very general.

Nonharmonious fluctuations of separate spectral components of the reflected signal are caused by target extent and start to appear when the wavelength of the difference frequency for the considered components becomes comparable with dimensions of the target. If the considered frequencies are sufficiently close, so that it is possible to disregard the difference of spectral properties of fluctuation at these frequencies, then, as it is easy to show, the correlation function of the signal reflected from an extended target, considered as a set of "brilliant" points, is recorded in the form

$$R(t_1,\ t_2;\ \tau)=P_0\rho(t_1-t_2)\,\mathrm{Re}\sum_{j,k=1}^{m} A_j A^*_k \rho_{jk}\, e^{i\omega_j(t_1-\tau)-i\omega_k(t_2-\tau)}, \qquad (7.9.5)$$

where $\rho(t)$ – correlation function of fluctuations, which was used in all the preceding cases;

τ – delay, determined by distance to a certain median point of target;

P_c – mean power of the reflected signal;

ρ_{jk} – coefficient of mutual correlation of oscillations with frequencies ω_j and ω_k

$$\rho_{jk}=\int_{-\infty}^{\infty}\sigma(x)e^{i(\omega_j-\omega_k)x}\,dx, \tag{7.9.6}$$

$\sigma(x)$ – normalized density of distribution of reflectors making up the target with respect to distance ($\rho_{jj} = 1$).

Functional $|Q(t, \tau)|^2$, determining the character of optimum operations and accuracy of range finding, is obtained for the given case by a certain generalization of results from the case of harmoniously fluctuating frequency components and has the following form (see (4.2.9) and (4.2.11)):

$$\begin{aligned}|Q(t,\tau)|^2=\sum_{j,k=1}^{m}A_jA^*_k\int_{-\infty}^{t}\int_{-\infty}^{t}y(s_1)\,y(s_2)\,e^{i\omega_j(s_1-\tau)-i\omega_k(s_2-\tau)}\times\\ \times h_{jk}(t-s_1)\,h'_{jk}(t-s_2)\,ds_1ds_2,\end{aligned} \tag{7.9.7}$$

where $h_{jk}(t)$ and $h'_{jk}(t)$ – pulse responses of filters, frequency responses of which satisfy relationship

$$H_{jk}(i\omega)H'^*_{jk}(i\omega)=V_{jk}(\omega), \tag{7.9.8}$$

and functions $V_{jk}(\omega)$ are determined from the following equation

$$\sum_{l=1}^{m}V_{jl}(\omega)\left[\delta_{lk}+hS_0(\omega)\,|A_l|^2\rho_{lk}\right]=hS_0(\omega)\,\rho_{jk}, \tag{7.9.9}$$

during derivation we use the practically always realized assumption that the difference of any two frequencies if great as compared to the width of the spectrum of fluctuations.

The discriminator of a range finder, carrying out exact or approximate formation of the derivative of $|Q(t, \tau)|^2$ with respect to τ, can be realized by any of the previously considered methods. In particular, with exact fulfillment of optimum operations the output of the discriminator is defined as

$$z(t,\Delta)=\frac{1}{N_0}\,\mathrm{Re}\sum_{j,k=1}^{m} i(\omega_j-\omega_k)A_jA^*_k\int_{-\infty}^{t}\int_{-\infty}^{t}h_{jk}(t-s_1)\times$$

$$\times K_{jk}(t-s_2)\,e^{i\omega_j(s_1-\tau)-i\omega_k(s_2-\tau)}y(s_1)\,y(s_2)\,ds_1ds_2 \qquad (7.9.10)$$

and can be realized by a complicated set of filters and phase detectors, where characteristic of the filters for all pairs of frequencies are different. The equivalent spectral density in the optimum case, determining potential accuracy of range finding is found from (7.2.14) and is determined by expression

$$\frac{1}{S_{\text{экв}}}=K_{\text{опт}}=$$

$$=h\,\mathrm{Re}\sum_{j,k=1}^{m}(\omega_j-\omega_k)^2\,|A_jA^*_k|^2\rho^*_{jk}\,\frac{1}{2\pi}\int_{-\infty}^{\infty}V_{jk}(\omega)\,S_0(\omega)\,d\omega. \qquad (7.9.11)$$

With harmonious fluctuations (ρ_{jk} = 1 for all j, k) (7.9.10)-(7.9.11) pass into the corresponding formulas of § 7.2 with coefficient $b - a^2$, determined by formula (7.9.2). With independently fluctuating signals (ρ_{jk} = 0 when $j \neq k$) $K_{\text{опт}} = 0$ and $S_{\text{экв}} = \infty$, i.e., range finding becomes impossible. Since squares of differences of frequencies in (7.9.11) enter into a sum with factors, proportional to coefficients of mutual correlation ρ^*_{jk}, those components for which $\rho_{jk} = 0$ drop out, and the presence of corresponding pairs of frequencies in no way affects accuracy of measurement of distance to a target.

The simplest results are obtained for a low signal-to-noise ratio h. In this case, as follows from (7.9.9),

$$V_{jk}(\omega)\approx h\rho_{jk}S_0(\omega), \qquad (7.9.12)$$

and frequency responses of all filters turn out to be identical with an accuracy to factor $\sqrt{\rho_{jk}}$. One possible block diagram of such a discriminator is shown in Fig. 7.36 for the case when σ(x) is an even function and ρ_{jk} are real numbers. The received signal enters m mixers, where it is mixed with radiated oscillations, delayed and shifted in frequency. After filtration oscillations ar-

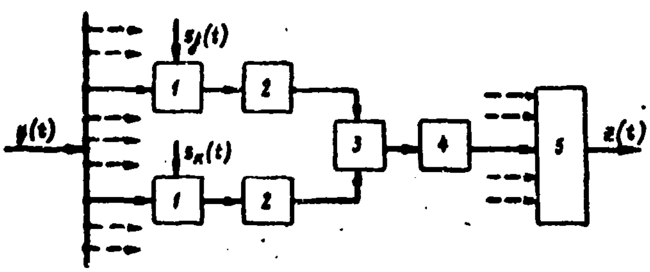

Fig. 7.36. Block diagram of a discriminator with multifrequency radiation: 1 — mixer; 2 — narrow-band filter; 3 — phase detector; 4 — amplifier with gain factor ρ_{jk}; 5 — adder. $\dot{s}_j = \mathrm{Re}\,A_j e^{i(\omega_0+\omega_{\text{пр}}+\omega_j)(t-\tau)}$.

fed to $\frac{m(m-1)}{2}$ phase detectors, outputs of which after multiplication by ρ_{jk} are summed and form the output voltage of the discriminator.

Formula (7.9.11) for h << 1 will be transformed to

$$\frac{1}{S_{onT}} = \frac{h^2}{2\pi} \int_{-\infty}^{\infty} S_0^2(\omega)\, d\omega \cdot \sum_{j,k=1}^{m} (\omega_j - \omega_k)^2 |\rho_{jk}|^2 |A_j A_k|^2. \qquad (7.9.13)$$

Coefficient of correlation ρ_{jk}, as follows from (7.9.6), depends on difference $|\omega_j - \omega_k|$, decreasing with increase of this difference. If ρ_{jk} decreases faster than $1/|\omega_j - \omega_k|$, there exists an optimum value of difference $|\omega_j - \omega_k|$, at which product $(\omega_j - \omega_k)^2\rho_{jk}$ is maximum. We can find the optimum form of spectrum, ensuring the greatest accuracy of range finding. It is obvious that the best result will be obtained if all differences of frequencies have optimum value, but in this case there can be only two frequencies. Power should be distributed equally between these frequencies, as one may see from (7.9.13).

The obtained result cannot, of course, serve as a basis for selection of the shape of the sounding signal, since here accuracy of measurement presents the latter with requirements of uniqueness and resolving power with respect to distance. However, this result permits us to conclude that the width of the spectrum of modulation of the signal should be taken close to $2\pi c/l$, where l is the linear dimensions of the target, and c is the speed of light.

§ 7.10. Analysis of Accuracy of Radar Range Finders

The above analysis of discriminators of various types and general results of Chapter VI permit us to investigate accuracy of range finders as a whole. As it was shown in Chapter VI, with linearized consideration and disregard of parametric fluctuations, total error of a tracking meter can be broken down into three components — fluctuation, dynamic and systematic. Calculation of the dependence of spectral density of noise at the discriminator output on mismatch leads to additional error, owing its origin to parametric fluctuations. Corresponding expressions for errors of measurement, which we shall use subsequently, are given in § 2, Chapter VI. In this section, from a series of examples of smoothing circuits and discriminators we shall calculate errors of range finding giving quantitative examples.

7.10.1. Influence of a System of Automatic Gain Control (AGC) on Discriminator Characteristics

The presence of an AGC system in general does not lead to essential change of our analysis of a discriminator. As it was shown in Chapter II, with good

approximation an AGC system can be described as a linear inertial amplifier of the received signal envelope with parameters which depend on signal level. Basically, its action reduces to normalization of the output signal of the preamplifier, covered by an AGC loop. Such an amplifier always is present in circuits of discriminators of distance and usually precedes the narrow-band filter. Its band width Δf_y, as a rule, substantially exceeds bandwidth of fluctuations of the signal (Δf_y = = $(5\text{-}20)\Delta f_c$) and at the same time is small as compared to spectrum width of modulation.

In two-channel discriminator circuits amplifiers are used in both channels, but the AGC system is common and works from the output of one of the amplifiers. Another factor which must be accounted for in circuits with automatic gain control is demodulation of amplitude fluctuations of the signal, leading in our equivalent consideration to a steep slope of low frequencies in the signal amplitude envelope. This factor leads to decrease of parametric fluctuations at the discriminator output and decreases errors of measurement for small noises. Furthermore, the AGC system, in general, produces modulation of the signal by noises, which pass through a feedback circuit. This phenomenon significantly increases the difficulty of analysis. However, with a demodulation band which is not too broad (AGC is not too high-speed), they can be ignored.

Although all these factors, in practice, do not influence discriminator characteristics, during the analysis of dynamic properties of a closed-loop measuring system they must be accounted for inasmuch as, due to the presence of an AGC system, its parameters (gain factor of the open loop and its fluctuations caused by parametric noises, the band of the closed system, etc) become functions of the signal-to-noise ratio and vary together with changes of it. In view of the great complexity in completely accounting for the influence of AGC (modulation of the signal by noise, cross modulation, nonlinear effects of high order, etc) and thanks to the fact that in normal circumstances these phenomena are weakly manifested, we shall limit ourselves to a very simple description of the AGC system and shall henceforth consider only changes of discriminator gain factor due to the normalizing action of AGC and decrease of the level of parametric fluctuations.

In accordance with the results of Chapter II for sufficiently great inertia of the filter in the feedback network the AGC system reduces the mean value of voltage at the output of the amplifier covered by the AGC loop to a constant level. As a result the gain factor of the receiver, and consequently, the discriminator gain factor, vary with change of the mean input signal level. The law of change of the

gain factor is determined from conditions of normalization of the output voltage, which for receivers with a linear and a square-law detector, respectively, have the following form:

$$\left.\begin{aligned} K_1 \overline{E(t)} &= \text{const}, \\ K_1^2 \overline{E^2(t)} &= \text{const}, \end{aligned}\right\} \tag{7.10.1}$$

where K_1 – gain factor of the amplifier covered by the AGC loop;

$E(t)$ – amplitude of input signal.

Since

$$\left.\begin{aligned} \overline{E(t)} &= \sqrt{\frac{\pi}{2}(2N_0\Delta f_y + P_c)}, \\ \overline{E^2(t)} &= 2(2N_0\Delta f_y + P_c), \end{aligned}\right\} \tag{7.10.2}$$

from either of these relationships it follows that

$$K_1 = K_{10}\frac{1}{\sqrt{1+\frac{1}{h}\frac{\Delta f_y}{\Delta f_c}}}, \tag{7.10.3}$$

where K_{10} – gain factor in the absence of noise.

In the considered circuits with square-law detectors or phase detectors, realizing multiplication of signals, the discriminator gain factor is proportional to K_1^2. Therefore,

$$K_Д = K_{Д0}\frac{1}{1+\frac{1}{h}\frac{\Delta f_y}{\Delta f_c}} = K_{Д0}\frac{1}{1+\frac{y}{h}}, \tag{7.10.4}$$

where $y = \Delta f_y / \Delta f_c$ – ratio of amplifier bandwidth to signal bandwidth;

$K_{Д0}$ – nominal value of discriminator gain factor, corresponding to the case of no noise.

The influence of all other discriminator parameters on gain factor, studied in the preceding paragraphs, is unchanged; relationship (7.10.4) should be understood in the sense that the gain factor of the open system depends on the signal-to-noise ration h according to the law (7.10.4), decreasing with strong noises and approaching a limit with increase of the signal. Expressions (7.10.3) and (7.10.4) are valid, of course, only for sufficiently large values of h, since for small signal levels it does not reach the delay level of AGC, the AGC system is opened and produces no normalization. Thus, for h smaller than a certain value h_0, corresponding to the delay level, K_1 = const and does not depend on the signal-to-noise ration.

Decrease of parametric fluctuations, caused by the demodulating effect of AGC. can be calculated with the help of results of Chapter II. As is clear from analysis of discriminators, parametric fluctuations are caused by random variations of the envelope of the useful signal, the spectral density of which in the low-frequency range at the output of an amplifier with AGC are determined by expression (2.7.23). Considering that the spectral density of parametric fluctuations is proportional to the spectral density of the signal amplitude envelope in the low-frequency range, from expression (2.7.23) we obtain the relationship between spectral densities of parametric fluctuations with ($S^{(1)}_{\text{пар}}$) and without ($S_{\text{пар}}$) AGC

$$S^{(1)}_{\text{пар}} = \frac{1}{n^2} S_{\text{пар}}, \qquad (7.10.5)$$

where n^2 – coefficient depending on parameters of AGC and the signal-to-noise ratio.

According to (2.7.23) this coefficient is presented in the form

$$n^2 = n_0^2 \frac{h}{h_0}, \qquad (7.10.6)$$

where h_0 – the signal-to-noise ratio which corresponds to the delay level $E(t) = E_0$;

$n_0 = bk_1E_0$ (b – slope of the controlled characteristic, k_1 – gain factor in the feedback network of AGC).

The value of n_0 in practice is normally great as compared with unity, and thus even for signals close to the delay level parametric fluctuations caused by change of signal amplitude substantially decrease.

7.10.2. Smoothing Circuits with Constant Parameters

Until recently, in smoothing circuits of radar range finders they almost exclusively used linear filters with constant parameters of a fairly low level. The most widely applied types of such filters are the single integrator and the double integrator with correction. Furthermore, when, for a number of reasons, it was impossible to obtain ideal integration, they use smoothing filters in the form of a single RC-circuit and two series-coupled RC-circuits with correction.

According to (6.2.20) fluctuation error of measurement in smoothing circuits with constant parameters is determined by the simple relationship

$$\sigma^2_{\phi x} = 2S_{\text{экв}} \Delta f_{\text{эф}}, \qquad (7.10.7)$$

where $S_{\text{экв}}$ – determined by the previous analysis of discriminators;

$\Delta f_{\text{эф}}$ – effective band width of the closed-loop servo system, determined by expression (6.2.19).

For smoothing circuits in the form of two RC-circuits with correction the

transfer function of the closed circuit is

$$H(p)=\frac{K(1+pT_3)}{(1+pT_1)(1+pT_2)}, \tag{7.10.8}$$

where K – gain factor of the smoothing filter;

T_1, T_2 – time constants of the RC-circuits;

T_3 – time constant of the correcting circuit.

The effective band width in this case is

$$\Delta f_{эф}=\frac{1}{2\pi}\int_0^{\infty}\left|\frac{K_Д H(i\omega)}{1+K_Д H(i\omega)}\right|^2 d\omega = \frac{1+\frac{K_Д K T_3^2}{T_1T_2}}{4\left[T_3+\frac{T_1+T_2}{K_Д K}\right]}\approx\frac{K_Д K T_3}{4T_1T_2}. \tag{7.10.9}$$

This approximate equality is valid for normal relationships of parameters of the servo system, since the gain factor of the open system $K_Д K$ usually is very great. Quantities in expression (7.10.8) have the following order:

$$K_Д K \sim 10^3 \div 10^5,\ T_1T_2 \sim 10 \div 100 \text{ sec},$$

where one of the time constants is larger than the other by a factor of 3-10, and $T_3 \sim 10^{-2}$ to 10^{-1} sec. Selection of the time constant of the correcting circuit is sometimes carried out in such a manner that for assigned $K_Д K$, T_1T_2 it ensures a minimum effective bandwidth

Here,

$$\left.\begin{aligned} T_3 &\approx \sqrt{\frac{T_1T_2}{2K_Д K}},\\ \Delta f^{*}_{эф} &= \frac{3\sqrt{2K_Д K}}{8\sqrt{T_1T_2}}. \end{aligned}\right\}$$

This selection is produced at the nominal gain factor of the open circuit, i.e., when $K_Д = K_{Д0}$. Then the dependence of effective bandwidth on the signal-to-noise ratio approximately has the form

$$\Delta f_{эф}=\Delta f_{эф0}\,\frac{2}{3}\,\frac{1+\frac{0.5h}{h+\mu}}{1+\left(1+\frac{\mu}{h}\right)\sqrt{\frac{2(T_1+T_2)^2}{K_{Д0}KT_1T_2}}}, \tag{7.10.10}$$

where $\Delta f_{эф0}$ – effective bandwidth with the nominal gain factor.

The dependence of $\Delta f_{эф}/\Delta f_{эф\,0}$ on h with $y = 3$, $T_1 = 4T_2$ and for various values of product $\Delta f_{эф\,0}T_2$ is shown in Fig. 7.37. With noises which are not too weak and sufficiently large $\Delta f_{эф\,0}T_2$ this dependence is not very essential.

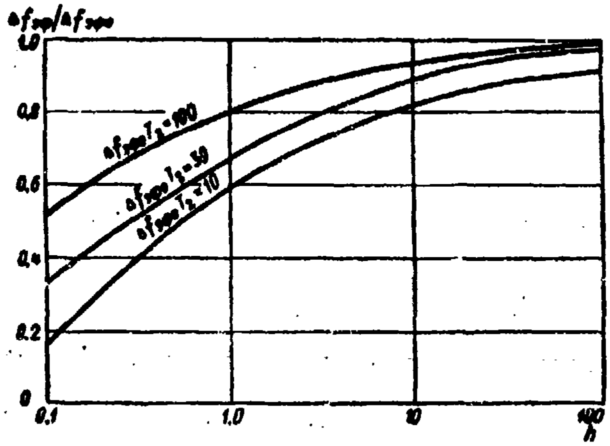

Fig. 7.37. Dependence of effective bandwidth of the servo system on the signal-to-noise ratio h.

Formula (7.10.9) also permits us to find effective bandwidth for other types of smoothing filters, enumerated above. In particular, for a double integrator with correction, i.e., when

$$H(p) = \frac{K_и(1 + pT_к)}{p^2}, \qquad (7.10.11)$$

the expression for $\Delta f_{эф}$ is obtained by driving T_1 and T_2 to infinity in such a manner that $K/T_1T_2 = K_И$, where $K_И$ – gain factor (measured) of the two integrators. Considering also that $T_3 = T_к$, we obtain

$$\Delta f_{эф} = \frac{1 + K_д K_и T_к^2}{4T_к}. \qquad (7.10.12)$$

Minimum $\Delta f_{эф}$ is attained when

$$T_к = \frac{1}{\sqrt{K_д K_и}} \qquad (7.10.13)$$

and is equal to $\sqrt{K_И K_Д}/2$. The dependence of effective bandwidth on the signal-to-noise ratio with selection of $T_К$ in accordance with (7.10.13) for $K_Д = K_{Д0}$ has the form:

$$\Delta f_{эф} = \frac{\Delta f_{эф0}}{2}\left(1 + \frac{h}{h + y}\right). \qquad (7.10.14)$$

For illustration in Fig. 7.38 there is shown the dependence of fluctuation error of measurement of delay, referred to pulse duration, on the signal-to-noise ratio h for a range finder with modulation of the signal by square pulses and a discriminator with two detuned channels for H(p) from (7.10.11), $\delta = 0.5$, $\tau_И$ and for different values of y and $\Delta f_{эф\,0}/\Delta f_c$. Since effective bandwidth $\Delta f_{эф}$ depends on h, the path of curves in Fig. 7.38 differs from dependence $S_{экв}(h)$. Bands of filters of the discriminator in accordance with (7.6.3) are assumed to be matched with signal spectrum width.

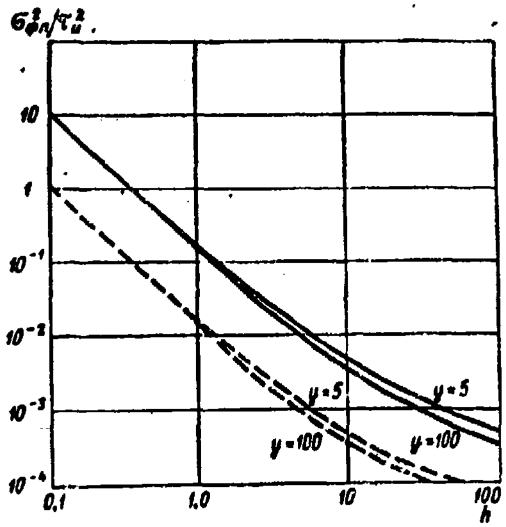

Fig. 7.38. Dependence of fluctuation error of measurement on the signal-to-noise ratio:
—— $\Delta f_{эф0}/\Delta f_c = 10^{-1}$;
- - - $\Delta f_{эф0}/\Delta f_c = 10^{-2}$.

By means of further simplifications from formula (7.10.9) we obtain the expression for effective bandwidth of a system with smoothing circuits in the form of one RC-filter

$$H(p)=\frac{K}{1+pT_1}, \tag{7.10.15}$$

and a single integrator

$$H(p)=\frac{K_и}{p}. \tag{7.10.16}$$

Corresponding expressions have form

$$\Delta f_{эф}=\frac{KK_д}{4T_1}, \tag{7.10.17}$$

$$\Delta f_{эф}=\frac{K_иK_д}{4}. \tag{7.10.18}$$

In both cases the dependence of $\Delta f_{эф}$ on the signal-to-noise ratio has the form

$$\Delta f_{эф}=\Delta f_{эф0}\frac{h}{h+y}. \tag{7.10.19}$$

This dependence is more sharply expressed.

As another example, in Fig. 7.39 we show the dependence of $\sigma^2_{\phi л}(b - a^2)\frac{\Delta f_c}{\Delta f_{эф0}}$ on h for an optimum discriminator with signal spectrum (7.3.11), various y and smoothing circuits of form (7.10.15) or (7.10.16).

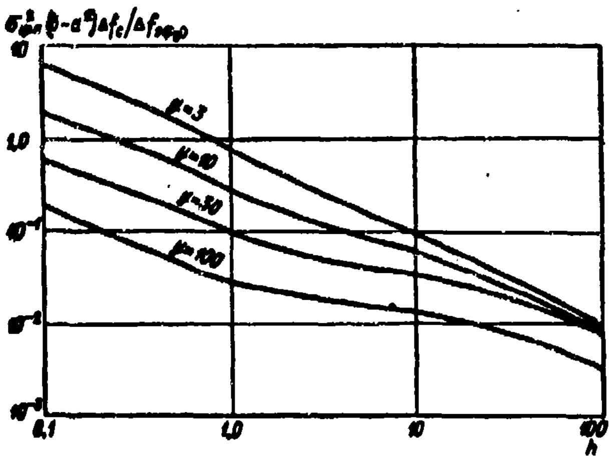

Fig. 7.39. Dependence of fluctuation error of measurement on the signal-to-noise ratio.

We shall now consider other components of error of measurement. In tracking meters there exists double interpretation of dynamic error of tracking. In the statistical approach to measured quantities by dynamic error, in accordance with Chapter VI, we understand the mean square of the component of error caused by random changes of measured parameter. Its magnitude is determined by formula (6.2.14). Along with this in such a consideration there is introduced

the concept of systematic errror, which is caused by inaccuracies of input of the mean value of the measured parameter, known a priori. Let us remember that in an optimum measuring system such insertion is obligatory, and the tracking loop is designed to process a priori unknown random changes of the parameter. The rationale for such input is obvious, too, for a normal tracking systems. The magnitude of systematic error is determined by formula (6.2.14).

In the absence of statistical evidence about the measured quantity determination of dynamic error in accordance with Chapter VI is impossible. In this case dynamic error is determined by the widely known method as the error of processing a certain model disturbance – an input varying linearly, by square-law, etc. Formal mathematical determination of such error, obviously, coincides with the expression for systematic error (6.2.14) with replacement of $\Delta\lambda(t)$ by the corresponding disturbance. Therefore, calculation of systematic error for various $\Delta\lambda(t)$ gives simultaneously the magnitude of dynamic error corresponding to the nonstatistical approach to the measured parameter.

Let us consider several examples of calculation of dynamic error in a system with smoothing circuits of form (7.10.8), (7.10.11), (7.10.15), and (7.10.16). If the random part of the measured distance changes in time as a stationary random process with spectral density $S(\omega)$, the stationary value of dynamic error, on the basis of (6.2.17), is

$$\sigma^2_{\text{дин}} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{S(\omega)\,d\omega}{|1+K_\lambda H(i\omega)|^2}. \qquad (7.10.20)$$

Stationary random changes of distance correspond to cases when the target carries out random maneuvers or experiences certain irregular disturbances, possessing a stationary character. Examples of such disturbances may be random oscillations of engine thrust, change of drag due to irregularity of the atmosphere, noises in drive assemblies of control systems of the target, and so forth. In the presence of sufficient damping such disturbances lead to stationary change of distance. If, in particular, spectral density $S(\omega)$ has the form

$$S(\omega) = \frac{2\sigma_0^2 T}{1+\omega^2 T^2}, \qquad (7.10.21)$$

where σ_0^2 – variance of the random component of the measured distance;

T – correlation time, and the smoothing filter is a single integrator; dynamic error is equal to

$$\sigma^2_{\text{дин}} = \frac{\sigma_0^2}{1+K_{\text{д}}K_{\text{и}}T} = \frac{\sigma_0^2}{1+4\Delta f_{\text{эф}}T}. \qquad (7.10.22)$$

This error will decrease with increase of product $K_{\text{д}}K_{\text{и}}\ T$, i.e., with broadening of the effective bandwidth of the system. With decrease of gain factor due to lowering of the signal-to-noise ratio dynamic error increases, approaching a limiting value, equal to the a priori variance of the measured quantity σ_0^2.

Likewise, with smoothing filter (7.10.11) and selection of $T_{\text{к}}$ in accordance with (7.10.13)

$$\sigma^2_{\text{дин}} = \frac{\sigma_0^2[1+(K_{\text{д}}K_{\text{и}}T^2)^{3/2}]}{1+K_{\text{д}}K_{\text{и}}T^2+K_{\text{д}}^2K_{\text{и}}^2T^4} = \sigma_0^2\,\frac{1+8\Delta f^3_{\text{эф}}T^3}{1+4\Delta f^2_{\text{эф}}T^2+16\Delta f^4_{\text{эф}}T^4}. \qquad (7.10.23)$$

In this case the magnitude of dynamic error also is determined by product $\Delta f_{\text{эф}}T$. The dependence of $\sigma_{\text{дин}}/\sigma_0$ on this product is shown in Fig. 7.40. The dependence of dynamic error on the signal-to-noise ratio can be estimated by formulas (7.10.14) and (7.10.23).

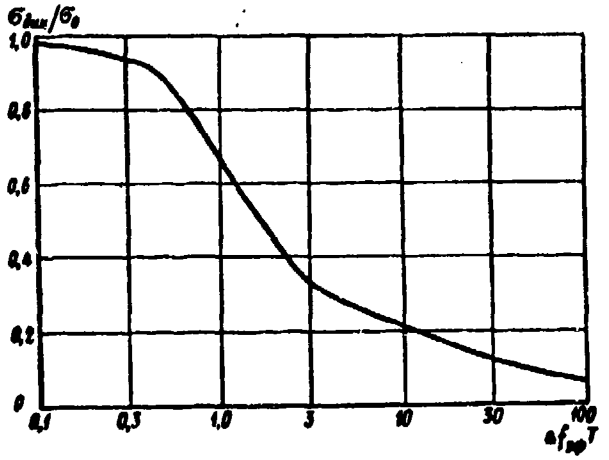

Fig. 7.40. Dependence of dynamic error on effective bandwidth.

Of essential interest is the case when the measured distance changes as a linear combination of known functions with random coefficients of the form

$$d(t) = \sum_{i=1}^{m} \mu_i f_i(t) + \overline{d(t)}, \qquad (7.10.24)$$

where $\overline{\mu_i} = 0$; $\overline{\mu_i\mu_k} = M_{ik}$; $\overline{d(t)}$ – mean value of distance.

Change of distance according to (7.10.24) is realized when the law of motion of the target is known with an accuracy of certain constant parameters, i.e.,

$$d(t) = F(t;\,\alpha_1,\,\ldots,\,\alpha_m) = F(t;\,\overline{\alpha}_1+\mu_1,\,\ldots,\,\overline{\alpha}_m+\mu_m),$$

where $\alpha_i (i = 1, \ldots, m)$ – unknown parameters with certain mean values $\overline{\alpha_i}$ and random deviations from them μ_i.

In the overwhelming majority of practically interesting cases these deviations are sufficiently small to permit presenting d(t) in the form of (7.10.24), where

$\overline{d(t)} = F(t; \overline{\alpha_1}, \ldots, \overline{\alpha_m})$, and $f_i(t) = \frac{\partial F(t; \overline{\alpha_1}, \ldots, \overline{\alpha_m})}{\partial \alpha i}$. Parameters α_i can have various meaning. With change of d(t) according to (7.10.24)

$$d^2_{\Delta m} = \sum_{i,k=1}^{m} M_{ik} \int_{t_0}^{t} v(t, \tau) f_i(\tau)\, d\tau \int_{t_0}^{t} v(t, \tau) f_k(\tau)\, d\tau = \\ = \sum_{i,k=1}^{m} M_{ik} \varepsilon_i(t) \varepsilon_k(t), \tag{7.10.25}$$

where $v(t, \tau)$ – pulse response, corresponding to the transfer function of error, which is determined by equation (6.2.13).

With smoothing circuits with constant parameters the Fourier transform from $v(t - \tau)$ is determined by the expression

$$V(i\omega) = \frac{1}{1 + K_{\text{д}} H(i\omega)}. \tag{7.10.26}$$

Expression (7.10.25) when $M_{ik} = 1$ simultaneously determines the square of dynamic error, corresponding to the nonstatistical approach, which takes place with influence on the servo system of a disturbance assigned by relationship (7.10.24) when $\mu_i = 1$ ($i = 1, 2, \ldots$), and also the square of systematic error, corresponding to the same difference between the real and introduced mean values of the measured distance.

With smoothing circuits with constants parameters and $t_0 = 0$, function $\varepsilon_j(t)$, obviously, is determined by relationship

$$\varepsilon_j(t) = \frac{1}{2\pi i} \int_{C - i\infty}^{C + i\infty} V(p) F_j(p) e^{pt}\, dp, \tag{7.10.27}$$

where $F_j(p)$ is the Laplace transform of function $f_j(t)$.

From formulas (7.10.26) and (7.10.27) it follows that with a smoothing filter in the form of a single integrator (7.10.16) function $\varepsilon_j(t)$ asymptotically for sufficiently large valves of t can be approximately presented in the form

$$\varepsilon_j(t) = \frac{\dot{f}_j(t)}{K_{\text{д}} K_{\text{и}}}, \tag{7.10.28}$$

where $\dot{f}_j(t)$ – derivative of $f_j(t)$.

This expression shows that stationary error in this case exists only if all $f_j(t)$ are functions with a bounded derivative. In particular, if $f_1(t) = a_0 + a_1 t$, $f_j(t) = 0$, $j > 1$, relationship (7.10.28) as $t \to \infty$ is exact and

$$\varepsilon_{\text{дин}} = \lim_{t\to\infty} \varepsilon_1(t) = \frac{a_1}{K_{\text{д}} K_{\text{и}}}. \qquad (7.10.29)$$

Dynamic error, determined by formula (7.10.29), is used most frequently in practice for evaluating the servo system.

Likewise, in a system with two integrators with correction function $\varepsilon_j(t)$ approximately is presented in the form

$$\varepsilon_j(t) \approx \frac{\ddot{f}_j(t)}{K_{\text{д}} K_{\text{и}}}, \qquad (7.10.30)$$

and, in particular, when $f_1(t) = a_0 + a_1 t + \frac{1}{2} a_2 t^2$ we obtain exact expression

$$\varepsilon_{\text{дин}} = \lim_{t\to\infty} \varepsilon_j(t) = \frac{a_2}{K_{\text{д}} K_{\text{и}}}, \qquad (7.10.31)$$

and there is no error from a linear disturbance in steady-state operating conditions. If the measured distance has components, growing faster than t in the first case, and faster than t^2 in the second, as $t \to \infty$ dynamic error increases without limit. With smoothing circuits in the form of a single RC-filter and two filters with correction, a stationary value of error exists only with a constant disturbance at the input. In both cases it is equal to

$$\varepsilon_{\text{дин}} = \frac{a_0}{1 + K_{\text{д}} K}. \qquad (7.10.32)$$

Relationships (7.10.28)-(7.10.32) show that in general the magnitude of dynamic error is inversely proportional to the gain factor of the open circuit of the servo system and increases with decrease of gain factor of the discriminator due to decrease of the signal-to-noise ratio. In Fig. 7.41 there is shown the dependence of $K_{\text{Д}}/K_{\text{Д}0}$ on h, characterizing change of dynamic error with change of the signal-to-noise ratio. The magnitude of error in the range of working values of h, depending upon selection of the magnitude of y, may vary by a factor of 1.2 to 10. With wider bands of the amplifier with AGC the influence of normalizing properties of the AGC

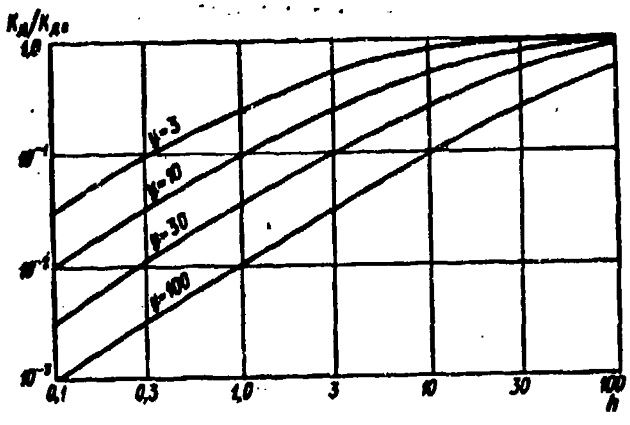

Fig. 7.41. Dependence of the gain factor of the discriminator on the signal-to-noise ratio.

system, naturally, are more noticeable.

Increase of the gain factor of the open loop has a different effect on magnitudes of fluctuation and dynamic error. Therefore, when designing smoothing circuits of a range finder there is required a definite compromise selection of their parameters. Let us consider for example the case of a double integrator with correction. Since the time constant of the correcting circuit does not affect the magnitude of dynamic error, it should be selected in accordance with (7.10.13), i.e., from the condition of minimum effective bandwidth. Then total error of measurement with a distance varying in time by square law will have the form

$$\sigma^2 = \sigma^2_{\text{фл}} + \sigma^2_{\text{дин}} = \sqrt{K_{\text{и}} K_{\text{д}}}\, S_{\text{экв}} + \frac{\overline{a_2^2}}{(K_{\text{и}} K_{\text{д}})^2}. \tag{7.10.33}$$

If mean square acceleration $\overline{a_2^2}$ is unknown, it can be replaced by a certain quantity given by tactical conditions. From (7.10.33) it follows that the optimum magnitude of gain is

$$(K_{\text{и}} K_{\text{д}})_{\text{опт}} = \left(\frac{4\overline{a_2^2}}{S_{\text{экв}}}\right)^{2/5}. \tag{7.10.34}$$

Total error here comprises

$$\sigma^2 = 1{,}68\,(\overline{a_2^2})^{1/5} S_{\text{экв}}^{4/5}, \tag{7.10.35}$$

where fluctuation error is approximately 4 times greater than dynamic. Inasmuch as $S_{\text{ЭКВ}}$ depends on h, the gain in the open circuit, according to (7.10.34), should vary with change of the signal-to-noise ratio. In particular, for optimum or nearly-optimum discriminators and a square spectrum of fluctuations of the signal, when $S_{\text{ЭКВ}}$ is proportional to $\frac{1+h}{h^2}$, the required law of change has the form

$$(K_{\text{и}} K_{\text{д}})_{\text{опт}} = K_0 \left(\frac{h^2}{1+h}\right)^{2/5}. \tag{7.10.36}$$

By proper selection of the amplifier band and parameters of the AGC system it is possible to achieve automatic approximation of this dependence in a sufficiently wide range of values of h.

In Fig. 7.42 is the dependence of the optimum gain factor on h and curves of change of gain factor for various y. All these curves are joined for point h = 10. From this figure it is clear that with selection y = 20 the required dependence of the gain factor of an open circuit with satisfactory accuracy is reproduced by an

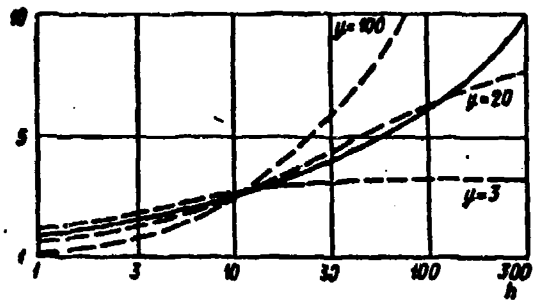

Fig. 7.42. Dependence of the optimum gain factor of an open system on the signal-to-noise ratio: ---- required dependence; ---- real dependence for various y.

AGC system in the range from h ~ 3 to h ~ 200.

For illustration of formulas (7.10.34)-(7.10.35) in Table 7.1 we give errors of range finding with a continuous signal frequency modulated by sinusoidal law with deviation $2\omega_m = 2\pi\cdot 10^6$ rad/sec for a range finder with a two-channel discriminator matched with spectrum (7.3.11) by a filter. Spectrum width Δf_c is taken equal to 20 cps. Values of errors correspond to various signal-to-noise ratios and different mean square accelerations developed by the target. In that same table there are given values of optimum gain factor of an open circuit.

Table 7.1

Mean square acceleration of the target $\sqrt{\overline{a_2^2}}$, m/sec²	σ, м	$K_Д K_И$, sec⁻²	σ, м	$K_Д K_И$, sec⁻²	σ, м	$K_Д K_И$, sec⁻²	σ, м	$K_Д K_И$, sec⁻²
	h=3		h = 10		h = 30		h = 100	
0,1	4,6	0,05	2,5	0,1	1,4	0,16	0,9	0,25
1,0	7,3	0,31	4	0,57	2,2	1,05	1,3	1,75
3,0	9,2	0,74	5	1,37	2,8	2,44	1,7	4
10	11,5	2,0	6,3	3,6	3,5	6,5	2,1	10,8
30	14,5	4,72	8	8,55	4,5	15,2	2,7	25,4

The presence of parametric fluctuations in the servo system leads to increase of errors of measurement. With constant smoothing circuits this increase is expressed by simple relationship (6.2.39), proceeding from which by (7.10.5) and previously obtained formulas for spectral density of parametric fluctuations (7.3.20), (7.3.21) and (7.4.14) we can be easily estimate the influence of these fluctuations. In the most interesting case of fairly large signal-to-noise ratios and for circuits of discriminators with detuned channels or differentiation of the reference signal, increase of error due to parametric fluctuations is determined by the following simple expression:

$$\sigma^2=\sigma_0^2\left(1+\frac{2\pi\Delta f_{эф}}{h^2\Delta f_c}\right), \qquad (7.10.37)$$

where σ_0^2 – variance of error;

$\Delta f_{эф}$ – bandwidth of the system, calculated without taking into account parametric fluctuations;

α = 0.5-1.0 – numerical coefficient (see Paragraph 7.3.2);

n – coefficient of suppression of fluctuations by the AGC system.

In cases of practical interest ratio $\Delta f_{эф}/\Delta f_c$ has an order of 10^{-3} to 1; therefore, already at n ~ 5-10 error is increased by no more than 10%, and in most cases this increase is, in general, immaterial.

7.10.3. Optimum Smoothing Circuits for a Randomly Varying Parameter.

In Chapter VI it is shown that when the measured parameter is a nondegenerate random process, stationary or with stationary increments, the smoothing filter is a filter with constant parameters, depending on statistical properties of the measured coordinate, the signal-to-noise ratio and parameters of modulation of the received signal. Let us consider one example of such a process. Let us assume that the radar range finder is designed for measurement of distance to a target moving with random uncorrelated accelerations with spectral density B_2 (quantity B_2 is numerically equal to the mean square of the speed developed by the target in 1 sec and has dimension [m^2/sec^3]). Here, the distance to the target is a random process in the form of the double integral of white noise of spectral density B_2, and the optimum smoothing filter according to results of Chapter VI is a double integrator with correction, the transfer function of which is given by formula (7.10.11). Moreover, in the optimum circuit

$$\left.\begin{aligned} K_и K_к &= \sqrt{B_2/S_{опт}(h)}, \\ T_к &= \sqrt[4]{4S_{опт}(h)/B_2}, \end{aligned}\right\} \qquad (7.10.38)$$

and total error of measurement is given by formula

$$\sigma^2_{опт} = \sqrt{2}\,(B_2 S^3_{опт}(h))^{1/4}. \qquad (7.10.39)$$

From (7.10.38) it follows that quantities $K_и$ and $T_к$ should change with change of h. We shall consider that these quantities are fixed, and change of the transmission factor of the open circuit occurs only due to normalizing properties of AGC. Furthermore, we shall assume that the real spectral density of the target differs from that used during synthesis and is equal to B_{20}.

The gain factor of the integrator and the time constant of the correcting circuit we select corresponding to some $h = h_0$. Then, total amplification in the open loop is equal to

$$K_и K_к = \sqrt{B_2/S_{опт}(h_0)}\,\frac{1+\frac{g}{h_0}}{1+\frac{g}{h}}, \qquad (7.10.40)$$

where $y = \Delta f_y / \Delta f_c$, and the effective bandwidth, according to (7.10.12), is

$$\Delta f_{\text{эф}} = \frac{1}{4T_{\text{к}}(h_0)}\left[1 + 2\frac{1+\frac{y}{h_0}}{1+\frac{y}{h}}\right]. \qquad (7.10.41)$$

Then, variance of fluctuation error

$$\sigma^2_{\text{фл}} = \frac{S_{\text{экв}}(h)}{2}\sqrt[4]{\frac{B_2}{4S_{\text{опт}}(h_0)}}\left[1 + 2\frac{1+\frac{y}{h_0}}{1+\frac{y}{h}}\right], \qquad (7.10.42)$$

and variance of dynamic error is obtained from (7.10.20) when $S(\omega) = B_{20}/\omega^4$,

$$\sigma^2_{\text{дин}} = \frac{B_{20}}{8}\left(\frac{4S_{\text{опт}}(h_0)}{B_2}\right)^{3/4}\left[\frac{1+\frac{y}{h}}{1+\frac{y}{h_0}}\right]^2. \qquad (7.10.43)$$

We select $h_0 = 5$, when for an exponential function of correlation of fluctuations $B_2/S_{\text{опт}}(h_0) = 23.7 B_2 b \Delta f_c / c^2$, which for instance, when $B_2 = 10^2 m^2/sec^3$, rms modulation spectrum width $\sqrt{b} = 2\pi \cdot 10^6\ sec^{-1}$, and width of the spectrum of fluctuati.ns $\Delta f_c = 30$ cps, gives the following value of parameters of the servo system:

$$K_{\text{к}} = 5{,}6\frac{1}{sec^{-2}},\ T_{\text{к}} = 0{,}6\ sec,\ \Delta f_{\text{эф}}(h_0) = 1{,}25\ \text{гц},$$

and error of measurement for $h = h_0$ is about 3.5 m. The dependence of the ratio of total error $\sqrt{\sigma^2_{\text{фл}} + \sigma^2_{\text{дин}}}$ to quantity $\sigma_{\text{опт}}$ from (7.10.39) for values of h_0, B_2, b, and Δf_c selected above, on h is shown for various values of B_{20} and two values of y(y = 20 and y = 100) in Fig. 7.43. In order to obtain an idea of the absolute values of error of measurement, in Table 7.2 there are given values of $\sigma_{\text{опт}}$ for various h with the same values of parameters B_2, b, and Δf_c.

Table 7.2

h	0,3	1	3	10	30	100
σ_{opt}, м	18	8,4	4,5	2,5	1,6	0,95

Figure 7.43 shows that in general the investigated radar range finder with a criterion for selection of its discriminator close to optimum sufficiently well approximates properties of an optimum system. With coincidence of assumed B_2 and

real B_{20} increase of error occurs practically only for the smallest values of h, which actually are not the working values. In the working range increase of error is several per cent.

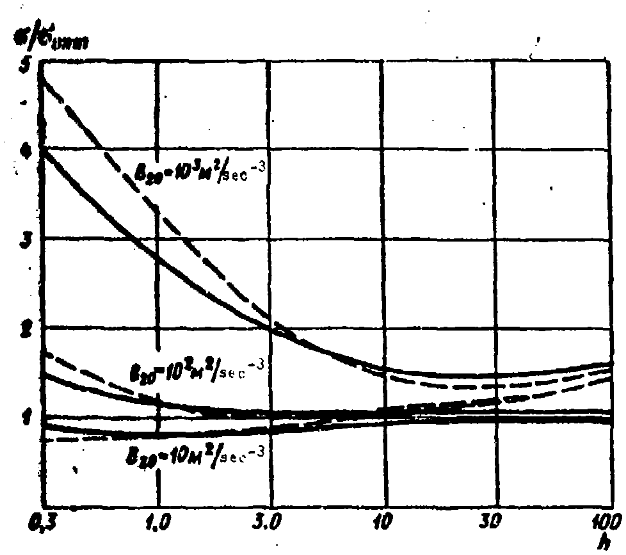

Fig. 7.43. Dependence of $\sigma/\sigma_{ОПТ}$ on h: ——y = 20; ---y = 100.

If the real value of B_{20} is less than the assumed, in the considered circuit for certain values of h, depending on selection of quantity y, error is less than in an optimum circuit, calculated for spectral density of accelerations B_2. This circumstance is not surprising, since measurement of distance of a target maneuvering less, naturally, should be produced with less error.

If, on the other hand, value of the spectral density of accelerations during synthesis is lowered, error of measurement for all values of h is greater than $\sigma_{ОПТ}$; however, in the working range of values of the signal-to-noise ratio this increase does not exceed a factor of 2-3. Selection of y, as comparison of curves in Fig. 7.43 shows, is not very critical. Best results in general are obtained for smaller y (in this case it is reasonable to select y approximately within the range from 10 to 30); however, if there is a danger that during synthesis the value of the spectral density of accelerations is understated and the range finder is designed, basically, for work at a comparatively high signal-to-noise ratio, it is somewhat better to increase y, expanding the bandwidth of the preamplifier.

7.10.4. Smoothing Circuits with Variable Parameters

In Chapter VI it is shown that in many cases requirements of optimality of a meter lead to the necessity of application of smoothing circuits with variable parameters. Most characteristic is the case when the measured quantity is presented in form (7.10.24). The pulse response of optimum smoothing circuits in this case is given by formula (6.8.53), and error of measurement is presented in the form of (6.8.54). The smoothing filter here is a set of variable-gain amplifiers, integrators and generators of known functions of time with controlled gain.

A characteristic peculiarity of a meter designed for a law of change of the

parameter of type (7.10.24) is the asymptotic tendency of error of measurement to zero, which is a consequence of the quasi-regularity of change of the measured quantity. Smoothing circuits of the meter are constructed here in such a way that the range finder, with the help of suppression of output signals of the discriminator, with the passage of time more and more shifts to work from memory, and change of the output quantity become smoother and smoother, ever more exact than change of the measured distance.

The pulse response of an optimum smoothing filter designed for measurement of distance which changes according to the law (7.10.24), according to (6.8.53), is determined in the following way:

$$G(t, \tau)=\sum_{i, k=1}^{m} A_{ik}(\tau) f_i(t) f_k(\tau), \tag{7.10.44}$$

where $A_{ik}(t) = A_{ik}(t;\, h)$ – matrix elements;

$$\mathbf{A}(t;\, h)=[\mathbf{M}^{-1}+\mathbf{U}(t;\, h)]^{-1}; \tag{7.10.45}$$

$$\mathbf{U}(t;\, h)=\left\|\frac{1}{S_{\text{опт}}(h)}\int_0^t f_i(s) f_k(s)\, ds\right\|. \tag{7.10.46}$$

According to (7.10.44)-(7.10.46) parameters of the optimum smoothing filter are functions of the signal-to-noise ratio and should be corrected with change of it. Let us assume first that such correction is produced, and we also assume that the discriminator is sufficiently close to optimum, and change of its gain factor is by the law required in an optimum meter, $K_{\text{Д}}(h) = 1/S_{\text{ОПТ}}(h)$. Then, according to equation (6.8.53) pulse response of the closed-loop servo system is

$$g(t, \tau)=\frac{1}{S_{\text{опт}}(h)}\sum_{i, k=1}^{m} A_{ik}(t) f_i(t) f_k(\tau), \tag{7.10.47}$$

and pulse response of error, according to (6.2.13), is

$$v(t, \tau)=\delta(t-\tau)-g(t, \tau). \tag{7.10.48}$$

If real changes of distance correspond to law (7.10.24), used during the synthesis of smoothing circuits, then, as it was shown in Chapter VI, total error of measurement is expressed by the simple dependence

$$\sigma^2(t)=G(t, t). \tag{7.10.49}$$

Corresponding examples of calculation of errors of measurement of a general character were already considered in Chapter VI. Here we give one more concrete

example. Let us assume that the target moves evenly toward the radar, so that the distance to it is equal to

$$d(t) = d_0 + \Delta d - (V + \Delta V)t, \tag{7.10.50}$$

where d_0 – distance at the time of tracking lock-on, known with an accuracy of error of lock-on Δd of the order of the magnitude of the resolution capability with respect to distance.

Target velocity V is also known with certain error ΔV, where errors Δd and ΔV are normal and uncorrelated. Designating $\overline{\Delta d^2} = \sigma_0^2$ and $\overline{\Delta V^2} = \sigma_1^2$, we obtain, according to (6.8.63), for the pulse response of the smoothing circuit the expression

$$G(t, \tau) = \frac{\sigma_0^2\left(1 - \frac{\sigma_1^2}{S_{\text{опт}}(h)}\frac{\tau^2}{6}\right) + \sigma_1^2 t\tau\left(1 + \frac{\sigma_0^2}{S_{\text{опт}}(h)}\frac{\tau}{2}\right)}{1 + \frac{1}{S_{\text{опт}}(h)}\left(\sigma_0^2\tau + \sigma_1^2\frac{\tau^3}{3}\right) + \frac{\sigma_0^2\sigma_1^2\tau^4}{12S_{\text{опт}}^2(h)}}, \tag{7.10.51}$$

which when $\tau = t$ also determines variance of total error of measurement. In formula (7.10.51) and subsequently we have in mind equivalent spectral density, recomputed to values of distance with coefficient $c^2/4$, where c – velocity of light. This variance changes in time both due to the obvious dependence of $G(t, t)$ on t, and also due to change of the signal-to-noise ratio in time, which occurs according to the law

$$h(t) = h_0 \frac{1}{\left(1 - \frac{Vt}{d_0}\right)^4}. \tag{7.10.52}$$

Considering the spectrum of signal fluctuations square, and using expression (7.2.15) for $S_{\text{опт}}(h)$, we obtain the following expression, characterizing change of error of measurement in time:

$$\frac{\sigma^2(t)}{\sigma_0^2} = \frac{1 + \frac{h^2(t)}{1 + h(t)}\frac{8\Delta f_c t^3 b\sigma_1^2}{3c^2} + \frac{\sigma_1^2 t^2}{\sigma_0^2}}{1 + \frac{h^2(t)}{1 + h(t)}\frac{8\Delta f_c t b\sigma_0^2}{c^2}\left(1 + \frac{\sigma_1^2 t^2}{3\sigma_0^2}\right) + \frac{16h^4(t)\Delta f_c^2\sigma_0^2\sigma_1^2 t^4 b^2}{3[1 + h(t)]^2 c^4}}. \tag{7.10.53}$$

The dependence of $\sigma(t)/\sigma_0$ on $\Delta f_c t$ for various $x_0 = d_0\Delta f_c/V = T_0\Delta f_c$, where T_0 – time to target impact, and various $y = \frac{\sigma_1}{\sigma_0 \Delta f_c} = \sqrt{\overline{\Delta V^2}/\overline{\Delta d^2}\Delta f_c^2}$, is shown in Fig. 7.44. The curves are constructed for the signal-to-noise ratio at maximum

range $h_0 = 3$ and for product $\frac{4\upsilon\sigma_0^2}{c^2} = 0.5$.

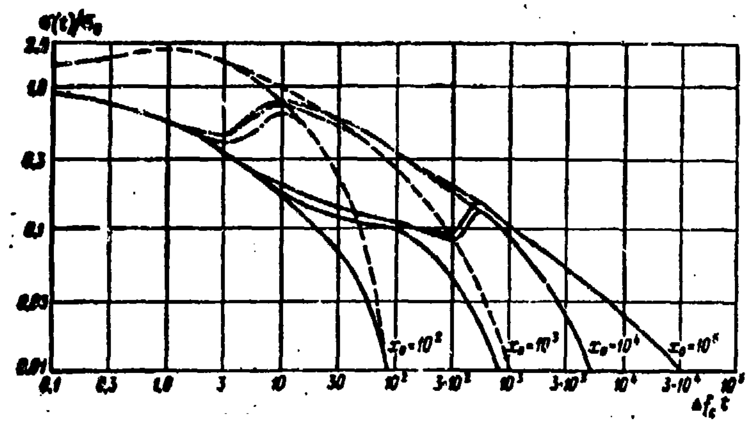

Fig. 7.44. Dependence of $\sigma(t)\sigma_0$ on $\Delta f_c t$:

——$y = 10^{-3}$; ---$y = 10^{-1}$; -·-·-$y = 10$.

Curves of Fig. 7.44 show that error of measurement sufficiently rapidly decreases in time. Here, there first occurs decrease of error, then a certain increase of it, and then decrease to zero. Maximum error is attained faster, the larger $y = \sigma_1/\sigma_0\Delta f_c$, i.e., the greater the variance of velocity V, and is expressed more strongly, the larger the magnitude of x_0, i.e., the smoother the change of the signal-to-noise ratio. For small values of x_0 and y there is no maximum, and in general its magnitude is greater, the larger y. Here, for small y the magnitude of this maximum is less than one, and for large y it may exceed one; however, even at $y = 10$ it does not exceed two.

Expression (7.10.53) characterizes total error of measurement only when the mean value of measured distance is inserted in the circuit of the servo system. If such insertion is not specially anticipated and there is introduced only value d_0, which occurs automatically upon lock-on, there appears additional error. Let us find the systematic error, appearing due to the difference between the real and inserted values of distance. Function $g(t, \tau)$ in this case, as it is simple to prove, is

$$g(t, \tau) = \frac{\sigma_0^2\left(1 - \frac{\sigma_1^2 t^3}{6S_{out}}\right) + \sigma_1^2 t\tau\left(1 + \frac{\sigma_0^2 t}{2S_{out}}\right)}{1 + \frac{1}{S_{out}}\left(\sigma_0^2 t + \frac{1}{3}\sigma_1^2 t^3\right) + \frac{1}{12S_{out}^2}\sigma_0^2\sigma_1^2 t^4}. \qquad (7.10.54)$$

Considering that difference $\Delta\lambda(t)$ in this case is equal to Vt, with the help of

(6.2.14) and (7.10.48) we obtain the following expression for systematic error:

$$\sigma_{\text{сист}} = Vt \frac{1 + \frac{\sigma_0^2 t}{2S_{\text{опт}}}}{1 + \frac{1}{S_{\text{опт}}}\left(\sigma_0^2 t + \frac{1}{3}\sigma_1^2 t^3\right) + \frac{1}{12S_{\text{опт}}^2}\sigma_0^2\sigma_1^2 t^4}. \tag{7.10.55}$$

This expression shows that systematic error for small t grows as Vt, and then rapidly decreases. Asymptotically, it changes as

$$\sigma_{\text{сист}} \approx \frac{6VS_{\text{опт}}}{\sigma_1^2 t^2},$$

and its maximum is attained for times t of the order of $1/\Delta f_c$, i.e., practically during the time equal to the inertia of the discriminator. The dependence of systematic error, related to magnitude d_0, on time, with the same conditions as for Fig. 7.44, is shown in Fig. 7.45. This analysis shows that in this case the influence of systematic error is of little substance.

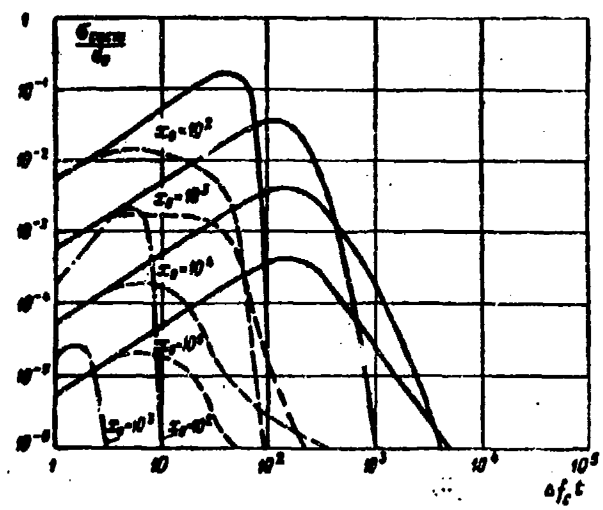

Fig. 7.45. Dependence of systematic error of range finding in a system with variable parameters, related to magnitude d_0, on $\Delta f_c t$:
——y = 10^{-3}; ---y = 10^{-1}; -·-·-y = 10.

In practice sometimes it is not possible to correctly determine statistics of the magnitude to be determined and it is necessary to make more or less arbitrary assumptions. Here, it may be that smoothing circuits are calculated for one law of change of the parameter, but in reality another takes place. Furthermore, it may be that the real law of change of parameter is so complicated that from considerations of technical convenience it is necessary consciously to replace it by a simpler one. The general case of measurement of a nonstationary parameter, varying according to the law

$$\lambda(t) = \sum_{k=1}^{n} \alpha_k \varphi_k(t) + \overline{\lambda(t)}, \tag{7.10.56}$$

where $\overline{\alpha_k} = 0$; $\overline{\alpha_i \alpha_k} = A_{ik}$, by a system whose smoothing circuits are designed for law of change (7.10.24) is considered in Chapter VI. We shall limit ourselves to a simple example, when smoothing circuits are designed for law of change of distance (7.10.50) with variances σ_0^2 and σ_1^2, and their pulse response has form (7.10.51), while, in fact, distance changes according to law

$$d(t) = d_0 + \Delta d_1 - (V + \Delta V_1)t + \frac{1}{2}\Delta a t^2, \tag{7.10.57}$$

where Δd_1, ΔV_1, Δa – independent normal random variables with variances $\overline{\Delta d_1^2} = \sigma_{01}^2$, $\overline{\Delta V_1^2} = \sigma_{11}^2$, $\overline{\Delta a^2} = \sigma_{21}^2$.

Fluctuation error remains the same as in the case of coincidence of the assumed and real laws, and is determined by expression (6.2.14)

$$\sigma_{\text{фл}}^2 = S_{\text{опт}} \int_0^t g^2(t, \tau)\, d\tau, \tag{7.10.58}$$

and its magnitude for sufficiently large t practically coincides with the magnitude of total error σ^2 of (7.10.49) and (7.10.53), calculated earlier.

Let us find dynamic error of range finding. According to (6.2.14)

$$\sigma_{\text{дин}}^2 = \sum_{i=0}^{2} \frac{\sigma_{i1}^2}{(i!)^2}\left[t^i - S_{\text{опт}} \int_0^t g(t, \tau)\, \tau^i d\tau\right]^2 = \frac{\sigma_{01}^2\left(1 - \frac{1}{6}\frac{\sigma_1^2 t^3}{S_{\text{опт}}}\right)^2 + \sigma_{11}^2 t^2\left(1 + \frac{\sigma_0^2 t}{2S_{\text{опт}}}\right)^2 + \frac{\sigma_{21}^2 t^4}{4}\left(1 + \frac{\sigma_1^2 t^3}{12 S_{\text{опт}}} + \frac{\sigma_0^2 \sigma_1^2 t^4}{72 S_{\text{опт}}^2}\right)^2}{\left[1 + \frac{1}{S_{\text{опт}}}\left(\sigma_0^2 t + \frac{1}{3}\sigma_1^2 t^3\right) + \frac{1}{12}\frac{\sigma_0^2 \sigma_1^2 t^4}{S_{\text{опт}}^2}\right]^2}. \tag{7.10.59}$$

From this expression it follows that different errors in the hypothesized statistics differently affect dynamic error. Inaccuracy of knowledge of corresponding variances does not lead to change of the character of the dependence of dynamic error on time. If, for instance, σ_{21}^2, i.e., acceleration of the target, is absent, components of dynamic error in position and speed change correspondingly by a factor of σ_{01}/σ_0 and σ_{11}/σ_1.

The expression for error in speed (component with coefficient σ_{11}^2), obviously, coincides with the expression for systematic error (7.10.55) studied above with replacement of V by σ_{11}. We already proved that its influence on total error of

measurement is immaterial.

Error in position (component with coefficient σ_{10}^2) also decreases fairly rapidly in time. Asymptotic error in position is determined by $2\sigma_{01}S_{0\Pi T}/\sigma_0^2 t$.

In the conditions considered above, and for $t \approx \frac{10}{\Delta f_c}$, its values are given in Table 7.3. Influence of this error is also immaterial.

Table 7.3

x_0	10^2	10^3	10^4	10^5
Error in position	$0.055\sigma_{01}$	$0.085\sigma_{01}$	$0.09\sigma_{01}$	$0.09\sigma_{01}$

More essential is error caused by incorrect selection of the degree of the polynomial describing change of distance. If σ_{21}^2 differs from zero, dynamic error increases without limit in time. Its asymptotic value is given by expression

$$\sigma_{x\,\text{дин}} \approx \frac{\sigma_{21} t^2}{12}. \tag{7.10.60}$$

This means that understating of the degree of the approximating polynomial is permissible only when error of approximation in the whole interval of observation is included in limits assigned by tactical requirements.

The opposite error, connected with overstatement of the degree of the approximating polynomial, does not present any danger. From formula (7.10.59) it follows that dynamic error consists of the sum of the squares of components, corresponding to various degrees of the polynomial, and if any of the coefficients is in reality equal to zero, dynamic error will only be less than that on which we counted. Certainly, here, dynamic and fluctuation errors will increase as compared to their values with correct selection of the degree of the polynomial, but this increase is not very substantial, since both errors decrease without limit with passage of time.

Let us consider one more example of smoothing circuits with variable parameters, when forced change of their parameters with change of the signal-to-noise ratio is absent, and there occurs only change of the gain factor of the open loop due to the AGC system. We shall limit ourselves to the simple case when the measured distance changes according to the law

$$d(t) = \mu f(t), \quad \overline{\mu^2} = \sigma_\mu^2. \tag{7.10.61}$$

Considering that the smoothing filter is designed for a certain signal-to-noise

ratio h_0, we obtain

$$G(t,\tau)=\frac{\sigma_0^2 f(t)f(\tau)}{1+\frac{\sigma_0^2}{S_{\text{опт}}(h_0)}\int\limits_0^{\tau} f^2(s)\,ds}. \tag{7.10.62}$$

Then, from equation (6.2.12),

$$g(t,\tau)= K_{\text{д}}(h)\,\sigma_0^2 f(t)f(\tau)\frac{\left[1+\frac{\sigma_0^2}{S_{\text{опт}}(h_0)}\int\limits_0^{t} f^2(s)\,ds\right]^{K_{\text{д}}(h)S_{\text{опт}}(h_0)-1}}{\left[1+\frac{\sigma_0^2}{S_{\text{опт}}(h_0)}\int\limits_0^{t} f^2(s)\,ds\right]^{K_{\text{д}}(h)S_{\text{опт}}(h_0)}}. \tag{7.10.63}$$

Selecting at point $h = h_0$ the optimum value of the gain factor $K_{\text{д}}(h_0) = 1/S_{\text{опт}}(h_0)$, we obtain the final expression

$$g(t,\tau)=\alpha^2 K(h)f(t)f(\tau)\frac{\left[1+\alpha^2\int\limits_0^{\tau} f^2(s)\,ds\right]^{K(h)-1}}{\left[1+\alpha^2\int\limits_0^{t} f^2(s)\,ds\right]^{K(h)}}, \tag{7.10.64}$$

where $K(h) = K_{\text{д}}(h)/K_{\text{д}}(h_0)$; $\alpha^2 = \sigma_0^2/S_{\text{опт}}(h_0)$.

Here, fluctuation and dynamic errors are determined by relationships

$$\sigma^2_{\text{фл}}=\frac{S_{\text{экв}}(h)K^2(h)\alpha^2 f^2(t)}{2K(h)-1}\,\frac{\left[1+\alpha^2\int\limits_0^{t} f^2(s)\,ds\right]^{2K(h)-1}-1}{\left[1+\alpha^2\int\limits_0^{t} f^2(s)\,ds\right]^{2K(h)}}, \tag{7.10.65}$$

$$\sigma^2_{\text{дин}}=\sigma_0^2 f^2(t)\frac{1}{\left[1+\alpha^2\int\limits_0^{t} f^2(s)\,ds\right]^{2K(h)}}. \tag{7.10.66}$$

Asymptotic change of fluctuation error when $\alpha^2\int\limits_0^t f^2(s)ds \gg 1$ coincides with the case of optimum selection of parameters of smoothing circuits, and dynamic error decreases faster when $K(h) > 1$ and slower when $K(h) < 1$.

The dependence of variance of total error of measurement $\sigma^2_{\text{вых}} = \sigma^2_{\text{фл}} + \sigma^2_{\text{дин}}$, related to the a priori variance of the measured distance $\sigma_0^2 f^2(t)$, on the quantity

$z = \alpha^2 \int_0^t f^2(s)ds$ when $h_0 = 10$ and $y = \Delta f_y / \Delta f_c = 30$ is shown in Fig. 7.46.

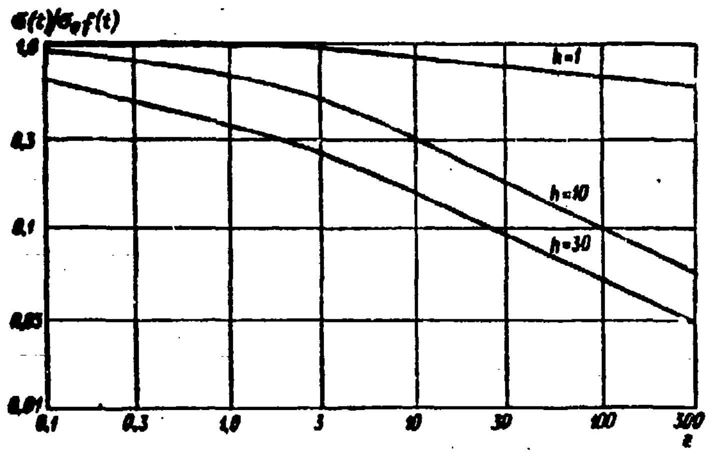

Fig. 7.46. The influence of nonoptimality of smoothing of circuits of a range finder on accuracy of measurement.

Quantity z can be considered dimensionless time. The physical meaning of it is that it is the ratio of variance of measured distance, averaged for time t, $\frac{1}{t}\sigma_0^2 \int_0^t f^2(s)ds$ to variance of the efficient estimate of a constant parameter during the time t for a signal-to-noise ratio h_0, equal to $S_{опт}(h_0)/t$. For instance, when $f(t) = t^n$ quantity $z = \sigma^2 t^{2n+1}/(2n + 1)S_{опт}(h_0)$, and when $f(t) = \sin(\omega_0 t + \varphi)$, $z \approx \frac{\sigma_0^2}{2S_{опт}(h_0)}$.

During construction of the graphs of Fig. 7.46 we assumed that the discriminator of the range finder is sufficiently close to optimum, so that $S_{экв}(h) \approx S_{опт}(h)$. As can be seen from the figure, curves corresponding to various values of h do not differ much from each other, and on the whole error of measurement sufficiently rapidly decreases with increase of z.

§ 7.11. Nontracking Radar Range Finders

In the practice of radar measurements they sometimes use nontracking range finders. They are applied usually when there is not required high accuracy of measurement or when there is required measurement simultaneously of coordinates of a large number of targets and application in the radar station of a large number of advanced, but complex tracking meters is connected with excessively great technical complications. Such situations are encountered in early warning and target acquisition radars [1, 26, 27].

In nontracking meters the estimate of the distance to the target is produced from the realization of the received signal in a comparatively small time interval.

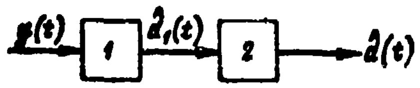

Fig. 7.47. Functional diagram of a nontracking range finder: 1 – unit, producing rough estimate of distance $\hat{d}_1(t)$; 2 – smoothing circuits.

Further data, obtained as a result of the estimate, either are used directly, or are additionally smoothed to decrease fluctuation errors of measurement. A functional diagram of the construction of a nontracking range finder is shown

in Fig. 7.47. Estimation of the distance to the target can be made by various devices or by the operator, reading data on the distance to the target from the screen of the radar set.

If calculation of the estimate is produced by a method, sufficiently close to optimum, for instance, the method of maximum likelihood, its variance is determined by formulas (7.2.22) for the case $\Delta f_c T \gg 1$, and (7.2.25) for the case $\Delta f_c T \ll 1$, where T is the time during which the estimate is formed. Quantities $\sigma^2_{э\Phi}$ from (7.2.22) and (7.2.25) characterize accuracy of measurement then the estimate of distance is used directly. If there is applied additional smoothing of data obtained by the estimator unit, variance of the estimate characterizes intensity of components of interferences at the input of smoothing circuits. Here, if the time for formation of the estimate T and the time of correlation of the received signal are sufficiently small as compared to inertia of the smoothing circuits, these components can be considered white noise with spectral density

$$S_{вх}=\sigma^2_{э\Phi}T. \tag{7.11.1}$$

With rapid fluctuations of the signal it is obvious that S_{BX} coincides with $S_{\text{оп т}}$. Therefore, tracking and nontracking meters with identical structure of smoothing circuits (in the case of a tracking meter we have in mind the pulse response of the closed-loop servo system) will give identical accuracy of measurement.

Actually, all real estimator units give estimates which are less than efficient. Furthermore, real estimates are not unbiased. All these circumstances can lead to considerable worsening of accuracy of measurement by nontracking systems. Nevertheless, for a number of applications of radar results obtained here may be satisfactory. Let us consider more concretely several examples of nontracking meters.

7.11.1. Range Finder with Frequency Modulation

In radio altimeters and certain other cases there is applied a range finder constructed according to the block diagram of Fig. 7.48 [27, 31]. A transmitter radiates the frequency modulated signal. The received signal is mixed with the signal of the transmitter, is passed through a low-frequency filter, and is fed to a frequency measuring unit. If the frequency is modulated according to the law $\Delta\omega(t)$, frequency of the signal at the mixer output varies according to the law

$$\omega_1(t)=\Delta\omega(t)-\Delta\omega(t-\tau).$$

For small τ frequency $\omega_1(t) \approx \tau\frac{\partial\Delta\omega(t)}{dt}$; therefore, knowing the law of change of frequency and measuring the frequency of the signal at the mixer output, it is

possible to measure dalay of the reflected signal. For any periodic frequency modulation the magnitude of $\frac{\partial \Delta\omega(t)}{\partial t}$ changes in time. For instance, with triangular and sawtooth modulation function $\omega_1(\tau)$ varies in accordance with Fig. 7.49. From the figure it is clear that, depending upon the magnitude of τ, the whole shape of curve $\omega_1(t)$ changes; for instance, for sawtooth modulation not only the level of flat sections, but also their location changes. Usually in nontracking radio range finders with frequency modulation the influence of delay τ on the shape of the curve is not taken into account, and measurement of distance is based on measurement of frequency in the flat section (Fig. 7.49).

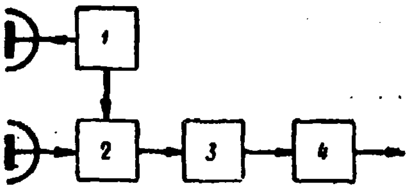

Fig. 7.48. Block diagram of a nontracking range finder with frequency modulation: 1 — transmitter; 2 — mixer; 3 — low-frequency filter; 4 — frequency measuring unit.

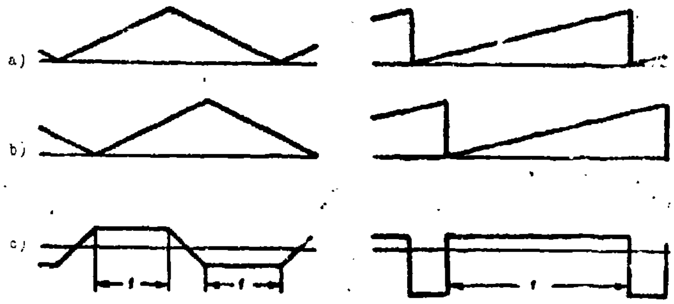

Fig. 7.49. Change of frequency in time; a) sounding signal; b) received signal; c) signal at mixer output; 1 — section utilized for measurement.

With triangular modulation, for instance, bandwidth of the filter is selected such that frequencies, corresponding to the interval of time from kT_r to $kT_r + \tau$, in general, do not pass through the filter. Such construction of a meter is permissible, obviously, only when the required range of change of delay is considerably less than the period of repetition of the signal. Otherwise, there will take place excessively large power losses, which are proportional to τ/T_r. Subsequently, we shall consider, in accordance with the way this is done in practice, that $\tau_{макс}/T_r \ll 1$, and we shall ignore losses connected with this.

With fulfillment of condition $\tau_{макс}/T_r \ll 1$ the considered range finder with triangular and sawtooth modulation, from the point of view of analysis of accuracy, is equivalent to a frequency meter. Fluctuation error of measurement is determined here by the following expressions for variance:

for sawtooth FM

$$\sigma_\tau^2 \approx \frac{T_r^2}{4\omega_m^2}\sigma_\omega^2, \qquad (7.11.2)$$

for triangular FM

$$\sigma_\tau^2 \approx \frac{T_r^2}{16\omega_m^2}\sigma_\omega^2, \tag{7.11.3}$$

where σ_ω^2 – variance of error of measurement of frequency;

$2\omega_m$ – frequency deviation.

During calculation of σ_ω^2 one should consider that coherence of the signal from period to period is disturbed. We explain this by the example of sawtooth modulation. Since the filter does not pass frequencies corresponding to negative values of $\omega_1(t)$ (Fig. 7.49), the signal at the filter output in every period has a new value of phase, depending on the duration of the negative section, equal to τ. If this advance of phase from period to period is not compensated, signals in various periods will be incoherent. Compensation of phase advance can be carried out only using the measured value of delay, which, obviously, signifies anew return to a tracking meter. In the absence of compensation coherence of the signal is disturbed, which leads to lowering of accuracy of measurement. Analogous phenomena take place also for any other law of frequency modulation. This means that for any method of measurement of frequency accuracy of the considered range finder will be worse than the accuracy of a tracking range finder.

It is possible to show that with an optimum method of measurement of frequency and low signal-to-noise ratios accuracy of the considered range finder is approximately identical to the precision of an incoherent tracking range finder using a signal with the same parameters.

Actually, as is shown in Chapter IX, variance of the efficient estimate of frequency, formed over one period T_r for not too large a signal-to-noise ratio, is approximately equal to

$$\sigma_\omega^2 = \frac{6}{T_r^2 q^2}(1+q), \tag{7.11.4}$$

where q – the ratio of energy of the signal for the period to the spectral density of noise ($q = P_c T_r/2N_0$).

Then, for instance, for sawtooth FM, according to the formula (7.11.2)

$$\sigma_\tau^2 = \frac{3}{\omega_m^2}\frac{1+q}{2q^2}. \tag{7.11.5}$$

With triangular FM measurement should be produced by half-periods $T_r/2$, and we again obtain the same expression for σ_τ^2. Quantity σ_τ^2 from (7.11.5) characterizes accuracy of measurement during single measurement. If measurements are produced

over many periods, then quantity σ_τ^2 determines the spectral density of fluctuation error

$$S_{\text{экв}} = \sigma_\tau^2 T_r = \frac{3T_r}{\omega_m^2}\frac{1+q}{2q^2}.$$

Since $\omega_m^2/3$ is equal to parameter $b = -C''(0)$ for the considered forms of modulation, $S_{\text{экв}}$ coincides with the expression for equivalent density of an incoherent range finder using a pulse signal with the same mean power and with linear change of frequency with deviation within limits of the pulse $2\omega_m \gg 1/\tau_M$ (see Chapter VIII).

In reality, due to the specific dependence of accuracy of measurement of frequency on the signal-to-noise ratio [for large q formula (7.11.4) is invalid; σ_ω^2 here approaches a finite quantity], actual accuracy of a nontracking meter will be still worse. Furthermore, in practice in such range finders there is not produced measurement of frequency by a method even faintly similar to optimum, but there is carried out counting of zeroes of output voltage of the filter. Since the passband of this filter, designed for the range of measured distances, considerably exceeds the frequency of repetition, and counting of zeroes is produced by nonlinear transformations, then due to worsening of the signal-to-noise ratio, connected with expansion of the passband, accuracy of measurement sharply decreases. Simultaneously with increase of fluctuation error, the method of counting zeroes leads to the appearance of systematic errors of measurement.

7.11.2. Use of a Range Finder with a Fast Tracking Loop for Forming an Estimate

In the beginning of this section we said that if the estimate of distance in a nontracking range finder is close to the efficient one, its accuracy is close to the accuracy of a tracking range finder with identical structure of its smoothing circuits. Optimum estimator units can be synthesized just as discriminators. However, a satisfactory solution of the problem of optimum formation of an estimate is application of a high-speed tracking range finder with very simple structure of its smoothing circuits. The output of this range finder should be smoothed in inertial circuits, matched in their characteristics with the law of change of the measured distance. As a whole such a range finder is "nontracking" in the sense that for control of tuning of the discriminator in it there is used, not the final output value of distance, but a certain intermediate value, which may repeat the true value of distance with considerably greater errors.

Requirements on the high-speed range finder ensue from requirements on the

optimum estimate. This estimate should be unbiased and efficient. From the requirement of unbiasedness there follow requirements of the absence of systematic and dynamic errors of tracking. These requirements will be carried out sufficiently satisfactorily if inertia of the high-speed range finder is small with respect to the rate of change of the measured distance.

The requirement of efficiency leads to the requirement of optimality of the discriminator of the high-speed range finder from the point of view of minimum equivalent spectral density. Then the output of the high-speed range finder in the absence of noises will repeat the law of change of the measured distance, and fluctuation error will be practically equal to the variance of the efficient estimate, built from the realization of the signal during the time $T = 1/2\Delta f_{эф}$, where $\Delta f_{эф}$ – bandwidth of the high-speed range finder. These variances for cases when the speed of action of the closed-loop system is low or high as compared to the time of correlation of fluctuations ($\Delta f_{эф}/\Delta f_c \ll 1$ and $\Delta f_{эф}/\Delta f_c \gg 1$), are determined by formulas (7.2.22) and (7.2.25), respectively. They characterize accuracy of measurement when the resultant estimate of distance is used directly.

If, however, subsequent smoothing is used, these variances characterize the intensity of interferences at the input of the smoothing circuits. Moreover, due to the low inertia of the closed-loop meter forming the estimate these components may be considered white noise with spectral density

$$S_{вх} = \frac{\sigma^2_{эф}}{2\Delta f_{эф}}, \tag{7.11.6}$$

which, for rapid fluctuations coincides with quantity $S_{опт}$ or $S_{эк}$ in the case of a nonoptimal discriminator, and when $\Delta f_{эф}/\Delta f_c \gg 1$ gives

$$S_{вх} = \frac{4N_0^2\Delta f_{эф}}{(b-a^2)P_c^2}\left(1 + \frac{P_c}{4N_0\Delta f_{эф}}\right). \tag{7.11.7}$$

Further analysis of the meter does not differ, in principle, from that performed above. Dynamic and fluctuation errors of the meter as a whole are determined by the very same formulas, in which by $g(t, \tau)$ we mean the pulse response of the open-loop smoothing filter. Pulse response $v(t, \tau)$, as before, is determined by formula (7.10.42). During calculation of fluctuation errors one should use the magnitudes of spectral density from (7.11.6) and (7.11.7). With optimum selection of the smoothing filter and a comparatively low level of noises errors of range finding, obviously, in this case coincide with errors of a tracking meter. With a high level of noises due to the high speed of operation of the preliminary meter,

forming the estimate, in the considered system breakoffs of tracking become more probable, the influence of parametric fluctuations increases, and as a whole errors of measurement increase. Furthermore, due to the high speed of operation application of a single-channel discriminator with switching of reference signals is hampered, the influence of all sorts of intermittent interferences is increased, etc., which, due to inevitable nonlinearities and parametric connections, may lead to swinging of the system, its excitation and loss of the target in the unit forming the estimate. All this, in general, leads to undesirability of application of such meters when we require from the radar range finder the very highest accuracy of measurement.

§ 7.12. Range Finder with Discrete Measurement

In a number of cases requirements on the width of the spectrum of modulation are dictated, not by permissible errors of range finding, but by the resolution capability. Modulation in certain cases is so broad-banded that potential accuracy of range finding, calculated from the concept of a point target, cannot be realized. The resolution range here is comparable to or even less than dimensions of the target, and accuracy of range finding is determined already practically by dimensions of the target. In these conditions it is sometimes senseless to apply complex range finders with special discriminators, and it is possible to limit ourselves to simpler systems.

Usually radars are equipped with detection channels designed for a certain range of distances, where often detection is produced simultaneously over the whole range of distances, for which there is a set of parallel channels of identical type, each of which is designed for a certain distance. Methods of construction of such channels are considered in Chapters IV and V.

With optimum processing the output voltage of such a channel coincides with the logarithm of the likelihood ratio of the received signal for the given distance. It is obvious that the unit of target detection channels, in one way or another present in a radar, can be used for range finding and also for simultaneous measurement of the distance to many targets. This unit can be used for various tasks, including creation of discrimination characteristics of normal range finders. Here we consider a case where taking of data on the distance to a target from the unit of detection channels is realized discretely with accuracy of the number of the channel. Possibilities of increase of the accuracy of determination of the distance already were considered in Chapters III and IV.

One very simple functional circuit of such a range finder, designed for

measurement of the distance to one target, is shown in Fig. 7.50. The received signal passes through the unit of range detection channels, at the output of which is a unit which selects at definite intervals of time of duration T_0 the channel whose output voltage is maximum. Then, there follows a unit for transformation of the number of the channel into a discrete distance, and there occurs smoothing of the obtained data by linear smoothing circuits. If there arises the task of tracking several targets, the selection unit should determine the number of channels in which voltage attains a maximum, the maximum of the remaining channels, etc. It is also possible to select all channels in which output voltage exceeds a certain threshold, determined by the level of noises, and to carry out smoothing of all data by separate smoothing filters. The circuit of Fig. 7.50 can be modified and turned into a servo system. Tracking here will be carried out not by zero of the discrimination characteristic, but by the channel number. It is possible to carry out such tracking, for instance, in the following way.

Fig. 7.50. Functional circuit of a range finder with discrete measurement: 1 – unit of detection channels; 2 – unit for selection of maximum voltage at the output of the unit 1; 3 – converter of channel number into distance; 4 – smoothing circuits.

After preliminary determination of the channel with the target, from its output voltage the output voltages of the two adjacent channels are subtracted. If one of these differences is less than a threshold selected in a certain definite form, as the measure of distance there is selected the channel for which this difference is less than the threshold, and its voltage is compared with voltages of the two adjacent channels. Discrete data are smoothed by the usual method, and they control switching of the channels, subsequently selecting the number of the channel with the target. For low signal-to-noise ratios, when errors of measurement can be great and exceed the magnitude of one discrete value of distance, one should anticipate comparison of voltage in the selected channel not only with adjacent, but with more distant channels.

Let us consider in more detail with very simple assumptions a nontracking meter of discrete type with a fundamental circuit of the type of Fig. 7.50. We shall consider that adjacent channels of the detection unit do not overlap, so that signals at their output are orthogonal and statistically independent. Furthermore, we assume that selection of number is carried out in such a time T_0, at which voltages in any of the channels at the end of adjacent intervals of duration T_0 are statistically independent. We note that formation of the logarithm of the likelihood

ratio may occur, not for all time T_0, but during a certain part of it.

On these assumptions the input signal of the smoothing filter is a sequence of random variables d_k discretely varying with period T_0, which can take one of the discrete values $d^{(1)}$, $d^{(2)}$, ..., $d^{(m)}$, corresponding to the 1st, 2nd, ..., m-th channel. Quantities $d^{(1)}$, $d^{(2)}$, ..., $d^{(m)}$ in every period T_0 have the probability distribution $p_{ij}^{(k)}$, depending on the real position of the target in the k-th period of T_0. Probability $p_{ij}^{(k)}$ is the conditional probability that in the k-th period of T_0 maximum voltage will be in the i-th channel on the condition that the target in this period is in the j-th channel.

The mean value and variance of the input signal of the smoothing filter are determined by the following relationships:

$$\overline{d_k(j)} = \sum_{i=1}^{m} d^{(i)} p_{ij}^{(k)}, \tag{7.12.1}$$

$$\sigma_k^2(j) = \sum_{i=1}^{m} d^{(i)2} p_{ij}^{(k)} - \left(\sum_{i=1}^{m} d^{(i)} p_{ij}^{(k)} \right)^2. \tag{7.12.2}$$

Both quantities are calculated under the condition that the target is at distance $d^{(j)}$.

If smoothing circuits are a discrete filter, fluctuation error of measurement will be determined by expression

$$\sigma^2_{\text{флх } n} = \sum_{k=1} g_{nk}^2 \sigma_k^2, \tag{7.12.3}$$

where g_{nk} – response of the filter at the time nT_0 to a disturbance applied at time kT_0.

Correspondingly, dynamic error in the nonstatistical approach is equal to

$$\varepsilon_{\text{дин } n} = d_{0n} - \sum_{k=1}^{n} g_{nk} \sum_{i=1}^{m} d^{(i)} p_{ij}^{(k)}, \tag{7.12.4}$$

where d_{0n} – true distance at time nT_0.

With the statistical approach, when we know the probability distribution of real distance at any moments of time, we must find the mean square dynamic error $\varepsilon_{\text{дин } n}$.

If inertia of the smoothing filter is great as compared to the interval of quantization T_0, it may be continuous, and then fluctuation error, obviously, is given by the following formula:

$$\sigma^2_{\phi л} = T_0 \int_0^t g^2(t, \tau)\sigma^2(\tau)\, d\tau, \tag{7.12.5}$$

where $\sigma^2(t)$ — value of variance from (7.12.2) at time $t = kT_0$.

The expression for dynamic error varies analogously.

Probabilities $p_{ij}^{(k)}$ with the accepted assumptions about the independence of output voltages of various channels, as it is simple to prove, are determined by the following expressions:

$$\begin{gathered} p_{jj}^{(k)} = \int_{-\infty}^{\infty} w_{сш}(x)\Big[\int_{-\infty}^{x} w_{ш}(z)\, dz\Big]^{m-1} dx, \\ p_{\substack{ij \\ i \neq j}}^{(k)} = \int_{-\infty}^{\infty} w_{ш}(x) \int_{-\infty}^{x} w_{сш}(z)\, dz \Big[\int_{-\infty}^{x} w_{ш}(z)\, dz\Big]^{m-2} dx \end{gathered} \tag{7.12.6}$$

and, as follows from these expressions, depend on k only through the value of j, corresponding to true distance at moment kT_0. Functions $w_{сш}(x)$ and $w_{ш}(x)$ are probability densities for output voltage of the detection channel under the condition that on it there act a signal mixed with noise and noise alone, respectively (see Chapters IV and V). If, at the output of each of the detection channels, as in practice, there is also a certain threshold unit allowing us to eliminate from consideration those channels, voltage in which in the given period does not attain its trigger level C, expressions (7.12.6) are somewhat changed — the lower limit of integration over x is changed in them from $-\infty$ to c. Transforming the second of expressions (7.12.6) by integration in parts, it is easy to prove the validity of the following relationship:

$$p_{\substack{ij \\ i \neq j}}^{(k)} = \frac{1}{m-1}\left[1 - p_{jj}^{(k)}\right] \tag{7.12.7}$$

which gives us the possibility to limit ourselves to determination only of probability $p_{jj}^{(k)}$.

With fluctuations which are slow as compared to the period of duty of a detection channel in one period distributions $w_{сш}(x)$ and $w_{ш}(x)$ are equal, respectively, (see Chapter IV), to

$$\begin{gathered} w_{сш}(x) = \frac{1}{1+\mu} e^{-\frac{x}{1+\mu}}, \\ w_{ш}(x) = e^{-x}, \end{gathered} \tag{7.12.8}$$

where μ is the ratio of energy accumulated in one period T_0 of the signal to the spectral density of noise.

Then, from formula (7.12.6), we have

$$p_{jj}^{(k)} = \sum_{k=0}^{m-1} C_{m-1}^{k} (-1)^{k} \frac{1}{1+k(1+\mu)} \approx 1 - \frac{m-1}{m\mu}. \qquad (7.12.9)$$

The approximate expression is valid when $\mu \gg 1$, which is usually realized in practice. Analogously, probability $p_{jj}^{(k)}$ can be calculated with certain approximations also for fluctuations which are rapid as compared to the period of duty of the detection channel. Probabilities of erroneous selection of the maximum on the basis of (7.12.7) are equal to

$$p_{\substack{ij \\ i \neq j}}^{(k)} \approx \frac{1}{m\mu}. \qquad (7.12.10)$$

Due to the equality of all probabilities $p_{ij}^{(k)}$ when $i \neq j$, formulas (7.12.1) and (7.12.2) can be simplified and reduced to the form

$$\overline{d_k(j)} = \frac{1 + mp_{jj}^{(k)}}{m-1} d^{(j)} + \frac{mN}{m-1}(1 - p_{jj}^{(k)})\Delta d, \qquad (7.12.11)$$

$$\overline{d_k^2(j)} = \frac{1 + mp_{jj}^{(k)}}{m-1} d^{(j)2} + (1 - p_{jj}^{(k)}) \frac{m\Delta d^2}{m-1}\left[N^2 + \frac{m^2-1}{6}\right], \qquad (7.12.12)$$

where Δd — interval of distances covered by one detection channel;

N — distance, expressed in intervals Δd, corresponding to the middle of the range covered by the unit of detection channels.

From formulas (7.12.11) and (7.12.12) it follows that when $m \gg 1$

$$\sigma_k^2 = p_{jj}^{(k)}(1 - p_{jj}^{(k)})(d^{(j)} - N\Delta d)^2 + (1 - p_{jj}^{(k)}) \frac{m^2\Delta d^2}{6}. \qquad (7.12.13)$$

A characteristic feature of the considered range finder is the dependence of errors of measurement on the magnitude of the real distance [see formulas (7.12.2) and (7.12.13)]. This peculiarity is inherent to a certain extent in all nontracking meters, but in the given case it is manifested especially distinctly. Fluctuation error reaches its minimum if the target is in the middle of the range occupied by the detection unit, and increased with shift of the target towards its edges. As follows from (7.12.13), this increase occurs by square law. With probability $p_{jj}^{(k)}$ close to one, variance of errors on the edges of the range is doubled as compared to the minimum magnitude. Here, from expression (7.12.13) it follows that variance on the edge of the range, with accuracy to coefficient $1 - p_{jj}^{(k)}$, coincides with variance

of a uniform a priori distribution of width $\Delta d_0 = m\Delta d$.

If, in particular, probability $p_{jj}^{(k)}$ is determined by formula (7.12.10), the magnitude of variance is

$$\sigma^2(t) = \frac{1}{\mu}\left[(d(t) - d_0)^2 + \frac{1}{6}\Delta d_0^2\right], \qquad (7.12.14)$$

where d_0 – distance, corresponding to the middle of the range ($d_0 = N\Delta d$);

$d(t)$ – true distance at time t.

Then, for great inertia of the smoothing filter and constancy of the measured distance $d(t) = d$, and considering that $\mu = P_c T_0/2N_0$, we obtain in steady-state operating conditions the following expression for variance of fluctuation error:

$$\sigma^2_{\text{фл}} = T_0 \sigma^2 \int_0^\infty g^2(t - \tau)\, d\tau = T_0 \sigma^2 2\Delta f_{\text{эф}} =$$
$$= \frac{N_0 \Delta f_{\text{эф}}}{P_c}\left[(d - d_0)^2 + \frac{1}{6}\Delta d_0^2\right]. \qquad (7.12.15)$$

This expression shows that a range finder of the considered type has essentially worse accuracy than a tracking range finder. The magnitude of fluctuation error with such a method of range finding increases by ratio $\Delta d_0/\Delta d$ and is no longer determined by the magnitude of the resolution capability, but by the magnitude of the a priori interval.

§ 7.13. Range Finding in a System with a Variable Repetition Frequency

As it was shown in Chapter IV, to increase noise immunity with respect to passive interferences it is useful to increase the frequency of repetition. Here, the advantages of unmodulated radiation (speed resolution) can be combined with selection of the target with respect to distance. However, when using sufficiently high frequencies of repetition there may arise ambiguity with respect to distance and a number of difficult circumstances connected with the necessity of suppressing powerful signals from closely located local objects and the necessity of work of the receiver and transmitter on one antenna. In the last case, when the transmitter is operating its signal proceeds to the input of the receiver, and even if the tuning frequency of the receiver, due to Doppler shift, is different from the transmitter frequency, due to transmitter noises, occupying a wide frequency range, at the input of the receiver during this time there will be powerful interference, essentially exceeding signal level [40]. This circumstance leads to the necessity of cutoff of the receiver during radiation, which in turn causes the appearance of "blind" zones with respect to

distance. The number of these zones with a high frequency of repetition may be great, and the total extent depends on the off-duty factor. In particular, with an off-duty factor equal to two, half the range is covered by "blind" zones.

With movement of the target the signal from it alternately passes through "blind" zones. Therefore, for a significant part of the time it is inaccessible for observation, and the radar works with interruption equal to the time of passage through the "blind" zone. In such interruptions there are power losses and essential impairment of accuracy of measuring of distance and other coordinates. In order to eliminate these losses, it is useful to produce an abrupt or smooth change of the frequency of repetition, reducing to a minimum the times when the target is in a "blind" zone. Most useful is continuous change of the frequency of repetition, which permits us in principle to shift the interval between "blind" zones right behind the target and reduce the time of signal cutoff to zero. Here, by change of frequency we realize target tracking.

In the course of such tracking of the target the system should change its period of repetition $T_r(t)$ in such a manner that error is minimal:

$$\varepsilon(t)=\left(n+\frac{1}{2}\right)T_r(t)-\tau(t), \tag{7.13.1}$$

where $\tau(t)$ – changing magnitude of delay;

$T_r(t)$ – period of repetition;

n – integer.

This system can consist of a discriminator and smoothing circuits the same type as in an ordinary range finder. It is possible to show that the optimum system, ensuring minimum mean square error $\overline{\varepsilon^2(t)}$, remains in this case the same as the circuit of an optimum range finder. The difference is that in the given case, not the magnitude of delay, but $(n + \frac{1}{2})T_r(t)$ should be controlled. We note that it is sometimes more convenient technically to control proportionally to control voltage, not the period, but the frequency of repetition f_r.

It is easy to prove that with use of the error signal for control of f_r and linear smoothing circuits the system does not ensure the required quality of tracking: with movement of the target the equation of the system does not have a solution in the form

$$f_r(t)=\frac{n+\frac{1}{2}}{\tau(t)}+\xi(t),$$

where $\xi(t)$ – small error which does not increase.

In practice the quantity permitting control is not $(n + \frac{1}{2})T_r(t)$, but the period of repetition $T_r(t)$. In this connection there arises the necessity of controlling the gain factor of the open circuit proportionally to $1/(n + \frac{1}{2})$. Such control is especially necessary if the distance changes in the course of tracking by a factor of several times. Inasmuch as linear change of the period is possible only in a limited range, in this case jumps of period T_r by a nominal amount after passage by the target through several "blind" zones with respect to distance are inevitable. Thus, quantity n changes in the course of tracking, and in order to keep the speed of operation of the range tracking system constant it is necessary to change the gain factor of the open circuit K_0.

To realize control of K_0 and to properly, i.e., without losing the target, choose moments of jumps it is necessary to measure the distance to the target. Unambiguous measurement of distance can be effected with the use of outputs of the period control and speed tracking systems. The latter always is present in coherent single-target radars. Let $\varepsilon(t) = 0$; then from (7.13.1) $(n + \frac{1}{2})T_r(t) = \tau(t) = \frac{2d(t)}{c}$. Differentiating both parts of this equality, we obtain

$$\left(n+\frac{1}{2}\right)\dot{T}_r(t) = -\frac{2V(t)}{c} = -\frac{\lambda}{c}\, f_{\text{Д}}(t), \tag{7.13.2}$$

where $V(t)$ – speed of approach;

$f_{\text{Д}}(t)$ – Doppler frequency.

Eliminating n, we can obtain

$$d(t) = -\frac{1}{2}\frac{T_r(t)}{\dot{T}_r(t)}\lambda f_{\text{Д}}(t) = -\frac{T_r(t)}{\dot{T}_r(t)}\, V(t). \tag{7.13.3}$$

Calculation of range by formula (7.13.3) can be done by analog or digital computers. To decrease fluctuations of voltage proportional to $\dot{T}_r(t)$ one can use additional smoothing circuits after differentiation of voltage at the output of the period control system.

Let us find the magnitude of fluctuation error of range finding. We shall consider fluctuations of functions $T_r(t)$, $\dot{T}_r(t)$ and $V(t)$ small as compared with their mean values. Then error of unambigous measurement of distance is

$$\begin{aligned}\Delta d(t) &= d(t)\left[\frac{\Delta T_r(t)}{T_r(t)} - \frac{\Delta\dot{T}_r(t)}{\dot{T}_r(t)} + \frac{\Delta V(t)}{V(t)}\right] = \\ &= \frac{c}{2}\left[\varepsilon_{\Phi}(t) + \frac{d(t)}{V(t)}\left(\dot{\varepsilon}_{\Phi}(t) + \frac{2\Delta V(t)}{c}\right)\right],\end{aligned} \tag{7.13.4}$$

where $\varepsilon_{\Phi}(t)$ – fluctuation error of the system for range tracking of a target;

$\Delta V(t)$ – fluctuation error of the speed tracking system.

Random variables ε_{Φ} (t), $\dot{\varepsilon}_{\Phi}(t)$, and $\Delta V(t)$ can be considered uncorrelated: ε_{Φ} (t) and $\dot{\varepsilon}_{\Phi}$ (t) are uncorrelated with one another in view of the symmetry of the spectrum of fluctuation error ε_{Φ} (t) with respect to zero frequency; while ε_{Φ} (t) and $\dot{\varepsilon}_{\Phi}$ (t) are not correlated with $\Delta V(t)$ in view of the unconnectedness of coding of distance and speed in the received signal.

As a result the variance of fluctuation error is equal to

$$\sigma^2(t)=\frac{c^2}{4}\left[\sigma_\varepsilon^2+t_0^2(t)\left(\sigma_{\dot{\varepsilon}}^2+\frac{4}{c^2}\sigma_V^2\right)\right], \tag{7.13.5}$$

where $t_0(t) = d(t)/V(t)$ — time remaining until target impact.

Error of tracking of distance σ_ε^2 can be calculated from the results of the preceding paragraphs; error of measurement of speed can be defined by formulas of Chapter IX, and $\sigma_{\dot{\varepsilon}}^2$ is equal to

$$\sigma_{\dot{\varepsilon}}^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2 S_{\text{экв}}\,|H_1(i\omega)|^2\,|H_2(i\omega)|^2\,d\omega, \tag{7.13.6}$$

where $H_1(i\omega)$ — frequency response of the range tracking system;

$H_2(i\omega)$ — frequency response of the filter intended for decreasing fluctuations $\dot{T}_r(t)$;

$S_{\text{экв}}$ — equivalent spectral density of the discriminator of the range tracking system.

Quantity σ_V during speed tracking can comprise 0.1-0.5 m/sec or more, and $\sigma_{\dot{\varepsilon}}^2$ with correct choice of smoothing circuits will coincide in order with $\frac{4}{c^2}\sigma_V^2 \approx$ $\approx 4\cdot 10^{-18}$ to 10^{-16}. Then, for instance, when $t_0 = 10$ min

$$\frac{c}{2}\sqrt{t_0^2\left(\sigma_{\dot{\varepsilon}}^2+\frac{4}{c^2}\sigma_V^2\right)}\approx 25-100 \text{ min.}$$

This means that error obtained due to inaccuracy of measurement of speed and magnitude $\dot{T}_r(t)$ may significantly exceed error of tracking for distance σ_ε.

Thus, in the considered system accuracy of unambiguous range finding is determined, basically, not by the usual tracking range finder, tracking the target, but by a computer.

§ 7.14. Influence of Interferences on Coherent Radar Range Finders

As it was shown in Chapter I, application of contemporary radar means may be accompanied by the influence on them of natural and man-made external interferences. The presence of such interferences impedes work of the radar and, in particular, decreases accuracy of range finding, and in certain cases generally leads to cessation of work of a radar range finder [69-70]. All this creates the necessity to consider

the influence of external interferences on range finders.

In the case of coherent radio range finders such consideration is facilitated by the very principle of their construction. Due to the presence in discriminators of range finders of narrow-band elements, preceding amplitude or phase detection, any interference having a random character and a spectrum width relatively large as compared to the transmission bandwidth of the narrow-band filters turns out to be statistically equivalent to normal white noise with the corresponding spectral density. Therefore, the influence of interferences can be described by means of corresponding increase of the level of noises at the input of the receiver of the range finder.

Analysis of the influence of interferences leads then to determination of the spectral density of such equivalent white noise and calculation of the new signal-to-noise ratio. In other respects all results obtained in the preceding paragraphs do not change. It is necessary only to substitute in all formulas for the previously used signal-to-noise ratio h its new magnitude

$$h_{\text{пш}}=\frac{P_c}{2\Delta f_c\,(N_0+N_{\text{п}})}=\frac{h}{1+\frac{N_{\text{п}}}{N_0}}, \tag{7.14.1}$$

where $h_{\text{пш}}$ – signal-to-noise ratio in the presence of noise and interference;

$N_{\text{п}}$ – spectral density of white noise, which is equivalent to interference recalculated at the input of the receiver of a coherent radio range finder.

Let us consider now examples of the most common external interferences.

7.14.1. Noise Interference

Active noise interference, radiated from the tracked or another target, by virtue of its usual broad-banded nature, obviously, is equivalent to white noise without any reservations. The spectral density of it at the input of the receiver is determined from the power of the jamming transmitter $P_{\text{п}}$ and width of the spectrum of interference $\Delta f_{\text{п}}$ by the distance formula

$$2N_{\text{п}}=\frac{P_{\text{п}}G_{\text{пп}}G_{\text{пр}}\lambda^2}{(4\pi)^2\,\Delta f_{\text{п}}d_{\text{п}}^2}, \tag{7.14.2}$$

where $G_{\text{пп}}$ – antenna gain of the station of interferences;

$G_{\text{пр}}$ – gain of the receiving antenna of the radar in the direction to the source of interference;

λ – wavelength;

$d_{\text{п}}$ – distance to the source of interference.

Decrease of the signal-to-noise ratio is determined by relationship

$$\frac{h_{пш}}{h}=\frac{1}{1+\frac{P_п G_{пп}}{2N_0\Delta f_п}\,\frac{G_{пр}\lambda^2}{(4\pi)^2 d_п^2}} \qquad (7.14.3)$$

and is stronger, the larger the ratio of the spectral density of radiated interference $P_{\Pi}\ G_{\Pi\Pi}\ /\Delta f_{\Pi}$ to the spectral density of internal noise and the larger the ratio of the area of the receiving antenna to the square of the distance to the source of interferences.

From formula (7.14.3) it follows that the signal-to-noise ratio $h_{\Pi Ш}$ is determined in many cases by the magnitude of external interference, and not the internal noises of the receiver. Therefor , for the case of a source of interference combined with the target the signal-to-noise ratio increases with approach to the target approximately proportionally to $1/d^2$, and not proportionally to $1/d^4$ as in the absence of interference. Curves of the dependence of the signal-to-noise ratio on $G_{\Pi\Pi}\ \lambda^2/(4\pi)^2 d_{\Pi}^2$ for various values of $P_{\Pi}\ G_{\Pi\Pi}\ /2N_0\Delta f_{\Pi}$ are shown in Fig. 7.51.

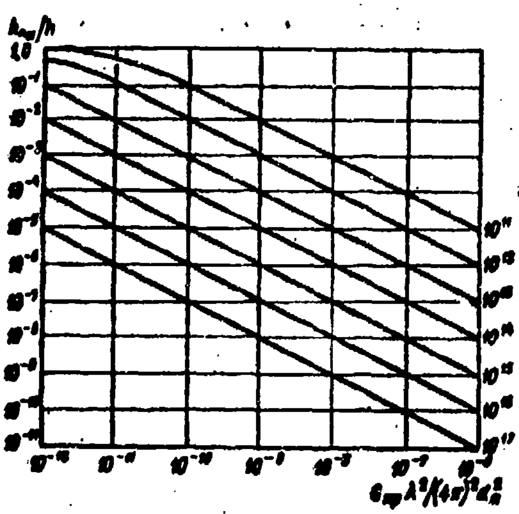

Fig. 7.51. Decrease of the signal-to-noise ratio under the influence of active noise interference.

7.14.2. Pulse Chaotic Interference

Properties of such interference were considered in Chapter I. Here, we additionally assume incoherence of pulse chaotic interference, i.e., we consider that separate pulses of interference have independent random initial phases. Then the signal of interference proceeding to the receiver input can be recorded in the form

$$n(t)=\sum_k u_п(t-t_k)\cos[(\omega_0+\omega_p)t+\varphi_k], \qquad (7.14.4)$$

where $u_{\Pi}\ (t)$ – function, describing the shape of the pulse of interference;

t_k – random moment of appearance of the k-th pulse;

φ_k – random phase of the high-frequency charge of the k-th pulse;

ω_p – frequency separation between carrier frequencies of the radar and the jamming station.

After multiplication of interference by a reference signal in the receiver (heterodyning and gating) it takes the form

$$n(t)\,u_\pi(t)\cos[(\omega_0+\omega_{np})t+\psi(t)]=$$
$$=A(t)\cos\omega_{np}t+B(t)\sin\omega_{np}t, \tag{7.14.5}$$

where A(t), B(t) – periodic nonstationary random processes with zero mathematical expectation and spectral density corresponding to the averaged correlation function at zero frequency, equal to

$$2N_\pi=\frac{\nu}{2\pi}\int_{-\infty}^{\infty}|U_\pi(i\omega)|^2\int_{-\infty}^{\infty}C(\tau)e^{i(\omega-\omega_p)\tau}d\tau\,d\omega, \tag{7.14.6}$$

where U_Π $(i\omega)$ – Fourier transform of u_Π (t);

$C(\tau)$ – autocorrelation function of the sounding signal of the radar;

ν – average frequency of interference pulses.

Quantity N_Π is the spectral density of equivalent white noise, which is contained in formula (7.14.1) and characterizes decrease of the signal-to-noise ratio. The permissibility of the idealization of pulse chaotic interference by white noise during action on a coherent radio range finder is determined by the relationship between the prolonged nature of the interference pulse and bandwidth of the narrow-band filter. If $\Delta f_\text{ф}\,\tau_\Pi \ll 1$, where τ_Π – duration of the interference pulse, then such idealization is fully justified.

Formula (7.14.6) allows certain simplifications with the proper assumptions. If, in particular, width of the spectrum of interference pulses is great as compared to the width of the spectrum of modulation, and detuning of the carrier ω_P is small as compared to the width of the spectrum of modulation, from expression (7.13.6) it follows that

$$2N_\pi\approx\frac{\nu}{2\pi}\int_{-\infty}^{\infty}|U_\pi(0)|^2\int_{-\infty}^{\infty}C(\tau)e^{i\omega\tau}d\tau\,d\omega=\nu|U_\pi(0)|^2C(0)=$$
$$=\nu|U_\pi(0)|^2=\nu\Big[\int_{-\infty}^{\infty}u_\pi(t)\,dt\Big]^2=\nu u_0^2\,\tau_\pi^2=\nu\tau_\pi^2\,P_\pi, \tag{7.14.7}$$

where P_Π – power of interference in the pulse,

u_0 – pulse amplitude.

In the other extreme case, when the width of the spectrum of the pulse of interference is small as compared to the width of the modulation spectrum,

$$2N_\pi\approx\nu\frac{1}{2\pi}\int_{-\infty}^{\infty}|U_\pi(i\omega)|^2d\omega\int_{-\infty}^{\infty}C(\tau)\,d\tau=\nu\tau_\pi P_\pi\frac{1}{\Delta f_{\text{эф}}}, \tag{7.14.8}$$

where $\Delta f_{m\ \text{эф}}$ — effective width of the spectrum of modulation, determined by the usual method and equal to $1/\int_{-\infty}^{\infty} C(\tau)d\tau$ on the basis of the relationship between spectrum width and correlation time.

In general, proceeding from the form of the integral in formula (7.14.6) one should expect that

$$2N_{\text{п}} = \frac{\nu\tau_{\text{п}}^{2}P_{\text{п}}}{\sqrt{1+\Delta f_{m\ \text{эф}}^{2}\tau_{\text{п}}^{2}}}. \qquad (7.14.9)$$

Effectiveness of the influence of pulse chaotic interference, except for the case when formula (7.14.8) is valid, depends not only on its mean power $\nu\tau_{\text{п}}\ P_{\text{п}}$, but also on pulse duration $\tau_{\text{п}}$, where with lengthening of the pulse its effectiveness increases.

Decrease of the signal-to-noise ratio under the influence of pulse chaotic interference on a coherent range finder can be characterized by an expression similar to (7.14.3),

$$\frac{h_{\text{пш}}}{h} = \frac{1}{1+\dfrac{\nu\tau_{\text{п}}^{2}P_{\text{п}}}{2N_{0}\sqrt{1+\Delta f_{m\ \text{эф}}^{2}\tau_{\text{п}}^{2}}}\ \dfrac{G_{\text{пр}}\lambda^{2}}{(4\pi)^{2}d_{\text{п}}^{2}}}. \qquad (7.14.10)$$

7.14.3. Return Interference

Effectiveness of return interference in reference to range finders essentially depends on its concrete form. It is obvious that the return signal of a station of interferences, radiated from an object without delay relative to the reflected signal, is not actually interference, but only helps to measure target position data.

If the responder is located not on the target, so that the distances between the radar and the target and between the radar and the station of interferences differ by more than the magnitude of the resolution capability, thanks to the selective ability of discriminator with respect to range the signal of return interference does not affect work of the measuring system, of course, under the condition that preliminary selecting of the required target was already performed. "Multiplication" of the return signal also cannot lead to essential increase of the effectiveness of return interference, since for this it is required that the period of repetition of the multiplied signal be commensurate with the magnitude of the range resolution capability of the radar, which for a sufficiently high resolution

capability requires huge mean power of interference.

Interference with multiplication can cause a range finder to malfunction only if the signal of interference in every period of repetition of modulation of the signal appears at different, but sufficiently small as compared to resolution capability, delays with respect to the signal from the target. In this case interference will pass to the discriminator output, and with fair effectiveness it will either suppress the signal from the target with the help of an AGC system, or due to shifts of the signal of the responder in range it will lead to swinging of the tracking system and breakoff of tracking. If, however, one of the multiplied return signals constantly coincides with the signal from the target, it only increases accuracy of measurement, introducing, possibly, a certain systematic error upon nonprecise coincidence of signals of the station of interferences and those reflected from the target.

Additional circumstances, decreasing effectiveness of return interference with multiplication, are:

1) the preference for complicated forms of modulation in coherent radars and the connected difficulties of undistorted multiplication of signals with such modulation;

2) difficulties of preserving coherence of the return signal, which is necessary for accumulation of interference in narrow-band filters.

An effective means of affecting radio range finders using discriminators with switching of reference signals can be any return interference with low-frequency modulation of a regular of random character. Beats of spectral components of this modulation with the frequency of switching can give low-frequency components which are capable, without essential suppression, of passing through a closed-loop tracking system and causing it to swing.

An effective means of combatting all range finders can be return interference with variable delay, which with sufficient intensity and a reasonably selected rate of change of delay distracts the range finder from the target and forces it to track the interference. Physically, the action of return interference with variable delay, obviously, is equivalent to a single disturbing target, moving relative to the selected target, and the question of the character and results of action of the interference belongs, in point of fact, to the problem of resolution of targets in radar, more exactly to the problem of dynamic resolution in the process of tracking by a servo system. Here, one should establish the possibility of transition of the servo system from tracking one target to tracking another, characteristics of th-

process of transition, the dependence of these characteristics on the intensity and and rate of movement of the interfering signal, etc.

Unfortunately, due to the necessity of solution of nonlinear nonstationary problems this question to this day has not been studied. It is possible to indicate only certain qualitative considerations about improvements of characteristics of range finders in the presence of such interferences. Rather obvious is the requirement of indication of the presence of an interfering signal, distinguishing the useful and interfering signals, and subsequent compensation of the latter by various methods. Further improvement of the quality of work of the range finder is possible using distinctions between laws governing change of distance to the target and distance corresponding to delay of the interfering signal known a priori and studied in the process of measurement. All these questions need systematic investigation.

7.14.4. Passive Interference

In Chapter IV it is shown that during coherent reception a signal from a passive interference after multiplication by the reference signal and with narrow-band filtration turns out to be, from the point of view of its influence on narrow-band filters and its further transformations, statistically equivalent to a stationary normal random signal whose spectral density on the basis of results of Chapter IV is equal to

$$S_{\text{П}}(\omega)=\frac{P_0}{2\Delta f_0}\cdot\frac{\sigma_{\text{П}}}{\sigma_{\text{Ц}}}\sum_{k=-\infty}^{\infty} f(\omega-\Delta\omega_{\text{Д}}+k\omega_r), \tag{7.14.11}$$

where $\sigma_{\text{Ц}}$ – reflecting surface of the target;

$\sigma_{\text{П}}$ – reflecting surface of the interference in the resolution volume of the radar;

$\Delta\omega_{\text{Д}}$ – difference between Doppler shifts of the signal from the target and from the interference;

$$\omega_r=2\pi f_r;$$

$f(x)$ – describes the form of the spectrum of interference, where $f(0) = 1$.

In deriving this formula it was assumed that Doppler shifts are small as compared to the width of the spectrum of modulation.

As follows from (7.14.11), the intensity of interference, besides the total reflecting surface of all dipole reflectors, essentially depends on the difference of Doppler shifts and the frequency of repetition. Summation in this formula emphasizes the fact that interference is dangerous not only when $\Delta\omega_{\text{Д}} \approx 0$, but also when $\Delta\omega_{\text{Д}} \approx k\omega_r$. Since in the overwhelming majority of practical cases the width of

the spectrum of interference is small as compared to the frequency of repetition, in formula (7.14.11) actually only one term is essential, the one for which difference $\Delta\omega_{\text{Д}}-k\omega_r$ is minimal. In many cases selection of the frequency of repetition is made so that this difference for all $k \neq 0$ is great, and in (7.14.11) one should consider only the zero term. It is obvious that such selection of the frequency of repetition in general ensures the greatest immunity for a coherent radar with respect to passive interference, since here there decrease to a minimum the number of situations in which to the input of the narrow-band filter there proceed the most intense components of the spectrum of interference, corresponding to a maximum of function

$$f(\omega - \Delta\omega_{\text{Д}} + k\omega_r) \text{ when } \Delta\omega_{\text{Д}} - k\omega_r \approx 0.$$

The width of the spectrum of each of the bands of the signal reflected from the cloud of interferences is determined by irregularity of movement of the dipoles and the transmitter wavelength, and in the case of a moving radar also by the speed of the radar with respect to the cloud of interferences and the width of its radiation pattern. Depending upon wavelength, for a motionless radar the width of the spectrum is 20-100 cycles, and for a moving radar it is from tens to thousands of cycles per second. Such a width of the spectrum, other things being equal exceeds the width of the spectrum of the reflected signal and the matched transmission bandwidth of the narrow-band filters. Therefore, with certain approximations in coherent radio range finders passive interference also is equivalent to white noise with spectral density

$$2N_{\text{п}} = \frac{P_c}{2\Delta f_c}\frac{\sigma_{\text{п}}}{\sigma_{\text{ц}}} f(\Delta\omega_{\text{Д}}), \tag{7.14.12}$$

where the frequency of repetition is assumed sufficiently great as compared to all possible values of $\Delta\omega_{\text{Д}}$. The signal-to-noise ratio decreases here by a factor of

$$\frac{h_{\text{пп}}}{h} = \frac{1}{1 + h\frac{\sigma_{\text{п}}}{\sigma_{\text{ц}}} f(\Delta\omega_{\text{Д}})} \text{ times.} \tag{7.14.13}$$

Ratio (7.14.13) essentially depends on the ratio of the difference of Doppler shifts $\Delta f_{\text{Д}} = \Delta\omega_{\text{Д}}/2\pi$ to the width of the spectrum of interference Δf_{Π}. For large $\Delta f_{\text{Д}}/\Delta f_{\Pi}$ function $f(\Delta\omega_{\text{Д}})$ takes small values, and we achieve tuning away from passive interference at the expense of frequency selection during coherent reception. For small $\Delta f_{\text{Д}}/\Delta f_{\Pi}$, when there is no tuning, function $f(\Delta\omega_{\text{Д}}) \approx 1$ and the discriminator of the range finder is affected by the most intense components of the spectrum of interference. Here, by virtue of the relation existing in real situations

$h\sigma_{\Pi}/\sigma_{\text{Ц}} \gg 1$, the signal-to-noise ratio sharply decreases and takes value $h_{\Pi\text{Ш}} = \sigma_{\text{Ц}}/\sigma_{\Pi}$, depending only on the reflecting surfaces of the target and the interference. The magnitude of $\sigma_{\text{Ц}}/\sigma_{\Pi}$ is frequently considerably less than unity. This means that with coincidence of speeds of the target and of the interference the meter with high probability may be caused to malfunction. Speeds of the target or, correspondingly, the angles between the speeds of the target and the interference with respect to the radar at which the difference of Doppler frequencies of the target and the interference are close to zero are sometimes called "blind." Their location and magnitude depend on the concrete situation and are specially calculated in every case interesting us.

7.14.5. Intermittent Interference

Any of the active interferences may be radiated, not continuously, but from time to time interrupted at preselected or random moments of time. Interruption is used for concentration of high power of interference in definite intervals of time, and also for causing parametric action on tracking systems of the radar and swinging them, achieving breakoff of tracking.

The frequency of interruptions may be most diverse, and depending upon it the results of the interference on the range finder will differ somewhat. With fast-intermittent (as compared to the inertia of smoothing circuits of the range finder) interference it generally turns out to be equivalent to continuous interference of the same mean power. For instance, with periodic interruption with times of the action and absence of interference T_1 and T_2 the power of radiation of interference should be multiplied by $T_1/(T_1 + T_2)$.

However, one should note that such equivalence of intermittent interference to continuous interference requires taking special measures to eliminate different parasitic factors in the radar receiver.

In particular, there should be taken measures to preventing overload in the limited range of the linear amplifiers of the receiver, so that after cessation of the action of interference the signal is passed by these amplifiers without distortion or suppression.

A second important element of the receiver, requiring special attention, is the AGC system. After powerful interference is switched on, the AGC system should rapidly develop its level, lowering amplification of the receiver, fixed before this from the signal level. Otherwise there will be different conditions during transmission of the useful signal and of interference, and the signal-to-noise ratio

will decrease as compared to the case of a linear receiver without automatic gain control.

After the interference is turned off, the AGC system, conversely, should rapidly increase amplification of the receiver, restoring its sensitivity to the signal. Only when the first process is fast as compared to the time of action of the interference and the second is fast as compared with the time of its absence is there no worsening of the signal-to-noise ratio as compared to the case of continuous interference of the same mean power.

A certain exception, for any frequency of interruption of interference, is a range finder using a discrimimator with switching of reference signals. Application of such discriminators is generaly undesirable from the point of view of the action of intermittent interferences, since beats of the frequency of switching with the frequency of interruption of interference may yield harmonics passing through the servo system and leading to its excitation.

With slow interruption of interference all processes in the range finder during the time of its action or absence can be established, and on the whole the influence of interference is characterized by the fact that the range finder works alternately, first with signal-to-noise ratio h, (when the interference is turned off) and then with signal-to-noise ratio $h_{пш}$ (when the interference is on). Under the action of interference errors of measurement increase and can exceed permissible levels, and with sufficiently great interference strength there may occur breakoff of tracking.

Another characteristic feature is the fact that as a result of the dependence of the gain factor of the open loop of a range finder on the signal-to-noise ratio due to the normalizing properties of the AGC system during interruption of interference the range finder as a whole experiences a parametric influence. With sufficiently complicated smoothing circuits this influence will lead to parametric excitation of the tracking system and total disruption of its operation.

With an arbitrary frequency of switching of the interference calculation of its influence on a range finder requires special, rather complex investigation. Analyzing the servo system, one should consider that the gain factor of the discriminator and the equivalent spectral density of noise at the input of the range finder are functions of time; $K = K(h(t))$ and $S_{экв}(t) = S_{экв}(h(t))$, taking values corresponding to the signal-to-noise ratio h at those moments of time when there is no interference and to the signal-to-noise ratio $h_{пш}$ when there is interference. We will indicate only the method of solution of the problem of calculation of dynamic and fluctuation errors in this case.

The pulse response of the closed-loop system $g(t, \tau)$ and the related pulse response $v(t, \tau)$ are determined as before by equations (6.2.12) and (6.2.13), in which, however, one should consider th[illegible] gain factor $K_д$ is a function of time. Besides this, during calculation of [illegible]ation error $S_{экв}$ no longer can be removed from under the integral sign, and one should consider its dependence on time. For instance, in the case of a very simple smoothing filter in the form a single integrator the solution of equation (6.2.12) has the form

$$g(t, \tau)=K_и K_д(t)\exp\left\{-\int_\tau^t K_и K_д(s)\,ds\right\}. \quad (7.14.14)$$

From this expression it follows that for any regular law of interruption of interference parametric excitation of a system with one integrator does not occur, since the pulse response remains limited. In this respect the considered example is not very informative. With chaotic interruptions with a certain probability even a first-order system may be excited. Fluctuation error in the considered example is determined by the following expression:

$$\sigma^2_{фл}(t)=\int_0^t g^2(t, \tau)\, S_{экв}(\tau)\,d\tau=$$

$$=K_и^2 K_д^2(t)\int_0^t e^{-2\int_\tau^t K_и K_д(s)\,ds} S_{экв}(\tau)\,d\tau, \quad (7.14.15)$$

from which in this particular case there follow all the conclusions about the relative character of the influence of fast-intermittent and slowly intermittent interferences. Analysis of the effect of interferences on range finders with smoothing circuits of higher order can, in principle, be conducted analogously; however, it requires solutions of differential equations of the corresponding order with variable coefficients.

§ 7.15. Nonlinear Phenomena in Tracking Range Finders

The whole preceding analysis was based on a linearized presentation of the tracking radar range finder. Here, we considered that the signal at the discriminator output is proportional to the average current mismatch, and noises in most cases were considered independent of mismatch. In fact, as follows from the results of the first paragraphs of this chapter, the discrimination and fluctuation characteristics of discriminators of coherent range finders meet these requirements only with limited ranges of change of mismatch, which take place only with comparatively high

signal-to-noise ratios. In real conditions, due to the influence of interferences, which essentially decrease the magnitude of the signal-to-noise ratio, these requirements can not be satisfied, and it is necessary therefore to consider nonlinearity of the discriminator.

Subsequently we shall limit ourselves to a simplified situation, when it is possible to consider the fluctuation characteristic constant in required limits and equal to quantity $S_{экв}$ used everywhere earlier. This assumption is justified by the fact that calculation of nonlinearity of the discriminator is most interesting for comparatively small signal-to-noise ratios, and here irregularity of the fluctuation characteristic become less and less noticeable. Furthermore, due to demodulation of signal fluctuations by the AGC system there occurs additional leveling of the fluctuation characteristic, so that in real coherent receivers it does not differ greatly from a constant magnitude even for comparatively large signal-to-noise ratios. The discrimination characteristics of range finders is expressed by formulas for $\overline{z(t, \Delta)}$, obtained in § 7.2-7.5, according to which, for detunings δ which are not very great, the discrimination characteristic of practically any of the range finders considered above is

$$\dot{m}(\Delta) \doteq -KC(\Delta)C'(\Delta), \tag{7.15.1}$$

where K – proportionality factor.

Henceforth, we shall consider only real $C(\Delta)$; therefore, in (7.15.1) we omit the sign of the real part.

The general approach to investigation of nonlinear phenomena in tracking measuring systems and certain results of more concrete content in reference to systems with a smoothing filter in the form of a single integrator are presented in Chapter VI. Basic attention there was paid to phenomenon of breakoff of tracking, by which we understood mismatch between the real and measured values of the measured coordinate reaching certain boundary-points of the discrimination characteristic. We calculated basic statistical characteristics of breakoff – the probability of reaching a boundary-point in a certain time t and the mean time to the first reaching of a boundary-point – by which it is possible to judge the critical magnitude of equivalent spectral density and, correspondingly, of the signal-to-noise ratio, already impermissible for normal work of the measuring system.

Along with such an approach to the investigation of nonlinear phenomena in a tracking meter, based on the assumption of the absence in it of steady-state operating conditions, we may also use a solution with reflecting screens at the points

corresponding to limitation of the output of the measuring system. Such limitation practically always exists in tracking meters, and its approximate calculation with the help of reflecting boundaries, valid for a signal-to-noise ratio which is not too small, permits us to obtain the stationary distribution of probabilities and with its help the errors of measurement in steady-state operation. We shall consi... several examples of calculation of various characteristics for concrete forms of discriminators of range finding meters.

7.15.1. Average Time to Breakoff of Tracking

Using formulas (6.3.26) and (7.15.1), it is possible with the help of simple transformations to obtain the following exact expression for the mean value of the time of the first attainment of a boundary-point of the discrimination characteri... tic, equal to Δ_0, under the condition that initial mismatch was zero:

$$T=\frac{1}{4\Delta f_{\text{эф}}\sigma_{\text{д}}^{2}}\int_{0}^{\Delta_0}dy\int_{0}^{y}dx\,e^{\frac{C^2(y)-C^2(x)-\frac{2Vb}{K_{\text{н}}K_{\text{д}}}(y-x)}{2b\sigma_{\text{д}}^{2}}}, \qquad (7.15.2)$$

where $\sigma_{Л}^{2} = 2\Delta f_{\text{эф}} S_{\text{экв}}$;

$\Delta f_{\text{эф}}$ — variance of fluctuation error and the effective bandwidth of the tracking system in the linearized consideration;

V — speed of the tracked target, assumed constant in time intervals ... the order of T;

$b = -C''(0)$ — mean square width of the spectrum of modulation of the sounding signal.

Considering that function $C(\Delta)$ actually depends on $\sqrt{b}\Delta$, we replace $\sqrt{b}y$ by η and $\sqrt{b}x$ and ξ, designate the relative magnitude of variance of fluctuation error in the linearized system and the relative magnitude of dynamic error

$$b\sigma_{\text{д}}^{2}=\mu^{2},\quad \frac{\sqrt{b}V}{K_{\text{н}}K_{\text{д}}}=a$$

and introduce designation $C(\Delta) = a(\sqrt{b}\Delta)$. Then the final expression for the average time to breakoff can be given the form

$$T=\frac{1}{4\Delta f_{\text{эф}}\mu^{2}}\int_{0}^{\sqrt{b}\Delta_0}d\eta\int_{0}^{\eta}d\xi\exp\left\{\frac{a^2(\eta)-a^2(\xi)-2a(r_1-\xi)}{2\mu^2}\right\}, \qquad (7.15.3)$$

where one should consider that $a''(0) = -1$. With initial mismatch Δ differing from zero, the lower limit of integration over η is replaced by $\sqrt{b}\Delta$.

Analytic calculation of quadratures in (7.15.3) for practically interesting

functions a(x) is impossible; therefore the average time to breakoff must be calculated by numerical or graphical integration. From formula (7.15.3) it follows that the average time to breakoff, expressed in units of the time constant of the tracking system $1/\Delta f_{э\phi}$, depends only on the relative magnitude of fluctuation error of the linearized system, the boundary value of the discrimination characteristic and the relative magnitude of dynamic error. Qualitative investigation of this formula shows that the magnitude of T rapidly decreases with increase of dynamic error and variance of fluctuation error. In order to estimate the influence of dynamic error on breakoff, we shall consider the asymptotic case of high-level noises, when $\mu^2 = b\sigma_{л}^2$ is great (of the order of unity or larger). Then, from (7.15.3) it follows that

$$T \approx \frac{\sqrt{b}\Delta_0}{4\mu\Delta f_{э\phi}} \approx \frac{\Delta_0}{4\sigma_{л}\Delta f_{э\phi}} \approx \frac{\Delta_0}{V}, \qquad (7.15.4)$$

i.e., the average time to breakoff is equal to the time of movement of the target the magnitude of the interval from zero to the boundary-point of the discrimination characteristic.

From expression (7.15.3) we can obtain the approximate formula, valid for noises at not too low a level and zero dynamic error,

$$T = \frac{1}{8\Delta f_{э\phi}}\left(\frac{\Delta_0}{\sigma_{л}}\right)^2\left[1 + 0{,}08\left(\frac{\Delta_0}{\sigma_{л}}\right)^2 + 16\cdot 10^{-4}\left(\frac{\Delta_0}{\sigma_{л}}\right)^4\right], \qquad (7.15.5)$$

which was already given in Chapter VI. This expression depends only on the ratio of the width of the assigned zone of the discrimination characteristic to the fluctuation error of the linearized system. Graphs of this expression and the investigation of it given in Chapter VI show that the average time to breakoff rapidly increases with growth of $\Delta_0/\sigma_{л}$, where as the threshold value of this ratio, ensuring work of the tracking system for a sufficiently large number of its time constants, we can select quantity $\Delta_0/\sigma_{л} \simeq 5$ to 7. With further increase of this ratio the phenomenon of breakoff practically is not observed.

Assigning magnitude $\Delta_0 \sim 2/\sqrt{b}$, we obtain for the threshold value of μ a quantity of the order of 0.3-0.4, which permits us to find the permissible magnitude of the signal-to-noise ratio in a coherent range finder. In particular, with a discriminator close to optimal, when $S_{экв} \approx S_{опт}$, the critical signal-to-noise ratio is determined by expression

$$\frac{\Delta f_{э\phi}}{\Delta f_0} F(h_{кр}) \approx 0{,}1 \div 0{,}15, \qquad (7.15.6)$$

where F(h) is given by formula (7.2.15).

Dependence of the critical signal-to-noise ratio $h_{кр}$ on the ratio of bandwidths $\Delta f_c / \Delta f_{кр}$ for the case of an exponential correlation function for fluctuations is shown in Fig. 7.52. For other correlation functions the magnitude of $h_{кр}$ experiences only immaterial changes relative to the given law. The magnitude of $h_{кр}$, naturally, decreases with decrease of ratio $\Delta f_c / \Delta f_{эф}$, and for practically interesting values of this ratio it is in the range of 0.1 to 3.

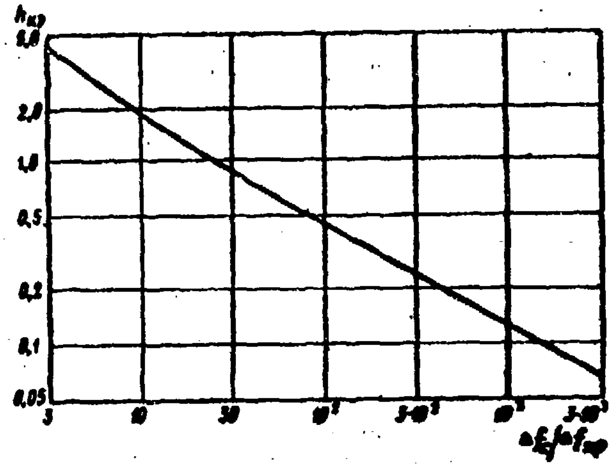

Fig. 7.52. Dependence of the critical signal-to-noise ratio on $\Delta f_c / \Delta f_{эф}$.

The dependence of the average time to breakoff, calculated by exact formula with the help of numerical integration, on μ is shown in Fig. 7.53 for $\sqrt{b}\Delta_0 = 2$ and a Gaussian autocorelation function for the sounding signal $a(x) = e^{-\frac{x^2}{2}}$.

This relationship very exactly coincides with that obtained by approximate formula (7.15.5).

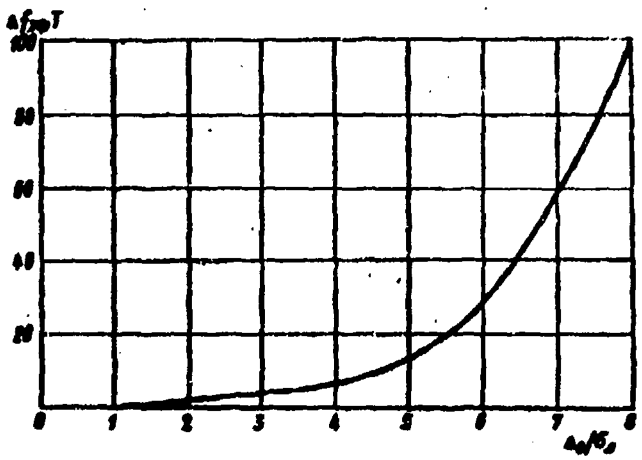

Fig. 7.53. Dependence of average time to breakoff on error of measurement in a linearized system.

Use of other approximations of a(x), corresponding to other methods of modulation of the sounding signal, does not lead to essential changes, since for all smooth functions of a(x) their behavior within limits of the basic maximum is approximately identical. Comparison of the curve of Fig. 7.53 with the curve in Fig. 6.7, constructed by the approximate formula, shows the good coincidence and the satisfactoriness of approximation of the expression for average time to breakoff by formula (7.15.5).

Let us consider for illustration one concrete example of analysis of phenomena

of breakoff of tracking in a coherent range finder. Let us assume that on the range finder there acts an active noise interference with spectral density $P_{\Pi} G_{\Pi\Pi} / \Delta f_{\Pi}$ and distance $d_{\Pi} = 300$ km. The receiving antenna of the radar has diameter 1.5 m, and the noise factor is 5. Let the signal-to-noise ratio h in the range finder receiver in the absence of interference by equal to 50. Then the signal-to-noise ratio in the presence of interference is

$$h_{\Pi\Pi} = \frac{50}{1 + 100 \frac{P_{\Pi} G_{\Pi\Pi}}{\Delta f_{\Pi}}},$$

where $P_{\Pi} G_{\Pi\Pi} / \Delta f_{\Pi}$ in watts/Mc.

Assuming that the discriminator of the range finder is close in characteristics to optimum, the spectrum of signal fluctuations is square with width $\Delta f_c = 30$ cycles, the effective bandwidth of the system is equal to 1 cycle, and taking boundary value of mistuning $\Delta_0 = 2/\sqrt{b}$, we obtain

$$\frac{\Delta_0}{\sigma_{\Pi}} = \frac{111}{\sqrt{(1 + 100 P_{\Pi} G_{\Pi\Pi}/\Delta f_{\Pi})(1 + 2 P_{\Pi} G_{\Pi\Pi}/\Delta f_{\Pi})}}.$$

This relationship permits us to find the average time to breakoff as a function of the spectral density of interference. This function is shown in Fig. 7.54, from which it is clear that a spectral density of interference near 1 watt/Mc in certain cases still gives an acceptable value of the average time to breakoff of the order of 50 sec. Slight decrease or increase of the spectral density of interference leads correspondingly to sharp increase or decrease of average time. With change of spectral density in any direction by a factor of two there occurs more than a tenfold change of average time to breakoff. These results emphasize the great criticality of a range finder to the level of interferences acting on them.

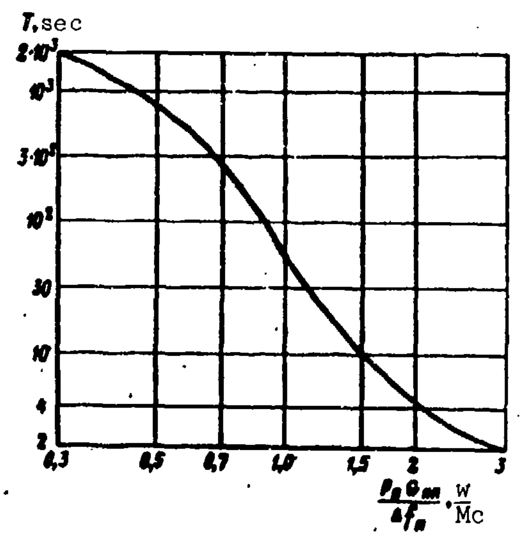

Fig. 7.54. Dependence of average time to breakoff on the spectral density of interference.

7.15.2. Fluctuation Error of Measurement Taking into Account Nonlinearity

As already noted above, breakoff of tracking, understood as the first time the magnitude of mismatch reaches a certain conditional point on the falling section of

the discrimination characteristic, cannot be, of course, an adequate description of the behavior of a tracking system during allowance for its nonlinearity.

Reaching a boundary-point, mismatch with noticeable probability may again take a small value, and the system will work normally, but, of course, with large errors. Therefore, the magnitude of mismatch, going beyond certain limits, still does not signify breakoff of tracking in the direct sense. Experimental study of nonlinear conditions of tracking meters also emphasizes the applicability of an approach based on the assumption of the absence in a nonlinear system of steady-state operating conditions.

In tracking range finders such an approach gives the correct answer for very large interferences and when tracking in comparatively small intervals of time is of interest. In examining the behavior of a system in large time intervals, due to the practically inevitable limitation on output in it, there sets in steady-state operating conditions, characteristics of which in a number of cases are of the greatest practical interest. Calculation of statistical characteristics of these conditions, in particular variance of fluctuation error, permits us to find the critical magnitude of the signal-to-noise ratio, at which fluctuation error still is within permissible limits. Comparison of this critical magnitude with the corresponding quantity, calculated in the preceding paragraph, permits us to characterize more fully nonlinear operating conditions of the meter and more completely find conditions when linearization of a tracking radar range finder is permissible.

Using formulas (6.3.22) and (7.15.[illegible]), it is possible to reduce the expression for variance of fluctuation error of measurement, found during idealization of the limitation at the system output by a reflecting screen at points $\pm\Delta_1$, to the following form:

$$\sigma^2 = \frac{1}{b} \frac{\int_{-\sqrt{b}\Delta_1}^{\sqrt{b}\Delta_1} y^2 e^{\frac{C^2(y)}{2\mu^2}} dy}{\int_{-\sqrt{b}\Delta_1}^{\sqrt{b}\Delta_1} e^{\frac{C^2(y)}{2\mu^2}} dy}, \qquad (7.15.7)$$

where all designations are the same as in Paragraph 7.15.1.

For low-level noises, when μ is small, from this expression it follows that

$$\sigma^2 = \frac{1}{b} \mu^2 = \sigma_x^2. \qquad (7.15.8)$$

For high-level noises, when $\mu \gg 1$,

$$\sigma^2 = \frac{\Delta_1^2}{3}, \tag{7.15.9}$$

i.e., the magnitude of variance here is determined only by the level of limitation on the output of the system, and the distribution of probabilities for mismatch becomes uniform.

Expression (7.15.7) allows us for certain forms of a sounding signal to perform analytic calculation. For instance, for a Gaussian $C(\Delta)$

$$b\sigma^2 = \frac{\frac{1}{3}(\sqrt{b}\Delta_1)^3 + \frac{\sqrt{\pi}}{4}\sum_{k=1}^{\infty}\frac{1}{(2\mu^2)^k k^3/_2 k!}}{\sqrt{b}\Delta_1 + \frac{\sqrt{\pi}}{2}\sum_{k=1}^{\infty}\frac{1}{(2\mu^2)^k k^1/_2 k!}}, \tag{7.15.10}$$

where it is assumed that $\sqrt{b}\Delta_1$ in accordance with real conditions is much larger than one. A similar expression can be obtained for cosinusoidal function $C(\Delta)$, and also for a series of other autocorrelation functions. Relative variance of error of measurement as a function of μ is shown in Fig. 7.55 for various values of the relative level of limitation $\sqrt{b}\Delta_1$. From formulas (7.15.7) and (7.15.10) and this figure it follows that already at $\mu = 0.3$ and at $\sqrt{b}\Delta_1 > 5$ to 10 the magnitude of error is determined to a significant degree by the level of limitation. For larger μ it is practically equal to the asymptotic value from formula (7.15.9).

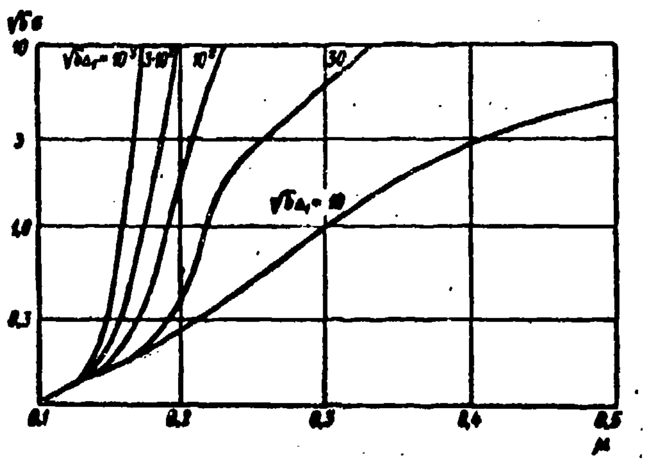

Fig. 7.55. Dependence of error of range finding on $\mu = \sqrt{b}\sigma_{\text{Д}}$ for various levels of limitation $\sqrt{b}\Delta_1$.

The curves of Fig. 7.55 show that the dependence of fluctuation error on the signal-to-noise ratio (error of a linearized system) also has a clearly expressed threshold character. Critical values of μ, corresponding, for instance, to fluctuation error $\sigma = 2/\sqrt{b}$, are in this case somewhat smaller than the same values found from the average time to breakoff of tracking (of the order of 0.15-0.3). In accordance with this permissible signal-to-noise ratios h increase somewhat. However,

this increase is not so considerable that one should take it into account.

Formulas (7.15.7) and (7.15.10) and the curves of Fig. 7.55 permit investigation of another interesting question — the limits of applicability of the linearized approach to a tracking range finder. From Fig. 7.55 it follows that equality of σ and $\sigma_{л}$ is preserved approximately to values $\mu \approx 0.12$-0.15. This range of values is very narrow so that at practically any level of limitation nonlinearity of the discriminator of a range finder starts to appear at the same signal-to-noise ratio. The magnitude of the signal-to-noise ratio at which linearized consideration of a tracking range finder with a discriminator close to optimum is still permissible is determined by this magnitude and can be found from equation

$$\frac{\Delta f_{эф}}{\Delta f_c} F(h_{кр}) = 0{,}017. \quad (7.15.11)$$

The dependence of $h_{кр}$ on the ratio of bandwidths $\Delta f_c / \Delta f_{эф}$ for the case of a square spectrum of fluctuations is shown in Fig. 7.56. For real relationships of bandwidth of the system and the width of the spectrum of fluctuations $h_{кр}$ varies from 0.2 to 6, which approximately exceeds by a factor of 2 the corresponding values calculated from the critical magnitude of the average time to breakoff. Thus, breakoff of tracking in the sense of the preceding paragraph occurs at approximately the same signal-to-noise ratios at which allowance for nonlinearities of the discriminator becomes necessary for calculating fluctuation error.

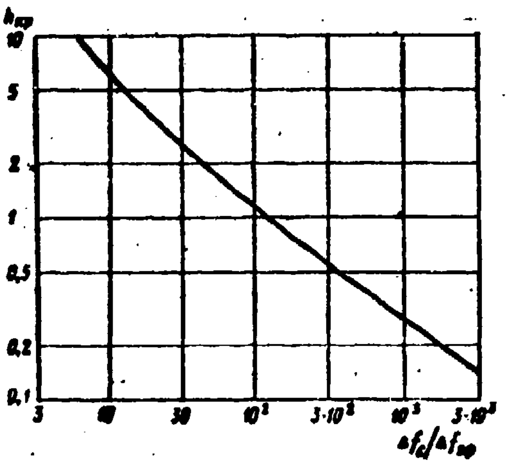

Fig. 7.56. Dependence of the critical signal-to-noise ratio on $\Delta f_c / \Delta f_{эф}$.

Steady-state operation of a tracking range finder in nonlinear conditions was studied experimentally. A range finder was simulated by an analog computer. In the simulation we reproduced a nonlinear discrimination characteristic and established limitation on output of the system. Analysis of results of the experiment showed satisfactory coincidence with the theoretically found probability distributions for mismatch obtained by solution of the boundary value problem for a Fokker-Planck equation with reflecting screens. In the course of the experiment we also studied characteristics of nonstationary conditions. We investigated change of the distribution of probabilities of mismatch with time and found the dependence of

fluctuation error on time. Data of the experiment, in general, well confirm conclusions of theoretical analysis, but for high-level noises there are certain divergences, caused by the approximate nature of the allowance for limitation on output. For illustration, in Fig. 7.57 are theoretically calculated and experimentally taken probability densities for mismatch in a range finder with one integrator with an effective bandwidth of the tracking system $\Delta f_{\text{эф}}$ = 1.5 cycles and the discrimination characteristic shown in Fig. 7.58. Curves correspond to the ratio of error of a linearized system to the half-width of the discrimination characteristic $\sigma_{\text{л}}/\Delta_0$, equal to 0.44 and 1.1.

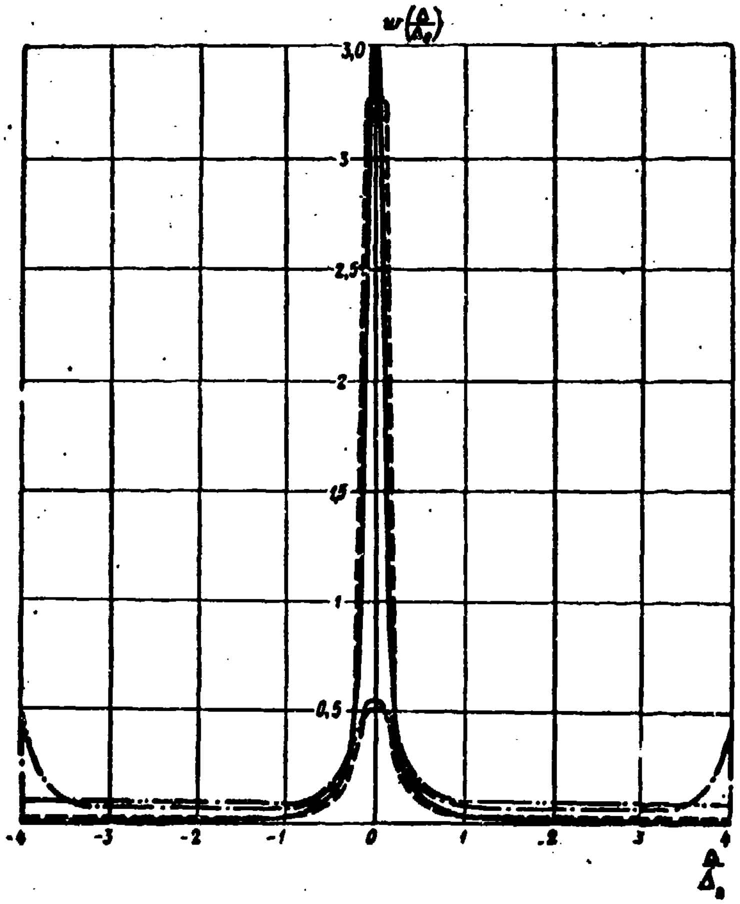

Fig. 7.57. Probability densities for mismatch: —— experimental curve for $\sigma_{\text{л}}/\Delta_0$ = 0.44; --- theoretical curve for $\sigma_{\text{л}}/\Delta_0$ = 0.44; -.-.- experimental curve for $\sigma_{\text{л}}/\Delta_0$ = 1.1; -..-..- theoretical curve for $\sigma_{\text{л}}/\Delta_0$ = 1.1.

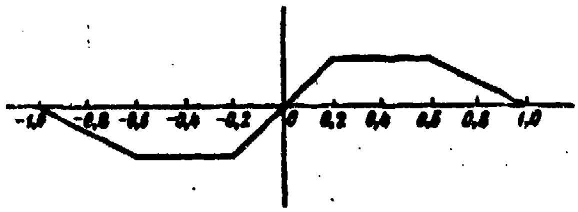

Fig. 7.58. Discrimination characteristic of the range finder in the experiment.

We shall consider one illustrating example of calculation of errors of range finding taking into account nonlinearity in a coherent range finder. We consider that the sounding signal of the radar is modulated by pulses close in form to Gaussian with linear frequency modulation, where frequency deviation due to the prolonged nature of the pulse is great as compared to the width of its spectrum. Then the autocorrelation function of the sounding signal is described by a Gaussian curve with parameter

$$b=\frac{2\omega_m^2}{\pi},\ \omega_m=\frac{a\tau_{и}}{2}, \qquad (7.15.12)$$

where a – rate of change of frequency;

$\tau_{и}$ – effective pulse duration.

Let frequency deviation $2\omega_m = 2\pi\cdot 10^6$ rad/sec. Then, $\sqrt{b} = 2.5\cdot 10^6$ sec^{-1}. We consider that limitation on output of the range finder is determined by natural factors – mismatch reaching a magnitude equal to the period of repetition of the signal. Considering the frequency of repetition equal to 25 kilocycles, we find $\sqrt{b}\Delta_1 = 100$. We consider that the signal-to-noise ratio $h_{пш}$ does not depend on distance and is equal to

$$h_{пш}=\frac{h_0}{\frac{P_п G_{пп}}{2\Delta f_п N_0}\cdot\frac{G_{пр}\lambda^2}{(4\pi d_0)^2}}, \qquad (7.15.13)$$

where h_0 – signal-to-noise ratio in the absence of interferences at distance d_0, corresponding to the moment of switching on of interference.

Assigning values $h_0 = 100$, $d_0 = 25$ km, the diameter of the antenna equal to 30 cm, and noise factor of the receiver of the radar equal to 10, we obtain

$$h_{пш}=\frac{\Delta f_п}{P_п G_{пп}},$$

where spectral density $P_п G_{пп}/\Delta f_п$ is expressed in watts/Mc. Assuming that the discriminator of the range finder is close to optimum, the width of the spectrum of the reflected signal is 100 cycles, and the bandwidth of the tracking system is 4 cycles, we obtain the following expression for variance of fluctuation error of the linearized system:

$$\mu^2=b\sigma_д^2=0{,}04\,\frac{P_п G_{пп}}{\Delta f_п}\left(1+\frac{P_п G_{пп}}{\Delta f_п}\right).$$

which gives the possibility of determining with the help of the graphs of Fig. 7.55 error of range finding taking into account non-linearity. Dependence of this error on the spectral density of the radiated interference is shown in Fig. 7.59. As follows from this figure, the dependence of accuracy of range finding on the spectral density of interference has a clearly expressed threshold character. With change of $P_{\Pi} G_{\Pi\Pi} / \Delta f_{\Pi}$ from 0.5 watts/Mc to 1 watt/Mc error increases by more than 60 times. The critical value of spectral density of interference comprises in the considered example about 0.6-0.7 watt/Mc.

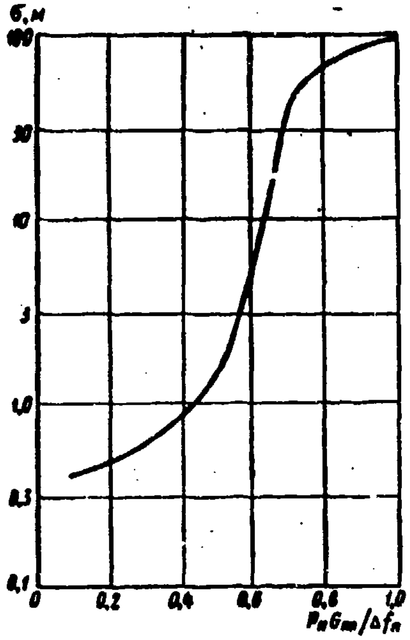

Fig. 7.59. Dependence of error of range finding on the spectral density of noise interference.

For high levels of interference accuracy of range finding becomes too low.

§ 7.16. Conclusion

This consideration of coherent range finders showed that in a significant majority of cases the above-described methods of realizing optimum operations permit us to obtain accuracy close to that which is potentially possible. This particularly pertains to discriminators of tracking radio range finders, which with sufficiently careful selection of their parameters, turn out to be close to optimal in their characteristics. Analysis in this chapter allowed us to find the influence on characteristics of a discriminator and on accuracy of range finding in general of parameters of modulation of the sounding signal, power features of the received signal and imperfections of its generation in the circuit of the radio range finder. Not dwelling on tiresome, detailed enumeration of the results of this chapter, we note only those directions in which there remain definite gaps and where there is required additional investigation.

Questions of the analysis of discriminators can be considered sufficiently exhaustively investigated. Here we should only consider in more detail the influence of the AGC system on discriminator characteristics for comparatively low signal-to-noise ratios, when the cross interaction of the signal with noise in the AGC system is substantial, and there occurs modulation of the signal by noises passing through the feedback circuit. Another question is more detailed consideration of the case of slow fluctuations, when during the time of processing of the signal in the

discriminator signal amplitude remains constant.

The case of fluctuations which are slow or comparable in speed with the rate of variation of parameters, in general, is of great interest from the point of view of synthesis of measuring systems and, in particular, of ranging meters. Very interesting is allowance for the non-Gaussian character of signal fluctuations, which in a number of cases takes place.

The next important question requiring investigation is consideration of the range finders in § 7.2 of the type using for measurement a unit of detection channels. Here, it would be useful to study the influence of overlapping of channels and nonorthogonality of their output signals, to calculate characteristics of such a meter allowing for disharmony of reference signals in the unit of detection channels with the sounding signal and change of the measured distance.

Many problems of concrete content still remain in the question of analysis of a range finder as a whole. It would be interesting to consider examples of smoothing circuits of more complex structure, especially optimum smoothing circuits, corresponding to a quasi-regular law of change of the measured distance. Here, the most important question is a sufficiently deep explanation of the influence of errors in the statistics used during synthesis as compared the real statistics. Especially interesting is investigation of cases when such errors have a qualitative character; however, interesting, too, would be more detailed investigation of the influence of quantitative errors. An important question is consideration of range finders with smoothing circuits in the form of filters with a finite memory. The requirement of finiteness of memory can become essential in optimum smoothing circuits with variable parameters when their structure is very complicated, and it causes great difficulties in technical realization.

Above we already discussed the importance of investigation of operating conditions of the tracking range finder, when in its selection zone there are several targets or a target and delayed signals of return interference. At present this, in point of fact, is an open domain. At the same time investigation of the process of resolution of targets by a tracking range finder permitted us to establish new laws, to find requirements on the sounding signal and on methods of construction of range finders.

And, finally, problems connected with the investigation of nonlinear conditions of tracking range finders also are very far from complete solution. In this area we need more precise definition both of already obtained results with consideration of a large number of quantitative examples, and also solution of new problems. It

would be very interesting to obtain other characteristics of nonstationary operating conditions, and also definitized characteristics of steady-state operating conditions with allowance for limiting. It is important to consider nonlinear phenomena in systems with smoothing circuits of higher order and with smoothing circuits in the form of filters with variable parameters.

All these questions are not, however, decisive, and the level of understanding of problems pertaining to methods of construction of range finders and characteristics of accuracy of their work already attained permits us to competently approach their practical development. Available knowledge gives development engineers the possibility in a well-founded and reasonable manner of selecting the structure both of the discriminator, and of smoothing circuits of a range finder, of correctly selecting their parameters and of fully intelligently making various simplifications and technical compromises, having estimated their influence beforehand.

CHAPTER VIII

RANGE FINDING WITH AN INCOHERENT SIGNAL

§ 8.1. Introduction

An incoherent pulse signal at present is widely used in various radars. Along with normal pulse modulation there are used signals of more complicated form with additional intrapulse modulation of various forms [35-39]. Properties of an incoherent signal are described in Chapter I. Theoretical synthesis of range discriminators for the case of an incoherent fluctuating signal is hampered by the fact that for such a signal the general form of the likelihood functional is unknown. Therefore, during solution of the problem of synthesis it is necessary to limit ourselves only to certain particular cases and to make definite assumptions, sometimes not covering all cases in practice. In contrast to this the problem of analysis of existing discriminators can be solved with the same degree of correctness as in the case of a coherent signal. This to an identical degree pertains both to the work of the range finder with only its own noises, and also to cases of the influence of various kinds of interferences.

As for problems of analysis of a range finder as a whole and synthesis of smoothing circuits, it is obvious that the form of the signal introduces nothing new. Actually, as soon as we have determined characteristics of the discriminator and we have assigned statistics of the measured distance, it already makes absolutely no difference to what these characteristics correspond. Therefore, after we have obtained a solution of the corresponding problems for coherent range finders, their solution for incoherent range finders reduces to trivial replacement of characteristics of the one discriminator by characteristics of the other. In connection with this the present chapter to a still lesser degree than the preceding one will deal

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with questions of analysis of range finders as a whole. Basic attention will be allotted namely to discriminators of incoherent range finders.

The present chapter is organized on the same plan as Chapter VII. First, we consider possibilities and certain particular results of synthesis of an optimum discriminator. Then in general form we investigate basic methods of creating a discrimination characteristic taking into account a series of inevitable technical deviations from optimality. Then, we consider different concrete forms of modulation, give examples of analysis of range finders as a whole, study the influence of interferences on incoherent distances and touch on nonlinear phenomena in such range finders. As will be clear subsequently, many results will permit broad analogies with, and will sometimes even coincide with the coherent case.

§ 8.2. Optimum Discriminator

To find operations of an optimum discriminator it is necessary to have the logarithm of the likelihood functional for an incoherent signal. In § 5.2 it is shown that it can be presented in the form of a certain function [see (5.2.3)] of the voltage envelope, corresponding to the value of the correlation integral formed in the j-th period of repetition of the signal, i.e.,

$$P(y/\tau)=C\int_0^{2\pi}\cdots\int_0^{2\pi}\exp\left\{\sum_{j,k}v_{jk}|f_j||f_k|e^{i(\alpha_j-\alpha_k)}\right\}\times p(\alpha_1,\ldots,\alpha_n)\,d\alpha_1,\ldots,d\alpha_n,$$

where α_k – value of the random phase in the k-th period;

$p(\alpha_1, \alpha_2, \ldots, \alpha_n)$ – distribution of probabilities for phases;

v_{jk} – elements of a matrix equal to the difference of matrices which are reciprocal to correlation matrices for noise and for a signal mixed with noise in the absence of modulation;

$$|f_j|=\left|\int_{(j-1)T_r}^{jT_r}y(s)\,u(s-\tau)\,e^{i\omega_0 s}\,ds\right|. \tag{8.2.1}$$

Here all designations are the same as in the preceding chapter.

Quantity $|f_j| = |f(jT_r)|$ obviously coincides with the value of envelope $|Q(t, \tau)|$ from (7.2.4), if pulse response $h(t - s)$ from (7.2.4) is

$$\begin{aligned}h(t-s)&=1 \text{ when } (j-1)T_r<t-s<jT_r,\\ h(t-s)&=0 \text{ when } t-s<(j-1)T_r,\ t-s>jT_r.\end{aligned} \tag{8.2.2}$$

This definition of pulse response $h(t - s)$ corresponds to the fact that due to properties of the incoherent signal its coherent processing is possible only in an interval of time equal to one period, and further accumulation of the signal can be carried out only incoherently. Pulse response (8.2.2) defines an integrator over

period T_r with clearing [?].

The form of function (5.2.3), determining the logarithm of the likelihood functional, can be made concrete only with additional assumptions about the character of fluctuations of the signal and the magnitude of the signal-to-noise ratio.

In § 5.2 it is shown that with certain conditions the logarithm of the functional of probability density reduces to the sum of the squares of envelopes $|f_j|$, taken in all periods corresponding to the time of observation. This takes place:

a) for small signal-to-noise ratios and arbitrary speed of fluctuations;

b) for fluctuations, fast as compared to period T_r, and an arbitrary signal-to-noise ratio;

c) for large signal-to-noise ratios and arbitrary speed of fluctuations, if the spectral density of fluctuations of the signal differs from square.

In general the sum of squares is replaced by a certain inertial nonlinear processing which considers the interperiod correlation of values of the envelope. Accumulated experience in designing systems based on simple summation of the squares of the envelope permits us to state that consideration of complicated systems considering interperiod correlation will lead only to slight improvement of their characteristics. This statement is based on the fact that with a sufficiently large signal-to-noise ratio incoherent range finders built without taking into account interperiod correlation ensure, as will be shown later, practically the same accuracy as coherent ones, and with a small signal-to-noise ratio, when impairment of accuracy in the incoherent case is considerable, simple processing is optimum.

From this point of view, complication of processing of the received signal, ensuing from calculation of interperiod correlation, is hardly desirable, since it increases technical complexity of realizing range finders without leading to essential improvement of accuracy. Therefore, subsequently, we shall consider that the logarithm of the likelihood functional is presented in the form of the sum of squares of envelope $|f_j|$, i.e.,

$$P(y/\tau)=C\exp\left\{K\sum_{j=1}^{n}|f_j|^2\right\}=C\exp\left\{K\sum_{j=1}^{n}\left|\int\limits_{(j-1)T_r}^{jT_r}y(s)\,u(s-\tau)\,e^{i\omega_0 s}ds\right|^2\right\}, \qquad (8.2.3)$$

where C – factor, not depending on delay τ;

$n = [T/T_r]$ – number of periods of the signal accumulated during time T;

K – a certain coefficient, depending on the signal-to-noise ratio.

Such a presentation is sufficiently accurate in the above-considered cases, and, of course, for $n = 1$, i.e., for the case of a single sending of the signal. In

accordance with expression (8.2.3) we shall consider only such a construction of discriminators which does not take into account interperiod correlation of the envelope. Namely thus are existing incoherent range discriminators built — processing of the signal for one period is carried out in them coherently, and inter-period accumulation is carried only with respect to the envelope without taking into account correlation of its values in various periods.

8.2.1. Operations of an Optimum Discriminator

In accordance with results of Chapter VI, output of the discriminator in this case is defined as

$$z(t)=K\frac{\partial}{\partial\tau}|f_j|^2=K\frac{\partial}{\partial\tau}\left|\int\limits_{(j-1)T_r}^{jT_r} y(s)\,u(s-\tau)\,e^{i\omega_0 s}\,ds\right|^2. \qquad (8.2.4)$$

It is obvious that z(t) — discrete random process, taking values $z_j = z(jT_r)$ in the j-th period. The concrete form of this process depends on the method of formation of correlation integral f_j, determining optimum processing of the signal in one period.

As we already noted above, this integral can be formed by two methods — the correlation method and by an optimum filter with frequency response matched with the spectrum of modulation. In the first case there is produced multiplication of the received signal by the sounding signal, shifted in frequency and in delay, and integration over the period with the help of filter (8.2.2), tuned to intermediate frequency. Actually, of course, integration can be carried out by any filter with a passband considerably larger than the frequency of repetition of the signal, and considerably smaller than the width of the spectrum of modulation. In this case the signal at the output of an optimum detection channel is a periodic sequence of squares of the envelopes of responses of this filter in every period under the influence at the input of the filter of a signal consisting of the product

$$y(t)\,u(t-\tau)\,e^{i(\omega_0+\omega_{np})t}.$$

During formation of z(t) in the range discriminator, the square of the envelope is replaced by the product

$$z(t)=-K2\operatorname{Re}\int\limits_{(j-1)T_r}^{jT_r} y(s)\,u(s-\tau)\,e^{i\omega_0 s}\,ds\times\int\limits_{(j-1)T_r}^{jT_r} y(s)\,u^{*\prime}(s-\tau)\,e^{-i\omega_0 s}\,ds, \qquad (8.2.5)$$

which corresponds to multiplication of the received signal in one channel by $u(t-\tau)$, and in the other by $u^{*\prime}(t-\tau)$ and subsequent multiplication in the phase detector of output voltages of these channels.

Thus, in this respect there is complete analogy with the coherent case. The whole difference reduces to difference of pulse responses of filters (8.2.2) and (7.2.5). The actual output voltage of the discriminator $z(t)$, as a function of time, can, as before, be presented by formula (7.2.7), where $h(t)$ — pulse response of any filter whose inertia is low as compared to the duration of the period, so that the response of this filter to an incoming pulse attenuates by the moment of appearance of the next pulse.

With the correlation method of processing creation of a discrimination characteristic is possible only in circuits preceding detection. Here we can use both an exact method — multiplication of the received signal in two channels by a delayed sounding signal and its derivative — and also approximate methods — either simultaneous, or alternate multiplication of the received signal by two reference signals detuned with respect to delay.

With the second method of processing a pulse signal the received signal is passed through a filter with pulse response $h(t) = u(-t)$ (such a filter is frequently called optimum for the given form of modulation, or "shortening," or matched with the spectrum of modulation). It is obvious here that the value of the output voltage envelope of this filter at time $t = (j - 1)T_r + \tau$ gives the needed value of $|f_j|$. These values should be selected by gating the envelope by a narrow gate pulse and then should be accumulated discretely.

Thus, in this case output voltage of the discriminator is a discrete random process consisting in the optimum case of infinitely narrow pulses appearing at moments $t_j = (j - 1)T_r + \tau$. Operations of the optimum discriminator in this case consist in transmission of the received signal through filters with pulse responses $u(-t)$ and $u^{*\prime}(-t)$, subsequent phase detection of output voltages of these filters and gating of the output voltage at moments $t_j = (j - 1)T_r + \tau$, where τ — output value of delay. Approximate formation of optimum operations in this case can be carried out in post-detector circuits, on the envelope, since the envelope of output voltage of a "shortening" filter gives the possibility of forming the likelihood functional for all values of τ. Creation of the discrimination characteristic here is carried out by two delay-detuned gates, tuned in accordance with the output value of delay. Gates can be fed simultaneously or alternately with a certain period of repetition. In practice, thanks to the great technical simplicity in real discriminators differentiation of the logarithm of the likelihood functional is carried out with respect to the envelope (see Chapter IV and § 7.6).

With optimum fulfillment of all operations in (8.2.4) and (8.2.5) both methods of construction of the discriminator are absolutely identical. This follows directly from (8.2.5), since both methods are the realization of one and the same mathematical expression. In real conditions, when the required operations are executed imperfectly, these methods can lead to somewhat differing results due to the varying influence of any kind of deviations from optimality. Therefore, subsequently, when necessary, we shall conduct parallel consideration of them.

8.2.2. Characteristics of an Optimum Discriminator

For calculation of characteristics of a discriminator, we record the input signal in the form

$$y(t)=\sqrt{P_c}\,\mathrm{Re}\sum_k E_k u_0(t-\tau_0-kT_r)\,e^{i\alpha_k+i\omega_0 t}+n(t), \tag{8.2.6}$$

where $E_k = E(kT_r)$ — the envelope of the reflected signal in the k-th period ($\overline{E_k^2} = 2$);

α_k — independent random phases, evenly distributed in the interval (0 to 2π);

τ_0 — true value of delay.

Function $u_0(t)$ is no longer periodic, but describes a single pulse and is normalized as before by unit mean power.

In all expressions describing real discriminators there is product $y(t)e^{i\omega_0 t}$; therefore, we introduce complex signal

$$\eta(t)=y(t)\,e^{i\omega_0 t} \tag{8.2.7}$$

and seek necessary statistical characteristics namely for it. Signal $\eta(t)$ according to (8.2.6) contains a low-frequency component and a component of frequency $2\omega_0$. With any fulfillment of the discriminator the last component does not pass to its output; therefore, in expressions for statistical characteristics of $\eta(t)$ we retain only low-frequency components, considering that integrals containing high-frequency components of the correlation function of signal $\eta(t)$ all the same turn into zero.

Let us find the low-frequency component of the correlation function. Dropping high-frequency components, we obtain

$$\overline{[\eta(t_1)\eta^*(t_2)]_{\text{l.f.}}}=\frac{P_c}{4}\sum_{j,k} u^*_{0j}(t_1)u_{0k}(t_2)\,\overline{E_j E_k}\,e^{-i(\alpha_j-\alpha_k)}+$$

$$+\overline{n(t_1)n(t_2)}\,e^{i\omega_0(t_1-t_2)}=\frac{P_c}{2}\sum_k u^*_{0k}(t_1)\,u_{0k}(t_2)+N_0\delta(t_1-t_2), \tag{8.2.8}$$

where $u_{0k}(t) = u_0(t - \tau_0 - kT_r)$.

Averaging in (8.2.8) for amplitudes and phases is conducted separately due to their independence. Furthermore, by virtue of the independence and uniformity of the distribution of phases α_k in different periods all members in the double sum turn into zero when $j \neq k$.

For calculation of the fluctuation characteristic and the equivalent spectral density we will need a fourth mixed moment for values of $\eta(t)$. As, too, during calculation of the correlation function, it is possible to show that

$$\overline{[\eta(t_1)\eta^*(t_2)\eta(t_3)\eta^*(t_4)} - \overline{\eta(t_1)\eta^*(t_2)}\,\overline{\eta(t_3)\eta^*(t_4)}]_{a\eta} = \frac{P_c^2}{4}\sum_{k,j}\rho_{k,j}^2 u_{0k}^*(t_1)u_{0k}(t_2)u_{0j}^*(t_3)u_{0j}(t_4) +$$
$$+\frac{P_cN_0}{2}\sum_k u_{0k}^*(t_1)u_{0k}(t_4)\delta(t_2-t_3) +$$
$$+\frac{P_cN_0}{2}\sum_k u_{0k}^*(t_3)u_{0k}(t_2)\delta(t_1-t_4) + N_0^2\,\delta(t_1-t_4)\,\delta(t_2-t_3). \tag{8.2.9}$$

When deriving formula (8.2.9), we consider that we are interested in integrals of this expression over all t_i, where integration over t_1 and t_2, and correspondingly over t_3 and t_4, will always be produced over the same period.

Let us find the expression for the discrimination characteristic. We note, first of all, that since in formulas (8.2.1), (8.2.3)-(8.2.5) integration is produced only over one period, in these formulas there actually enter values of the modulating function $u(t - \tau_0)$ and only in this period. Therefore,

$$\int_{(j-1)T_r}^{jT_r} u(s-\tau)y(s)e^{i\omega_0 s}\,ds = \int_{(j-1)T_r}^{jT_r} u_0(s-\tau-(j-1)T_r)y(s)e^{i\omega_0 s}ds. \tag{8.2.10}$$

Using the given circumstance, dropping immaterial numerical factor 2K in formula (8.2.5), replacing variable s by $s + \tau$ and introducing former designation $\Delta = \tau_0 - \tau$, we obtain

$$z_k(\Delta) = \overline{z(kT_r,\ \Delta)} = -\operatorname{Re}\int_{(k-1)T_r}^{kT_r}\int_{(k-1)T_r}^{kT_r} u_0(s_1-(k-1)T_r)u_0^{*\prime}(s_2-(k-1)T_r)\overline{\eta(s_1)\eta^*(s_2)}\,ds_1ds_2 =$$
$$= -\operatorname{Re}\Big\{\sum_j\frac{P_c}{2}\int_{(k-1)T_r}^{kT_r} u_0(s_1-(k-1)T_r)u_0^*(s_1-\Delta-$$
$$-jT_r)\,ds_1\int_{(k-1)T_r}^{kT_r} u_0^{*\prime}(s_2-(k-1)T_r)u_0(s_2-\Delta-jT_r)\,ds_2 +$$
$$+N_0\int_{(k-1)T_r}^{kT_r} u_0(s_1-(k-1)T_r)u_0^{*\prime}(s_1-(k-1)T_r)\,ds_1\Big\} =$$
$$= -\operatorname{Re}\left\{\frac{P_cT_r^2}{2}C(\Delta)C^{*\prime}(\Delta) + N_0T_rC^{*\prime}(0)\right\} = \frac{P_cT_r^2}{2}\operatorname{Re}C(\Delta)C^{*\prime}(\Delta), \tag{8.2.11}$$

where $C(\Delta)$ — autocorrelation function of one period of modulation, defined in Chapters I and VII [see, for instance, (7.2.9)].

Replacement of variables of integration in (8.2.11) without change of the limits of the integrals is permissible due to the periodicity of the integrands.

Comparison of expression (8.2.11) with (7.2.10) shows that the discrimination characteristic in this case coincides with the discrimination characteristic of an optimum coherent discriminator with the accuracy of a factor. Therefore, its dependence on the form of modulation will have the same character as before. The gain factor of the discriminator will be determined by expression

$$K_{\text{д}}=\frac{\partial z_k(\Delta)}{\partial\Delta}\bigg|_{\Delta=0}=-\frac{P_cT_r^2}{2}\,\mathrm{Re}\,[C(0)\,C^{*''}(0)+|C'(0)|^2]=\frac{P_cT_r^2}{2}(b-a^2), \tag{8.2.12}$$

coinciding with the accuracy of a constant factor with the expression for the gain factor in the coherent case.

Let us find now the expression for the fluctuation characteristic. Just as when calculating $\overline{z_k(\Delta)}$, we can obtain

$$R_{ik}=\overline{z_i(\Delta)\,z_k(\Delta)}-\overline{z_k(\Delta)}\,\overline{z_i(\Delta)}=$$
$$=\frac{N_0^2T_r^2}{2}\,\mathrm{Re}\,\{q^2\rho_{ik}^2\,[|C(\Delta)|^2\,|C'(\Delta)|^2+C^{*2}(\Delta)\,C'^2(\Delta)]+$$
$$+q\delta_{ik}\,[b\,|C(\Delta)|^2+|C'(\Delta)|^2+2iaC^*(\Delta)\,C'(\Delta)]+(b-a^2)\,\delta_{ik}\}, \tag{8.2.13}$$

where $\rho_{ik}=\rho((i-k)T_r)$, $\rho(t_1-t_2)$ — correlation function of fluctuations;

$q=P_c\,T_r/2N_0$ — the ratio of the energy of the received signal for a period to the one-sided spectral density of noise, already used earlier in Chapter V as the signal-to-noise ratio in the incoherent case;

δ_{ik} — Kronecker delta $\left(\delta_{ik}=\begin{cases}1 & i=k\\ 0 & i\neq k\end{cases}\right)$.

Expression (8.2.13) characterizes the fluctuation component at the discriminator output in the discrete consideration. In the majority of real cases inertia of the smoothing circuits is great as compared to the period of repetition of the signal. In these cases the discrete process at the discriminator output can be replaced by a continuous one. The rule of replacement follows from the presentation of the output quantity of the range finder. In the discrete case it is equal to

$$\sum_k g_{nk}z_k,$$

where $g_{nk}=g(nT_r,\,kT_r)$ — values of the pulse response of the smoothing filter at moments $t_n=nT_r$, $t_k=kT_r$.

If function $g(t, \tau)$ changes slowly as compared to T_r,

$$\sum_{k=n_0}^{n} g(nT_r, kT_r) z_k = \sum_{k=n_0}^{n} g(nT_r, kT_r) \frac{z_k}{T_r} \cdot T_r = \int_{t_0}^{t} g(t, \tau) z(\tau) d\tau,$$

where continuous process $z(t)$ is defined as

$$\int_{(k-1)T_r}^{kT_r} z(t) dt = z_k, \quad z(t_k) = \lim_{T_r \to 0} \frac{z_k}{T_r}, \quad t_k = kT_r.$$

In accordance with this, during continuous consideration the expression for the gain factor found above should be divided by T_r, and the one for the correlation function R_{ik}, by T_r^2, having defined the correlation function of the continuous process by relationship

$$R(t_i, t_k) = R(iT_r, kT_r) = \lim_{T_r \to 0} \frac{R_{ik}}{T_r^2}. \qquad (8.2.14)$$

Then on the basis of formulas (8.2.12)-(8.2.14) the correlation function of an equivalent fluctuating disturbance at the input of the range finder, determining equivalent spectral density, is equal to

$$\begin{aligned} R(t_1, t_2) = \lim_{T_r \to 0} \frac{R_{ik}}{K_A^2} = \frac{1}{2q^2 (b-a^2)^2} \operatorname{Re} \{q^2 \rho^2 (t_1 - t_2) \times \\ \times [|C(\Delta)|^2 |C'(\Delta)|^2 + C^{*2}(\Delta) C'^2(\Delta)] + T_r q \delta (t_1 - t_2) [b|C(\Delta)|^2 + \\ + |C'(\Delta)|^2 + 2iaC^*(\Delta) C'(\Delta)] + T_r (b - a^2) \delta (t_1 - t_2)\}, \end{aligned} \qquad (8.2.15)$$

where $t_1 = kT_r$, $t_2 = iT_r$,

$$\lim_{T_r \to 0} \frac{\delta_{ik}}{T_r} = \delta(t_1 - t_2).$$

Formula (8.2.15) determines the fluctuation characteristic of the discriminator:

$$\begin{aligned} S_{\text{экв}}(\Delta) = \int_{-\infty}^{\infty} R(t_1 - t_2) d(t_1 - t_2) = \frac{T_r}{2q^2 (b - a^2)^2} \Big\{ b - a^2 + q [b|C(\Delta)|^2 + |C'(\Delta)|^2 + \\ + 2iaC^*(\Delta) C'(\Delta)] + \frac{q^2}{2\pi T_r \Delta f_c^2} \int_{-\infty}^{\infty} S_0^2(\omega) d\omega [|C(\Delta)|^2 |C'(\Delta)|^2 + C^{*2}(\Delta) C'^2(\Delta)] \Big\}, \end{aligned} \qquad (8.2.16)$$

where as before $S_0(\omega) = \Delta f_c \int_{-\infty}^{\infty} \rho(\tau) e^{-i\omega\tau} d\tau$ – normalized spectral density of fluctuations of the reflected signal.

The dependence of functions $S_{\text{экв}}(\Delta)$ on characteristics of modulation completely coincides with the corresponding dependence for the coherent case (7.2.20), which already was investigated in Chapter VII. The dependence on the signal-to-noise ratio and characteristics of fluctuations of the signal is found differently. Let us

consider, first of all, the value of $S_{ЭКВ}(\Delta)$ for $\Delta = 0$, i.e., equivalent spectral density of the discriminator $S_{опт}$. From formula (8.2.16) it follows that

$$S_{опт} = \frac{T_r (1+q)}{2q^2 (b-a^2)}. \qquad (8.2.17)$$

This expression, obviously, coincides with expression (7.2.15) for the coherent case if it is considered that spectral density of fluctuations of the signal has the form of a square function with width $\Delta f_c = f_r = 1/T_r$. Such coincidence is completely natural, since with spectrum width $\Delta f_c = 1/T_r$ the coherent signal coincides with the incoherent one. The dependence of $S_{ЭКВ}$ on q was already investigated earlier in Chapter VII. It is obvious that the curve of Fig. 7.3 for a square spectrum of fluctuations describes simultaneously the dependence on q of quantity $S_{опт}$ from (8.2.17) with replacement of h by q. For large signal-to-noise ratios q, quantity $S_{опт}$ is equal to $N_0/P_c(b - a^2)$, which coincides with the limiting value of $S_{опт}$ for a coherent signal with large h. This means that with optimum construction and high signal levels coherent and incoherent range finders ensure identical accuracy.

For signal-to-noise ratios which are not very large incoherence of the signal leads to definite loss of accuracy, the magnitude of which for the case of a square spectrum of fluctuations of the signal can be determined by the following expression:

$$\frac{S_{опт\ неког}}{S_{опт\ ког}} = \frac{1+q}{\Delta f_c T_r + q} = \frac{h + \frac{1}{\Delta f_c T_r}}{h+1}, \qquad (8.2.18)$$

which is valid when $\Delta f_c T_r < 1$.

When $\Delta f_c T_r \geq 1$, this ratio turns into one. Formula (8.2.18) shows that for identical accuracy of coherent and incoherent range finders it is necessary that q is great as compared to unity, and h is great as compared with $1/\Delta f_c T_r$. The dependence of ratio (8.2.18) on h for various values $\Delta f_c T_r$ is shown in Fig. 8. . From this figure it follows that with high frequencies of repetition and values of the signal-to-noise ratio $h \sim 10$ to 100, the loss due to incoherence may be rather substantial. Presence of such loss is one more ground for selection in incoherent radars of a low frequency of repetition (see Chapter V).

Let us find now the spectral density of parametric fluctuations. From (8.2.16) it follows that

$$S_{пар} = \frac{1}{2} \frac{\partial^2 S_{экв}(\Delta)}{\partial \Delta^2}\bigg|_{\Delta=0} = \frac{1}{2\pi \Delta f_c^2} \int_{-\infty}^{\infty} S_0^2(\omega)\, d\omega = \frac{\alpha}{\Delta f_c}, \qquad (8.2.19)$$

i.e., it coincides with an accuracy of numerical factor α of the order of 0.5 to 1

with the magnitude of $1/\Delta f_c$ and thus, as in an optimum coherent range finder, does not depend on the form of modulation of the sounding signal. With a square spectrum of fluctuations $\alpha = 1$, and in the case of an exponential correlation function $\alpha = 0.5$.

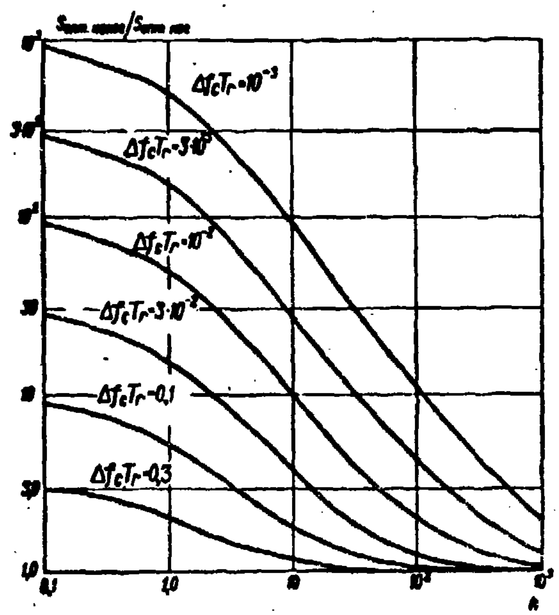

Fig. 8.1. Loss in accuracy of incoherent range finders.

Thus, the optimum incoherent discriminator both in the character of dependences of the discrimination and fluctuation characteristics on the form of modulation and the magnitude of mismatch, and also in the magnitude of equivalent spectral density with sufficiently large signal-to-noise ratios does not differ from a coherent one. Substantial difference in characteristics of accuracy can occur only for signal-to-noise ratios q which are small as compared to unity, where the loss in accuracy is approximately equal (for small $\Delta f_c T_r$) to quantity $(1 + q)/q$.

§ 8.3. Discriminators Using Correlation Processing of the Signal

Discriminators with correlation processing of the signal in methods of construction and calculation of characteristics are the closest to coherent discriminators. Considering analogies which exist here, we shall start our consideration with them. Approximate fulfillment of optimum operations during correlation processing can be realized by any of three methods: with the help of two delay-detuned channels, in the form of a single-channel discriminator with switching of reference signals, with the help of a two-channel discriminator using a reference signal.

8.3.1. Discriminator with Two Detuned Channels

The block diagram of a discriminator with two range-detuned channels is shown in Fig. 8.2. In its functional structure it coincides with the corresponding diagram of a coherent discriminator. The received signal is multiplied after transfer to intermediate frequency or before this transfer by two reference signals $u_1(t)$ and $u_2(t)$, which to some degree of accuracy repeat the law of modulation of the sounding signal and have a corresponding high-frequency filling. Signals $u_1(t)$ and $u_2(t)$ are delayed relative to the sounding signal by $\tau + \delta$ and $\tau - \delta$, respectively. After multiplication the signals are passed through filters with pulse response $h(t) \cos \omega_{пр} t$, are detected and subtracted, forming output voltage of the discriminator. The difference between the incoherent and the coherent cases is only the inertia of the filters, which in the incoherent case should be small as compared to the period of repetition of signal, in such a manner that responses of the filter to adjacent pulses do not overlap. In the optimum case this filter should represent an integrator for a period with clearing; however, as we shall prove subsequently, final results do not depend on the form of the characteristic of the filter if only it possesses integrating properties during a time equal to the pulse duration of the signal. Let us assume, first, that this condition is satisfied, i.e., that $\Delta f_{\Phi} \tau_{И} \ll 1$, where $\tau_{И}$ — pulse duration of the signal, and Δf_{Φ} — bandwidth of the filter. Then we shall consider the case when this condition is not satisfied.

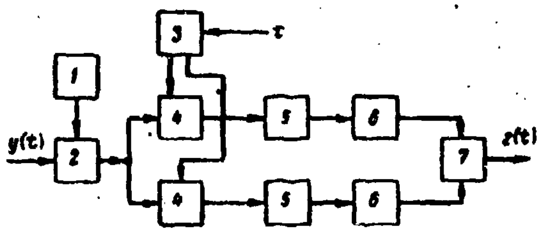

Fig. 8.2. Block diagram of a correlation discriminator with two delay-detuned channels: 1 — local oscillator; 2 — mixer; 3 — controlled generator of reference signals; 4 — multiplier; 5 — intermediate-frequency filter; 6 — detector; 7 — subtractor.

Output voltage of the discriminator, analogously to § 7.3, is presented in the form

$$z(t, \Delta) = \left| \int_{-\infty}^{t} h(t-s) u_1(s-\tau-\delta) y(s) e^{i\omega_0 s} ds \right|^2 - \left| \int_{-\infty}^{t} h(t-s) u_2(s-\tau+\delta) y(s) e^{i\omega_0 s} ds \right|^2, \quad (8.3.1)$$

where it is assumed that pulse responses of filters in both channels are identical. Due to nonstationariness of the input signal and small (as compared to the period) inertia of the filter $z(t, \Delta)$ is a nonstationary random process with periodic nonstationariness. Therefore, the discrimination and fluctuation characteristics are determined by expressions

$$\overset{\approx}{\overline{z(t, \Delta)}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \overline{z(t, \Delta)}\, dt, \tag{8.3.2}$$

$$S_{экв}(\Delta) = \frac{1}{K_{д}^2} \int_{-\infty}^{\infty} ds \lim_{T\to\infty} \frac{1}{T} \int_0^T \overline{[z(t, \Delta) z(t+s, \Delta)} - \\ - \overline{z(t, \Delta)}\ \overline{z(t+s, \Delta)}]\, dt, \tag{8.3.3}$$

where

$$K_{д} = \left.\frac{\partial \overset{\approx}{\overline{z(t, \Delta)}}}{\partial \Delta}\right|_{\Delta=0} \tag{8.3.4}$$

is the gain factor of the discriminator (see § 7.4).

Expanding expression (8.3.1) and substituting in it the expression for the correlation function of signal $\eta(t) = y(t)e^{i\omega_0 t}$ from (8.2.8), we obtain

$$\overline{z(t, \Delta)} = \int_{-\infty}^{t}\int_{-\infty}^{t} h(t-s_1)\, h(t-s_2)\, [u_1(s_1-\tau-\delta)\, u^*_1(s_2-\tau-\delta) - \\ - u_2(s_1-\tau+\delta)\, u^*_2(s_2-\tau+\delta)] \Big[\frac{P_c}{2} \sum_k u^*_0(s_1-\tau_0- \\ - kT_r)\, u_0(s_2-\tau_0-kT_r) + N_0\delta(s_1-s_2)\Big]\, ds_1\, ds_2 = \\ = \sum_k T_r h^2(t-kT_r) \Big\{\frac{P_c T_r}{2} [|C_{10}(\Delta-\delta)|^2 - \\ - |C_{20}(\Delta+\delta)|^2] + N_0(C_{11}(0) - C_{22}(0))\Big\}, \tag{8.3.5}$$

where functions $C_{ik}(x)$ are determined by relationship

$$\left.\begin{aligned} C_{i0}(x) &= \frac{1}{T_r}\int_0^{T_r} u_i(s+x)\, u^*_0(s)\, ds, \\ C_{ik}(x) &= \frac{1}{T_r}\int_0^{T_r} u_i(s+x)\, u^*_k(s)\, ds, \end{aligned}\right\} \tag{8.3.6}$$

i.e., are crosscorrelation and autocorrelation functions for the totality set of sounding and reference signals.

Thus, as in the preceding chapter, we shall be interested only in values of these functions within the limits of one period, since mismatch exceeding the period of repetition of the signal is not of practical interest.

Averaging (8.3.5) in time, for the discrimination characteristic we obtain the following expression:

$$\overset{\approx}{\overline{z(t, \Delta)}} = \{q\, [|C_{10}(\Delta-\delta)|^2 - |C_{20}(\Delta+\delta)|^2] + C_{11}(0) - \\ - C_{22}(0)\}\, N_0 \int_0^{\infty} h^2(t)\, dt = \{q\, [|C_{10}(\Delta-\delta)|^2 - \\ - |C_{20}(\Delta+\delta)|^2] + C_{11}(0) - C_{22}(0)\}\, N_0 \Delta f_{\Phi}. \tag{8.3.7}$$

where

$$\Delta f_{\phi}=\int_{0}^{\infty} h^2(t)\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(i\omega)|^2\,d\omega. \qquad (8.3.8)$$

and we assume that $H(i0) = 1$.

From comparison with (7.3.26) it is clear that according to the character of the dependence of the discrimination characteristic on the laws of modulation of the sounding and reference signals the considered discriminator is completely equivalent to the corresponding coherent discriminator. In particular, slope and null shift of the discrimination characteristic are equal

$$K_{\text{д}}=2qN_0\Delta f_{\phi}\,\mathrm{Re}\,[C'_{10}(-\delta)\,C^*_{10}(-\delta)-C'_{20}(\delta)\,C^*_{20}(\delta)], \qquad (8.3.9)$$

$$\Lambda_0=\frac{\overline{\overline{z(t,0)}}}{K_{\text{д}}}=\frac{C_{11}(0)-C_{22}(0)+q\,[|C_{10}(-\delta)|^2-|C_{20}(\delta)|^2]}{q\,\mathrm{Re}\,[C'_{10}(-\delta)\,C^*_{10}(-\delta)-C'_{20}(\delta)\,C^*_{20}(\delta)]}, \qquad (8.3.10)$$

which differs from (7.3.28) and (7.3.29) only by numerical factors. Null shift is absent for all q if

$$\left.\begin{aligned}&C_{11}(0)=C_{22}(0),\\ &|C_{10}(-\delta)|^2=|C_{20}(\delta)|^2,\end{aligned}\right\} \qquad (8.3.11)$$

i.e., if conditions of symmetry of reference signals relative to sounding signals and of equality of gains in both channels are satisfied. The dependence of the gain factor on detuning between the reference signals is precisely the same as in the coherent discriminator.

Let us find now the equivalent spectral density. We assume that conditions (8.3.11) are satisfied. Here, as in the coherent discriminator, the component of $S_{\text{экв}}$ which does not depend on the signal-to-noise ratio will turn into zero, and in the expression for $S_{\text{экв}}$ there will remain only terms with $1/q$ and $1/q^2$. Calculating correlation function $z(t, 0)$ with the help of (8.3.5), executing transformations similar to those which were used in deriving (7.3.5), averaging the obtained expression in time and integrating in accordance with (8.3.3), we can obtain the following expression:

$$\begin{aligned}S_{\text{экв}}=T_r\,\{2\,|C_{11}(0)|^2-|C_{12}(2\delta)|^2-|C_{21}(-2\delta)|^2+\\+2q\,[\,2C_{11}(0)|C_{10}(-\delta)|^2-\\-\mathrm{Re}\,C_{12}(-2\delta)\,C^*_{10}(-\delta)\,C^*_{20}(-\delta)-\\-\mathrm{Re}\,C_{21}(2\delta)\,C^*_{20}(\delta)\,C^*_{10}(\delta)]\}:\{4q\,\mathrm{Re}\,[C'_{10}(-\delta)\,C^*_{10}(-\delta)]\}^2.\end{aligned} \qquad (8.3.12)$$

From comparison of this expression with the corresponding expression for the coherent discriminator (7.3.30, it follows that expression (8.3.12) differs from it only by coefficients which depend on the signal-to-noise ratio. The structure of

these coefficients in this case is still simpler — they depend neither on the form of spectral density of fluctuations of the signal nor on the frequency response of the filter. For small detunings δ expression (8.3.12) approaches $S_{\text{опт}}$ from (8.2.17), and with arbitrary detuning the dependence on its magnitude and the form of the sounding and reference signals can be described just as in § 7.3 by introduction of the equivalent signal-to-noise ratio $q_{\text{экв}}$ and the equivalent width of the spectrum of modulation $b_{\text{экв}}$. Formula (8.3.12) here takes the form

$$S_{\text{экв}}=\frac{T_{\text{р}}(1+q_{\text{экв}})}{2q^2_{\text{экв}}b_{\text{экв}}}, \tag{8.3.13}$$

where

$$\frac{q_{\text{экв}}}{q}=2\,\frac{2C_{11}(0)\,|C_{10}(\delta)|^2-\operatorname{Re}C_{12}(-2\delta)\,C^*_{10}(-\delta)\,C^*_{02}(-\delta)-\operatorname{Re}C_{21}(2\delta)\,C^*_{20}(\delta)\,C^*_{10}(\delta)}{2C^2_{11}(0)-|C_{12}(2\delta)|^2-|C_{21}(2\delta)|^2}; \tag{8.3.14}$$

$$\frac{b_{\text{экв}}}{b}=\frac{2[\operatorname{Re}C'_{10}(-\delta)\,C^*_{10}(-\delta)]^2\,[2C^2_{11}(0)-|C_{12}(2\delta)|^2-|C_{21}(2\delta)|^2]}{b\,[2C_{11}(0)\,|C_{10}(-\delta)|^2-\operatorname{Re}C_{12}(-2\delta)\,C^*_{10}(-\delta)\,C^*_{02}(-\delta)-\operatorname{Re}C_{21}(2\delta)\,C^*_{10}(\delta)\,C^*_{20}(\delta)]^2} \tag{8.3.15}$$

coincide with the corresponding ratios $h_{\text{экв}}/h$ and $b_{\text{экв}}/b$ from § 7.3, which already were investigated in the preceding chapter. These relationships permit us with the help of graphs for $h_{\text{экв}}/h$ and $b_{\text{экв}}/b$ for various concrete forms of modulation from Chapter VII and formula (8.3.13) for Fig. 8.1 to calculate the magnitude of $S_{\text{экв}}$ for any case interesting us.

8.3.2. Discriminator with Switching of the Reference Signals

The block diagram of an incoherent discriminator with switching of reference signals differs from the corresponding coherent diagram (Fig. 7.7) only in the form of the filter h(t). In all other respects, as in the preceding case, all operations and the expression for the output voltage of the discriminator coincide with those given in § 7.4. Due to the great (as compared to $\Delta f_{\text{с}}$) bandwidth of the filter in the incoherent case there is possible selection of a frequency of switching of reference signals, rather high as compared to $\Delta f_{\text{с}}$. As shown in § 7.4, such selection is very desirable from the point of view of decreasing the component of equivalent spectral density which does not depend on the signal-to-noise ratio but is caused by incomplete correlation of the useful signal in adjacent half-periods of the frequency of switching. This component disappears only when the frequency

of switching is great as compared to Δf_c (see 7.4.7).

. In coherent discriminators possibilities of selection of the frequency of switching are limited by the passband of the filter, matched with the spectrum of fluctuations of the signal; rapid switching and elimination of the corresponding component of $S_{\text{ЭКВ}}$ are possible only with a wide filter passband Δf_{Φ}, so that the frequency of switching f_{Π} should satisfy condition $\Delta f_{\Phi} \gg f_{\Pi} \gg \Delta f_c$.

During coherent processing this condition can be satisfied only in rare cases, while in incoherent discriminators, due to the great bandwidth of the filter ($\Delta f_{\Phi} \geq 1/T_r$), fulfillment of this condition does not present any difficulty, and the frequency of switching can be only half the frequency of repetition of the signal. We assume that conditions on the speed of switching, leading to elimination of the component of equivalent spectral density which does not depend on the signal-to-noise ratio, are satisfied.

The gain factor of the discriminator with switching of reference signals and its discrimination characteristic, as also in the coherent case, coincide with an accuracy of coefficient 1/2 with the corresponding characteristics of a two-channel discriminator when $C_{10}(\delta) = C_{20}(\delta)$. This equality corresponds to the natural assumption of coincidence in form of the reference signals, alternately utilized as the "leading" and "lagging" signals. The condition of absence of systematic error here is the equality $|C_{10}(\delta)|^2 = |C_{10}(-\delta)|^2$. Then, the gain factor will be proportional to $\operatorname{Re}(C'_{10}(\delta)\, C^*_{10}(\delta))$.

Just as in Paragraph 8.3.1, we can obtain the following expression for equivalent spectral density:

$$S_{\text{ЭКВ}} = \frac{T_r}{4q^2} \cdot \frac{1 + 2q\,|C_{10}(\delta)|^2}{[\operatorname{Re} C'_{10}(\delta)\, C^*_{10}(\delta)]^2}. \qquad (8.3.16)$$

In particular, with coincidence of modulations of the sounding and the reference signals

$$S_{\text{ЭКВ}} = \frac{T_r}{4q^2} \cdot \frac{1 + 2q\,|C(\delta)|^2}{[\operatorname{Re} C'(\delta)\, C^*(\delta)]^2}. \qquad (8.3.17)$$

This quantity, depending upon the selection of δ, may differ substantially from $S_{\text{ОПТ}}$, where there exists a certain optimum value of detuning ensuring minimum $S_{\text{ЭКВ}}$. The physical reasons for such a dependence of $S_{\text{ЭКВ}}$ on detuning are the same as in the case of a coherent discriminator (§ 7.4); however, the actual optimum value $S_{\text{ОПТ}}$ is found somewhat differently. This quantity is determined from equation

$$q\left[\frac{\partial}{\partial\delta}|C(\delta)|^2\right]^2 - (1 + 2q\,|C(\delta)|^2)\,\frac{\partial^2}{\partial\delta^2}|C(\delta)|^2 = 0$$

and depends on signal-to-noise ratio q. Detailed investigation of the dependence of $S_{экв}$ on detuning δ and comparison of the considered discriminator with a two-channel one will be conducted later in examining concrete forms of modulation.

8.3.3. Discriminator with Differentiation of the Reference Signal

Such a discriminator in its principles of construction is the closest to the optimum one. Its block diagram coincides with the block diagram of the corresponding coherent discriminator (Fig. 7.2). The difference, as also in the preceding cases, is only in the bandwidth of the filter. Merits and deficiencies of such a method of realizing optimum operations already were discussed in Chapter VII. Incoherence of the signal in this respect introduces nothing new.

Designating reference signals in the two channels of discriminator $u_1(t)$ and $u_2(t)$ and assuming that the frequency responses of filters in both channels are identical, for output voltage of the discriminator we obtain an expression coinciding with (7.5.1). From the latter, by transformations of § 8.2, Paragraphs 8.3.1 and 8.3.2, and averaging in time we can find the following expressions for the gain factor and equivalent spectral density:

$$K_{д} = N_0 \Delta f_4 q \,\mathrm{Re}\,[C'_{10}(0)\,C^*_{20}(0) + C^*_{10}(0)\,C'_{20}(0)], \tag{8.3.18}$$

$$S_{экв} = T_r \frac{C_{11}(0)C_{22}(0) + \mathrm{Re}\,C_{12}^2(0) + q\,[C_{11}(0)\,|C_{02}(0)|^2 + C_{22}(0)\,|C_{10}(0)|^2 + 2\,\mathrm{Re}\,C_{12}(0)\,C_{02}(0)\,C^*_{01}(0)]}{2q^2\,[\mathrm{Re}\,(C'_{10}(0)\,C^*_{02}(0) + C^*_{10}(0)\,C'_{02}(0))]^2}, \tag{8.3.19}$$

where $C_{ik}(x)$ are determined by formulas (8.3.6), and we assume that there is no systematic error. The condition of its absence is

$$\mathrm{Re}\,C_{12}(0) = \mathrm{Re}\,C_{10}(0)\,C^*_{20}(0) = 0. \tag{8.3.20}$$

With optimum reference signals $u_1(t) = u(t)$ and $u_2(t) = u'(t)$ formula (8.3.19) changes into the expression for $S_{опт}$ (8.2.17). With arbitrary $u_1(t)$ and $u_2(t)$, just as in Paragraph 8.3.1, it is possible to introduce equivalent values of the signal-to-noise ratio $q_{экв}$ and the mean square width of the spectrum of modulation $(b - a^2)_{экв}$. If, in particular, reference signal $u_1(t)$ differs from the sounding signal, and $u_2(t) = u_1'(t)$, i.e., differentiation of the reference signal is produced exactly,

$$S_{экв} = T_r \frac{b_{11} - a_{11}^2 + q\,|C_{10}(0)|^2\,(b_{11} - 2a_{11}a_{10} + a_{10}^2)}{2q^2\,|C_{10}(0)|^4\,(\mathrm{Re}\,b_{10} - a_{10}^2)^2}, \tag{8.3.21}$$

where $a_{ik} = \mathrm{Re}\,C'_{ik}(0)$; $b_{ik} = -C''_{ik}(0)$.

Thus, consideration of incoherent discriminators with correlation processing of the received signal shows that the character of the dependence of the gain factor and equivalent spectral density on the method of construction of discriminator and the form of modulation of the sounding signal and characteristics of the reference signals remain the same as in the case of a coherent signal. The dependence on the magnitude of detuning in two-channel discriminators also does not change, and in discriminators with switching of reference signals the quantitative form of the dependence changes somewhat, although qualitatively all remains as before.

The structure of the formulas determining characteristics of incoherent discriminators does not differ from the corresponding coherent cases. Only, here, in formulas for $S_{\text{экв}}$ pertaining to coherent range finders all integrals containing the frequency response of the filter and spectral density of fluctuations are replaced by $1/T_r$, and the signal-to-noise ratio h is replaced by q. This gives us the possibility of using for calculation the characteristics of incoherent discriminators, calculation formulas and graphs already given in Chapter VII, by a simple change of scale.

A distinctive feature of incoherent discriminators with correlation processing is independence of their characteristics from the form of the particular characteristic of the filter and the magnitude of its passband, if only the latter satisfies relationships $1/\tau_{И} \gg \Delta f_{\Phi} \gg 1/T_r$.

§ 8.4. Discriminator with a "Shortening" Filter

In incoherent range finders using a pulse signal in many cases it is more rational to use the principle of optimum filtration of the received signal in a ("shortening") filter [4, 35, 38]. This gives us the possibility to form the likelihood functional and to measure the distance to many targets using the high-frequency part of the radar set common for all meters and the detection unit.

Even in single-target radars use of a "shortening" filter can lead to great technical simplicity as compared to correlation receivers.

8.4.1. Block Diagram of the Discriminator

A "shortening" filter, in principle, can be realized on a carrier of intermediate or low frequency. In practice, they most often use filtration at intermediate frequency. However, selection of the frequency at which the "shortening" filter is realized does not affect analysis of the discriminator and its results. For definitiveness in Fig. 8.3 we show the block diagram of a discriminator with a

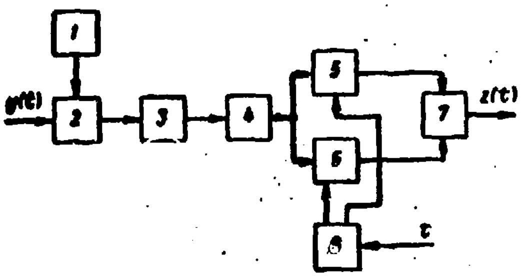

Fig. 8.3. Block diagram of a discriminator with a "shortening" filter: 1 — local oscillator; 2 — mixer; 3 — "shortening" filter; 4 — detector; 5 — gating stage; 6 — controlled generator of gates; 7 — subtractor.

"shortening" filter at intermediate frequency.

The received signal $y(t)$ after transfer to intermediate frequency is passed through a shortening filter and detected. Output voltage of the detector is fed to two gating stages, to which there are fed gating pulses (gates), detuned relative to the output value of delay by $\pm\delta$. Output voltages of the gating stages are subtracted, forming the output voltage of the discriminator.

The derivative of the logarithm of the likelihood funtional in the given circuit will be formed approximately by the two detuned gate pulses. This approximation will be all the better, the narrower the gate pulses and the less the detuning between them. At the limit they should be delta-shaped and located with detuning $\delta \to 0$. In practice gates have finite width and are detuned a finite quantity, where both have a magnitude of the order of the duration of the pulse at the detector output.

The considered circuit represents an analog of the correlation discriminator with two range-detuned channels. It is obvious that the corresponding analog for a circuit with switching of reference signals is a circuit in which gates are not fed simultaneously, but one-by-one during a certain time $T_{\Pi}/2$. Such a change in this case is not a technical simplification, and at the same time, as in correlation discriminators, it leads to certain loss depending on the degree of overlapping of the gates, their form and the ratio of the width of the spectrum of signal fluctuations to the frequency of switching.

The circuit with differentiation in this case requires two "shortening" filters with different characteristics, which, because of their complexity, is hardly an acceptable technical solution. Therefore, subsequently we shall consider only the block diagram of Fig. 8.3, which, as we shall prove subsequently, in a broad range of conditions gives results sufficiently close to those potentially attainable.

Pulse response $h_0(t)$ of the "shortening" filter in the block diagram of Fig. 8.3 should be sufficiently close to Re $u_0(-t)e^{i\omega_{\Pi p}t}$, where $u_0(t)$ — a function describing one period of modulation. Only in the absence of additional intrapulse modulation can this pulse response be presented, as for all filters considered earlier, in the form $h(t)\cos\omega_{\Pi p}t$, where $h(t)$ is the pulse response envelope.

In general, with intrapulse modulation this pulse response can be presented by low-frequency functions only in the form

$$h_1(t)\cos\omega_{\text{пр}}t + h_2(t)\sin\omega_{\text{пр}}t,$$

where $h_1(t)$, $h_2(t)$ – sine and cosine components of pulse response.

For instance, for the pulse described by function u_{a0}, with phase modulation according to the law $\psi_0(t)$ for the case of an optimum shortening filter

$$h_1(t) = u_{a0}(-t)\cos\psi_0(-t),\ h_2(t) = -u_{a0}(-t)\sin\psi_0(-t).$$

With arbitrary modulation the pulse response of the filter can be recorded in the form $\operatorname{Re} h(t)e^{i\omega_{\text{пр}}t}$, where $h(t)$ – the complex pulse response envelope of the filter. In the above-mentioned example

$$h(t) = u_{a0}(-t)e^{i\psi_0(-t)}.$$

Taking into account this circumstance we can record output voltage of the filter in the form

$$u_\Phi(t) = \operatorname{Re}\int_{-\infty}^{t} h(t-s)\, e^{i\omega_{\text{пр}}(t-s)} y(s)\cos(\omega_0 + \omega_{\text{пр}})s\,ds =$$

$$= \frac{1}{2}\operatorname{Re} e^{i\omega_{\text{пр}}t}\int_{-\infty}^{t} h(t-s)\,y(s)\,e^{i\omega_0 s}ds, \qquad (8.4.1)$$

where there are omitted components with doubled frequency $\omega_{\text{пр}}$. Then, on the basis of (8.4.1) we can describe output voltage of the discriminator by the following expression:

$$z(t) = \left|\int_{-\infty}^{t} h(t-s)\,y(s)\,e^{i\omega_0 s}ds\right|^2 v(t-\tau), \qquad (8.4.2)$$

where there are omitted immaterial numerical coefficients, and $v(t)$ – a function describing the sequence of gate pairs, so that

$$v(t) = \sum_k v_0(t - kT_r), \qquad (8.4.3)$$

τ – measured value of delay.

Function $v_0(t)$ must have negative and positive values, since only in this case is creation of a discrimination characteristic possible. In accordance with the block diagram of Fig. 8.3 function $v_0(t)$ is the difference of functions describing gating pulses fed to the various gating stages. For example, with square gates $v_0(t)$ has the form shown in Fig. 8.4. This form depends on the amplitudes of the gates

and detuning between them. By selection of the form of the gates and parameters of channels one should try to ensure that function $v_0(t)$ is odd.

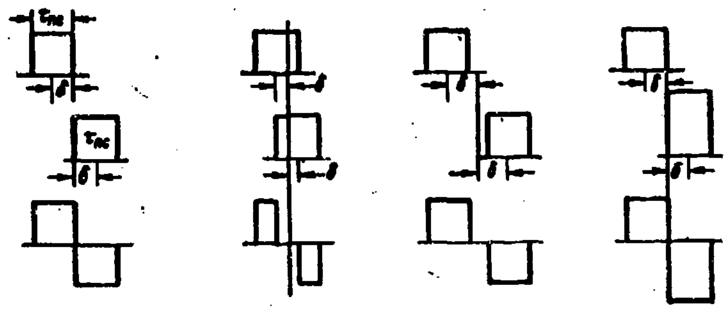

Fig. 8.4. The form of function $v_0(t)$ depending upon the amplitude of the gates and detuning between them.

8.4.2. Discriminator Characteristics

Let us find characteristics of the considered discriminator. Substituting in formulas (8.4.2) the expression for correlation function $q(t) = y(t)e^{i\omega_0 t}$ from (8.2.5), we obtain the mean value

$$\overline{z(t)} = \int_{-\infty}^{t}\int_{-\infty}^{t} h(t-s_1)\,h^*(t-s_2)\Big[\frac{P_c}{2}\sum_k u^*_0(s_1-\tau_0-kT_r)\times$$
$$\times\, u_0(s_2-\tau_0-kT_r)+N_0\delta(s_1-s_2)\Big]\,ds_1ds_2\cdot v(t-\tau)=$$
$$=\Big[\frac{P_c}{2}\sum_k |f(t-\tau_0-kT_r)|^2+N_0\int_0^{\infty}|h(t)|^2\,dt\Big]\cdot v(t-\tau), \quad (8.4.4)$$

where

$$f(t)=\int_0^{\infty} h^*(s)\,u_0(t-s)\,ds \quad (8.4.5)$$

is the result of passage of signal $u_0(t)$ through a filter with pulse response $h^*(t)$, which it is possible to treat the same as the function of crosscorrelation between the pulse response of the filter and the function describing the law of modulation of the signal in one period. Such treatment is useful for finding an analogy with the correlation method of processing a signal. In particular, with an optimum "shortening" filter $[h(t) = u_0(-t)]$

$$f(t) = T_r C(t), \quad (8.4.6)$$

where $C(t)$, as before, is the autocorrelation function of the signal. To produce the discrimination characteristic we average expression (8.4.4) in time. Then

$$\overline{\overline{z(t,\ \Delta)}}=\frac{1}{T_r}\int_0^{T_r}\overline{z(t,\ \Delta)}\,dt=\frac{P_c}{2T_r}\int_0^{T_r}|f(t-\tau_0)|^2 v_0(t-\tau)\,dt+$$

$$+N_0\int_0^{\infty}|h(t)|^2\,dt\,\frac{1}{T_r}\int_0^{T_r}v_0(t-\tau)\,dt=\frac{P_c}{2T_r}\int_{-\infty}^{\infty}|f(s)|^2 v_0(s+\Delta)\,ds+N_0\int_0^{\infty}|h(t)|^2\,dt\,\frac{1}{T_r}\int_{-\infty}^{\infty}v_0(t)\,dt. \quad (8.4.7)$$

Hence, the gain factor of the discriminator

$$K_{\text{д}}=\frac{P_c}{2T_r}\int_{-\infty}^{\infty}|f(s)|^2 v'_0(s)\,ds. \quad (8.4.8)$$

During calculation of discriminator characteristics it is often more convenient to use, not a time, but a frequency presentation of functions. Passing in expression (8.4.7) to spectra, we obtain

$$\overline{\overline{z(t,\ \Delta)}}=\frac{P_c}{2T_r}\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}U^*[i(\omega+x)]\times$$

$$\times H[-i(\omega+x)]\,U(ix)\,H^*(-ix)\,V_0(i\omega)\,e^{i\omega\Delta}\,d\omega dx+$$

$$+\frac{N_0}{T_r}\int_{-\infty}^{\infty}v_0(t)\,dt\,\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(i\omega)|^2\,d\omega, \quad (8.4.9)$$

where $U(i\omega)$, $H(i\omega)$, $V_0(i\omega)$ — Fourier transforms of the corresponding functions.

Likewise, the gain factor of the discriminator

$$K_{\text{д}}=\frac{P_c}{2T_r}\cdot\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i\omega U^*[i(\omega+x)]\,H[-i(\omega+x)]\times$$

$$\times U(ix)\,H^*(-ix)\,V_0(i\omega)\,d\omega dx. \quad (8.4.10)$$

From formula (8.4.7) it follows that the conditions of the absence of systematic error are the following:

$$\int_{-\infty}^{\infty}v_0(t)\,dt=0, \quad (8.4.11)$$

$$\int_{-\infty}^{\infty}|f(t)|^2 v_0(t)\,dt=0. \quad (8.4.12)$$

The first of these conditions consists of the requirement of equality of the areas of the positive and negative parts of $v_0(t)$, and the second is the requirement of orthogonality of the square of the shortened pulse envelope with function $v_0(t)$. If the gates have identical shape, condition (8.4.11) is the condition of identity of amplifications of gating stages. Then the second of conditions (8.4.12) reduces to the corresponding determination of the reference point for delay. With symmetric form of the shortened pulse envelope this reference point coincides with its center,

but with asymmetric form it should be determined in such a manner that (8.4.12) is satisfied. In practice conditions (8.4.11) and (8.4.12) usually are satisfied. Moreover, gates, as a rule, are selected identical, so that

$$v_0(t) = \nu(t+\delta) - \nu(t-\delta), \tag{8.4.13}$$

where $\nu(t)$ – function describing the form of the gate;

δ – detuning relative to their center.

Subsequently, we shall consider systematic error absent.

In discriminators of the considered type they very often use square gates of certain duration $\tau_{\Pi C}$. Detuning δ here can, without loss of generality, be considered larger than $\tau_{\Pi C}/2$, since when $\delta < \tau_{\Pi C}/2$ the form of function $v_0(t)$ coincides with the form of this function for $\delta > \tau_{\Pi C}/2$ with an accuracy of quantitative parameters (Fig. 8.4). With such a form of gates

$$v'_0(t) = -\delta\left(t+\delta+\frac{\tau_{\Pi C}}{2}\right) - \delta\left(t-\delta-\frac{\tau_{\Pi C}}{2}\right) + \delta\left(t+\delta-\frac{\tau_{\Pi C}}{2}\right) + \delta\left(t-\delta+\frac{\tau_{\Pi C}}{2}\right)$$

and the expression for the gain factor has the form

$$K_Д = \frac{P_c}{2T_r}\left\{\left|f\left(\delta-\frac{\tau_{\Pi C}}{2}\right)\right|^2 + \left|f\left(-\delta+\frac{\tau_{\Pi C}}{2}\right)\right|^2 - \left|f\left(\delta+\frac{\tau_{\Pi C}}{2}\right)\right|^2 - \left|f\left(-\delta-\frac{\tau_{\Pi C}}{2}\right)\right|^2\right\}. \tag{8.4.14}$$

In particular, for the most wide-spread case when gates are located end-to-end ($\delta = \tau_{\Pi C}/2$)

$$K_Д = \frac{P_c}{2T_r}\{2|f(0)|^2 - |f(\tau_{\Pi C})|^2 - |f(-\tau_{\Pi C})|^2\}. \tag{8.4.15}$$

Let us find now the correlation function of output voltage of the discriminator. Using formula (8.2.9) and assuming conditions of the absence of systematic error to be satisfied, for zero mismatch we obtain

$$\begin{aligned} R_s(t,\theta) = &\int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t+\theta}\int_{-\infty}^{t+\theta} h(t-s_1)h^*(t-s_2)h(t+\theta-s_3)\times \\ &\times h^*(t+\theta-s_4)v_0(t-\tau_0)v_0(t+\theta-\tau_0)\times \\ &\times[\overline{\eta(s_1)\eta^*(s_2)\eta(s_3)\eta^*(s_4)} - \\ &-\overline{\eta(s_1)\eta^*(s_2)}\,\overline{\eta(s_3)\eta^*(s_4)}]\,ds_1ds_2ds_3ds_4 = \\ = &\Big\{N_0P_0\,\mathrm{Re}\sum_k f^*(t-\tau_0-kT_r)f(t+\theta-\tau_0-kT_r)\times \\ &\times\int_0^\infty h^*(s)h(s+\theta)\,ds + N_0^2\left|\int_0^\infty h(s)h^*(s+\theta)\,ds\right|^2\Big\}\times \\ &\times v_0(t-\tau_0)v_0(t+\theta-\tau_0). \end{aligned} \tag{8.4.16}$$

Averaging this expression over time t, we obtain

$$\widetilde{\overline{R_z(\theta)}}=\frac{N_0^2}{T_r}\left|\int\limits_0^\infty h^*(s)\,h(s+\theta)\,ds\right|^2\int\limits_{-\infty}^{\infty}v_0(t)\,v_0(t+\theta)\,dt+$$

$$+\frac{N_0P_0}{T_r}\mathrm{Re}\int\limits_{-\infty}^{\infty}f^*(t)\,v_0(t)\,f(t+\theta)\,v_0(t+\theta)\,dt\times\int\limits_{-\infty}^{\infty}h^*(s)\,h(s+\theta)\,ds. \qquad (8.4.17)$$

Integrating the expression for $R_z(\theta)$ over θ and dividing by the square of the gain factor, we find the following expression for equivalent spectral density:

$$S_{\text{экв}}=T_r\left\{\int\limits_{-\infty}^{\infty}\left[\frac{2q}{T_r^3}\mathrm{Re}\int\limits_{-\infty}^{\infty}f^*(t)\,v_0(t)\,f(t+\theta)\,v_0(t+\theta)\,dt\times\right.\right.$$

$$\times\int\limits_0^\infty h^*(s)\,h(s+\theta)\,ds+\left|\frac{1}{T_r}\int\limits_0^\infty h^*(s)\,h(s+\theta)\,ds\right|^2\times$$

$$\left.\left.\times\int\limits_{-\infty}^{\infty}v_0(t)\,v_0(t+\theta)\,dt\right]d\theta\right\}:\left[\frac{q}{T_r}\int\limits_{-\infty}^{\infty}|f(s)|^2\,\sigma_0(s)\,ds\right]^2. \qquad (8.4.18)$$

Just as in formula (8.4.10), integration over time in (8.4.18) can be replaced everywhere by integration over frequency; then in the expression for $S_{\text{экв}}$ there will be only spectra of the corresponding functions. Producing the necessary transformations, consisting of multiple application of Parseval's formula, we arrive at the following expression for $S_{\text{экв}}$:

$$S_{\text{экв}}=\left\{2q\frac{1}{(2\pi T_r)^3}\mathrm{Re}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}U(ix)\,H^*(-ix)\,U^*(iy)\times\right.$$

$$\times H(-iy)\,V_0[i(\omega-x)]\,V_0^*[i(\omega-y)]\,H^*(-i\omega)\times$$

$$\times H(-i\omega)\,d\omega\,dx\,dy+\frac{1}{(2\pi T_r)^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}|V_0(i\omega)|^2\,H^*(i\omega)\,H(i\omega)\times$$

$$\left.\times H[i(\omega+x)]\,H^*[i(\omega+x)]\,d\omega\,dx\right\}:\left\{\frac{q}{(2\pi T_r)^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}i\omega U^*\times\right.$$

$$\left.\times[i(\omega+x)]\,H[-i(\omega+x)]\,U(ix)\,H^*(-ix)\,V_0(i\omega)\,d\omega\,dx\right\}^2. \qquad (8.4.19)$$

Let us consider the case when the duration of gates is small as compared to duration of the pulse at the detector output. Then function $v_0(t)$ can be presented in the form

$$v(t)=\delta(t+\delta)-\delta(t-\delta), \qquad (8.4.20)$$

where $\delta(t)$ – delta-function.

Here

$$\overline{z(\Delta)}=\frac{P_0}{2T_r}\{|f(-\delta-\Delta)|^2-|f(\delta-\Delta)|^2\} \qquad (8.4.21)$$

and the gain factor of the discriminator will be equal to

$$K_\text{д}=\frac{P_0}{2T_r}2\,\mathrm{Re}\,\{f'(-\delta)f^*(-\delta)-f'(\delta)f^*(\delta)\}. \qquad (8.4.22)$$

Substituting (8.4.20) in formula (8.4.18) for equivalent spectral density, we obtain

$$\begin{aligned}S_{\text{экв}}=T_r\Big\{&\frac{2q}{T_r^3}\mathrm{Re}\Big[(|f(-\delta)|^2+|f(\delta)|^2)\int_0^\infty|h(s)|^2ds-f^*(-\delta)f(\delta)\int_0^\infty h^*(s)h(s+2\delta)\,ds-\\&-f^*(\delta)f(-\delta)\int_0^\infty h^*(s)h(s+2\delta)\,ds\Big]+\frac{2}{T_r^2}\Big[\Big(\int_0^\infty|h(s)|^2ds\Big)^2-\\&-\Big|\int_0^\infty h^*(s)h(s+2\delta)\,ds\Big|^2\Big]\Big\}:\Big[\frac{2q}{T_r^2}\mathrm{Re}\,(f'(-\delta)f^*(-\delta)-f'(\delta)f^*(\delta))\Big]^2.\end{aligned} \qquad (8.4.23)$$

From comparison of this formula with (8.3.12) it follows that both expressions coincide if we set

$$u_1(t)=u_2(t)=h(-t). \qquad (8.4.24)$$

Thus, the considered discriminator with gates which are narrow as compared to the duration of the shortened pulse is completely equivalent to a two-channel discriminator with correlation signal processing, in which reference signals are identical and are determined in accordance with (8.4.24). In particular, with an optimum "shortening" filter h(t) = u(-t) and detunings δ, small as compared to the duration of the shortened pulse, formula (8.4.23) becomes expression (8.2.17) for $S_{\text{опт}}$. When h(t) = u(-t) (8.4.6) is satisfied, and

$$\int_0^\infty h^*(s)h(s-t)\,ds=T_rC(t). \qquad (8.4.2?)$$

Further investigation of the discriminator without specifying the form of modulation of the sounding signal and characteristics of the optimum filter is difficult. The obtained general results permit us to conduct such an investigation in any concrete case. We shall pursue this in subsequent paragraphs.

§ 8.5. Incoherent Pulse Radiation Without Additional Modulation

Simple pulse radiation, probably, is the most wide-spread in practice [1, 26, 27, 36]. The matched filter in this case can be called "shortening" only in quotes. I.

reality it is a normal i-f amplifier (UPCh) with a bandwidth, approximately matched with the spectrum width of the pulse. Requirements on the form of the frequency response of the UPCh are not very critical; in practice exact matching of it with the form of the pulse spectrum is not achieved [30, 41]. With a pulse signal without additional modulation most rational from the technical point of view is the application of discriminators with "shortening" filters; however, in principle, there can also be used correlation discriminators. We will consider, basically, discriminators with a "shortening" filter; however, we shall also touch on correlation discriminators.

8.5.1. Discriminator with a Broad-Band UPCh

We consider first one particular case of a discriminator using filtration, without specifying the form of the pulse.

Quite often bandwidth of the UPCh (УПЧ)

$$\Delta f_{УПЧ}=\int_0^\infty h^2(t)\,dt=\frac{1}{2\pi}\int_0^\infty |H(i\omega)|^2\,d\omega \tag{8.5.1}$$

considerably exceeds the matched bandwidth, so that $\Delta f_{УПЧ}\,\tau_И >> 1$ (pulse response of the filter h(t) during pulse radiation without additional modulation is real). Then, the pulse passes to the UPCh output without distortion, so that

$$f(t)=u_0(t) \tag{8.5.2}$$

and formulas for the gain factor and spectral density take the form

$$K_Д=\frac{P_0}{2T_r}\int_{-\infty}^{\infty}u_0^2(s)\,v'_0(s)\,ds, \tag{8.5.3}$$

$$S_{экв}=T_r\,\frac{2q\frac{1}{T_r}\int_{-\infty}^{\infty}u_0^2(t)\,v_0^2(t)\,dt+\int_{-\infty}^{\infty}v_0^2(t)\,dt\,\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(i\omega)|^4\,d\omega}{q^2\left[\frac{1}{T_r}\int_{-\infty}^{\infty}u_0^2(t)\,v'_0(t)\,dt\right]^2}, \tag{8.5.4}$$

where frequency response of the filter is normalized so that H(i0) = 1, and integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(i\omega)|^4\,d\omega$$

has the order of $\Delta f_{УПЧ}$ and is equal to $\Delta f_{УПЧ}$ in the case of a square frequency response, $\frac{1}{2}\Delta f_{УПЧ}$ with a frequency response of the UPCh coinciding with the frequency response of a single LRC-circuit, and $(1/\sqrt{2})\Delta f_{УПЧ}$ with a Gaussian frequency response.

For validity of formula (8.5.4) there is required, in general, not only condition $\Delta f_{УПЧ}\, \tau_{И} \gg 1$, but also the condition that bandwidth of the UPCh is great as compared to the width of the spectrum of gates. From expressions (8.5.3) and (8.5.4) by means of calculations we can obtain particular results for any concrete form of the signal pulse and of gates, in particular for square gates of duration $\tau_{ПС}$ and detuning $\delta = \tau_{ПС}/2$:

$$S_{экв} = T_r \frac{q \frac{1}{T_r} \int\limits_{-\tau_{пс}}^{\tau_{пс}} u_0^2(t)\, dt + \alpha \Delta f_{УПЧ} \tau_{пс}}{2q^2 \left[\frac{u_0^2(0)}{T_r} + \frac{u_0^2(\tau_{пс}) + u_0^2(-\tau_{пс})}{2T_r}\right]^2}, \qquad (8.5.5)$$

where α — coefficient of the order of 0.5-1, characterizing the difference of integral $\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} |H(i\omega)|^4\, d\omega$ from $\Delta f_{УПЧ}$.

8.5.2 Square Pulse

Let us consider first of all the case when the frequency response of the UPCh is matched with the spectrum of the pulse, and the gates are narrow as compared to its duration. Then

$$f(t) = \int\limits_0^{\infty} h(s)\, h(s-t)\, ds = T_r \left(1 - \frac{|t|}{\tau_и}\right), |t| < \tau_и \qquad (8.5.6)$$

and from formula (8.4.23) it follows that

$$S_{экв} = T_r \delta \tau_и \frac{1 + q\left(1 - \frac{\delta}{\tau_и}\right)}{2q^2 \left(1 - \frac{\delta}{\tau_и}\right)}. \qquad (8.5.7)$$

Just as in the case of a coherent discriminator, this expression approaches zero as $\delta \to 0$. Actually $S_{ЭКВ}$ seeks a value determined by the duration of the pulse edge. The dependence of $S_{ЭКВ}$ on detuning δ is identical with (7.6.3), describing the same dependence in a coherent discriminator.

With a broad-band UPCh and square gates of duration $\tau_{ПС}$ with detuning $\pm\delta$($\delta \geq \geq \tau_{ПС}/2 \geq \tau_И/2$) from formula (8.5.5) we obtain

$$S_{экв} = \frac{T_r \tau_и^2}{2q^2} (q + \alpha \Delta f_{УПЧ} \tau_{пс}). \qquad (8.5.8)$$

From this formula it follows that with finite duration of the gates ($\tau_{\Pi C} \gg$ $\gg 1/\Delta f_{У\Pi Ч}$) accuracy of range finding is finite for all δ and is determined by pulse duration. The dependence of $S_{ЭКВ}$ on q for different $\alpha \Delta f_{У\Pi Ч}\ \tau_{\Pi C}$ is shown in Fig. 8.5, from which it follows that with sufficiently large signal-to-noise ratios q the considered discriminator is uncritical to the width of the UPCh passband and duration of the gates.

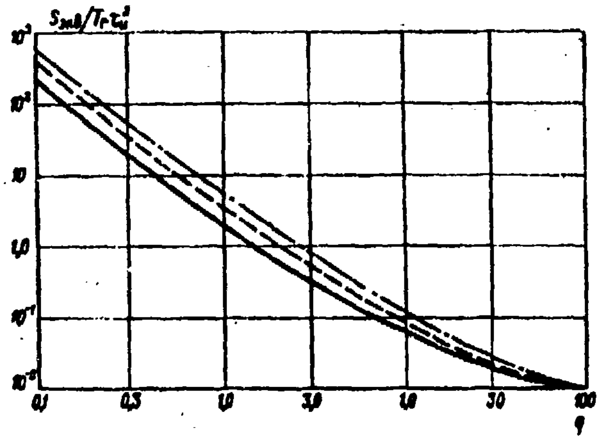

Fig. 8.5. Equivalent spectral density for a discriminator with a broad-band UPCh: —— $\alpha \Delta f_{У\Pi Ч}\ \tau_{\Pi C} = 3$; – – – $\alpha \Delta f_{У\Pi Ч}\ \tau_{\Pi C} = 6$; –·– $\alpha \Delta f_{У\Pi Ч}\ \tau_{\Pi C} = 10$.

Let us further consider the case when the frequency response of the UPCh is matched with the pulse spectrum and gates have finite width and a square shape. In accordance with how this is most frequently done in practice, we shall consider that gates are located end-to-end, i.e., $\delta = \tau_{\Pi C}/2$. The gain factor of the discriminator here is proportional to

$$\frac{2\tau_{\Pi C}}{\tau_И}\left(2-\frac{\tau_{\Pi C}}{\tau_И}\right) \qquad (8.5.9)$$

and reaches a maximum value at $\tau_{\Pi C} = \tau_И$. Further expansion of the gates has no sense, since pulse duration with respect to zeroes at the output of the matched UPCh is $2\tau_И$ and at $\tau_{\Pi C} > \tau_И$ the power of the useful signal at the discriminator output does not increase, and the power of noise components grows, inasmuch as integration of noise occurs over the whole large interval of time. Therefore we shall consider $\tau_{\Pi C} \leq \tau_И$. Substituting expression (8.5.6) and function $v_0(t)$, describing square gates, in formula (8.4.13) we obtain as a result of integration the following expression:

$$S_{экв} = \frac{T_r\tau_и^2}{2q^2}\left[a_1\left(\frac{\tau_{\Pi C}}{\tau_и}\right)q + a_2\left(\frac{\tau_{\Pi C}}{\tau_и}\right)\right], \qquad (8.5.10)$$

where a_1, a_2 — functions describing the dependence of components of spectral density on the ratio of the durations of gates and of the pulse. These functions have the form

$$
\left.
\begin{aligned}
a_1 &= \frac{\frac{4}{3}\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}-\frac{5}{3}\left(\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}\right)^2+\frac{8}{15}\left(\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}\right)^3}{\left(2-\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}\right)^2}, \qquad \frac{\tau_{\text{пс}}}{\tau_{\text{и}}}<\frac{1}{2},\\
a_1 &= \frac{\frac{43}{240}-\frac{4}{3}\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}+\frac{11}{3}\left(\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}\right)^2-\frac{10}{3}\left(\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}\right)^3+\left(\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}\right)^4}{\left(\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}\right)^2\left(2-\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}\right)^2}, \quad \frac{\tau_{\text{пс}}}{\tau_{\text{и}}}>\frac{1}{2},
\end{aligned}
\right\} \tag{8.5.11}
$$

$$
\left.
\begin{aligned}
a_2 &= \frac{\frac{4}{3}\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}-\left(\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}\right)^2}{\left(2-\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}\right)^2}, \qquad \frac{\tau_{\text{пс}}}{\tau_{\text{и}}}<\frac{1}{2},\\
a_2 &= \frac{\frac{1}{12}-\frac{2}{3}\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}+2\left(\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}\right)^2-\frac{4}{3}\left(\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}\right)^3+\frac{1}{3}\left(\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}\right)^4}{\left(\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}\right)^2\left(2-\frac{\tau_{\text{пс}}}{\tau_{\text{и}}}\right)^2}, \quad \frac{\tau_{\text{пс}}}{\tau_{\text{и}}}>\frac{1}{2}.
\end{aligned}
\right\} \tag{8.5.12}
$$

In particular, when $\tau_{\text{пс}}/\tau_{\text{и}} = 1$ coefficients a_1 and a_2 take values $a_1 = 0.18$ and $a_2 = 0.42$. For small $\tau_{\text{пс}}/\tau_{\text{и}}$ both coefficients have the order of $\tau_{\text{пс}}/\tau_{\text{и}}$. The dependence of a_1 and a_2 on $\tau_{\text{пс}}/\tau_{\text{и}}$ is shown in Fig. 8.6. For small signal-to-noise ratios q duration of the gate rather strongly affects accuracy of measurement. For large signal-to-noise ratios this influence is less essential if duration of the gates is not very small — in the domain $1/2 < \tau_{\text{пс}}/\tau_{\text{и}} < 1$ coefficient a_1 varies little. For small $\tau_{\text{пс}}/\tau_{\text{и}}$ the dependence of a_1 on $\tau_{\text{пс}}/\tau_{\text{и}}$ is approximately linear. Thus, with a matched UPCh it is useful to decrease duration of the gates; in spite of decrease of the gain factor of the discriminator quantity $S_{\text{экв}}$ decreases with decrease of $\tau_{\text{пс}}/\tau_{\text{и}}$. For very small $\tau_{\text{пс}}/\tau_{\text{и}}$ results of analysis, of course, do not reflect reality — we need allowance for the finite duration of the edge and nonlinearity of the discrimination and fluctuation characteristics.

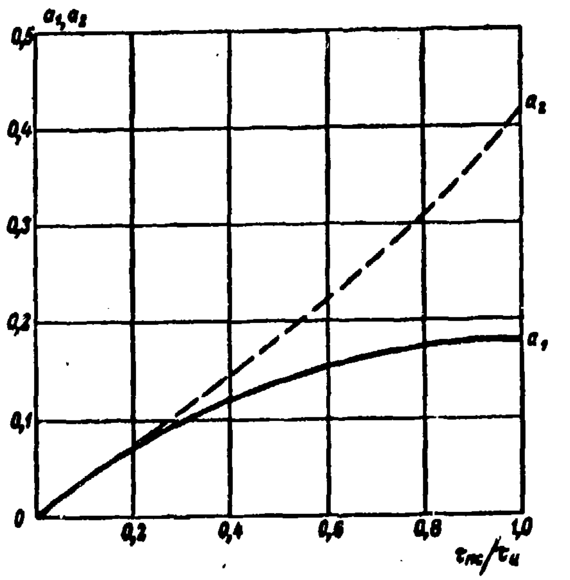

Fig. 8.6. Influence of ratio $\tau_{\text{пс}}/\tau_{\text{и}}$ for a discriminator with a matched UPCh on $S_{\text{экв}}$.

We shall briefly discuss discriminators with correlation processing. For a two-channel discriminator with detuned channels we need only repeat what was already said in examining coherent range finders (see Paragraphs 7.6.1 and 8.3.1). For a

discriminator with switching of reference signals of interest is the dependence of equivalent spectral density on detuning. Considering that the duration of the gate pulse, which in this case is the reference signal, $\tau_{ПС} \gtrless \tau_И$, on the basis of formulas (7.6.7) and (8.3.16) we obtain

$$S_{экв} = \frac{T_r \tau_c^2}{4q^2} \frac{1 + 2q \frac{\tau_и}{\tau_c}\left(1 - \frac{\delta}{\tau_и}\right)}{\left(1 - \frac{\delta}{\tau_и}\right)^2}. \qquad (8.5.13)$$

As follows from this expression, equivalent spectral density monotonically decreases with decrease of detuning; however, even as $\delta \to 0$ accuracy remains finite. With magnitudes of detuning close to $\tau_И$, spectral density increases without limit, which is caused by decrease of the gain factor. Here increase of $S_{ЭКВ}$ is faster than in a two-channel discriminator. An optimum for detuning, which existed in the coherent case, is absent here.

It is interesting to compare the accuracy of the discriminator with switching of reference signals with the accuracy of a two-channel correlation-type discriminator, for which quantity $S_{ЭКВ}$ with coinciding durations of the pulse and gate pulses is determined, obviously, by formula (8.5.7). The ratio of quantities $S_{ЭКВ}$ for both cases is equal to

$$\frac{S_{экв1}}{S_{экв2}} = \frac{1 + 2q\left(1 - \frac{\delta}{\tau_и}\right)}{2\frac{\delta}{\tau_и}\left(1 - \frac{\delta}{\tau_и}\right)\left[1 + q\left(1 - \frac{\delta}{\tau_и}\right)\right]}, \qquad (8.5.14)$$

where $S_{ЭКВ1}$ is expressed by formula (8.5.13) when $\tau_c = \tau_И$.

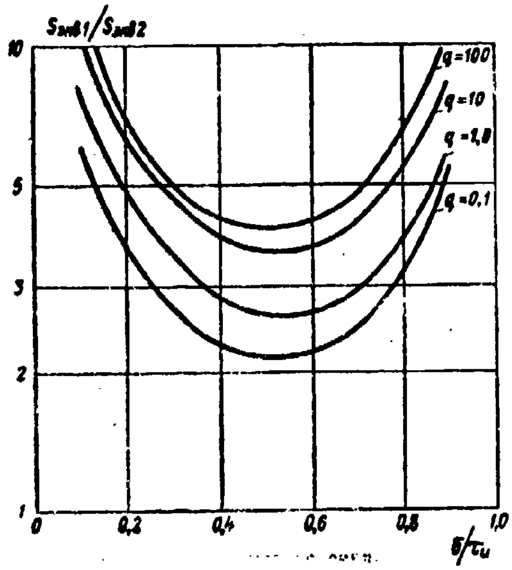

Fig. 8.7. The influence of detuning δ and the signal-to-noise ratio q on $S_{ЭКВ}$ for a discriminator with switching of reference signals.

For small q the discriminator with switching of reference signals gives a loss of at least one half, and for large q, at least one fourth. In both cases minimum loss is attained at $\delta = 0.5\tau_И$. The dependence of ratio (8.5.14) on detuning δ for various q is shown in Fig. 8.7.

8.5.3. Gaussian Pulse

Let us consider now the case when the form of the pulse is approximated a Gaussian curve of form (7.6.9). In the case of square gates and a UPCh with a

broad band as compared to $1/\tau_И$ on the basis of (8.5.5) we have

$$S_{экв} = T_r \tau_и^2 \frac{\alpha\Delta f_{УПЧ}\tau_{пс} + q\,[2\Phi(\sqrt{2\pi}\tau_{пс}/\tau_и) - 1]}{2q^2\,[1 - \exp(-\pi\tau_{пс}^2/\tau_и^2)]^2}, \qquad (8.5.15)$$

where $\Phi(x)$ — integral of probability.

For small $\tau_{ПС}$ this expression is valid if $\Delta f_{УПЧ}\tau_{ПС} > 3$ to 5. For larger $\tau_{ПС}/\tau_И$ and q quantity $S_{ЭКВ}$ is determined by asymptotic expression

$$S_{экв} \approx \frac{T_r \tau_и^2}{2q}, \qquad (8.5.16)$$

which is $\pi/2$ times larger than the magnitude of $S_{ОПТ}$ from (8.2.17) for large q (let us remember that for a Gaussian pulse $a = 0$, $b = \pi/2\tau_И^2$).

Analysis of formula (8.5.15) shows that duration of the gates should be selected sufficiently great as compared to $\tau_И$, since increase of $\tau_{ПС}/\tau_И$ increases the gain factor to a greater degree than the spectral density of output voltage of the discriminator. For very small $\tau_{ПС}/\tau_И$ the asymptotic expression for $S_{ЭКВ}$ has the form

$$S_{экв} \approx \frac{T_r \tau_и^6 \alpha\Delta f_{УПЧ}}{2\pi^2 \tau_{пс}^3 q^2}. \qquad (8.5.17)$$

Practically, ratio $\tau_{ПС}/\tau_И$ should be near 1.0. Further increase of duration of the gates already almost does not change the gain factor of the discriminator but increases the influence of the noise term in $S_{ЭКВ}$. The dependence of ratio $S_{ЭКВ}/S_{ОПТ}$ on q for different $\alpha\Delta f_{УПЧ}\tau_{ПС}$ and $\tau_{ПС}/\tau_И$ is shown in Fig. 8.8.

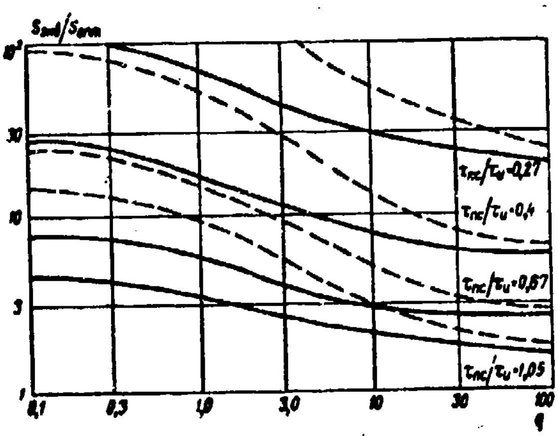

Fig. 8.8. The dependence of ratio $S_{ЭКВ}/S_{ОПТ}$ on q for the case of a Gaussian pulse and a broad-band UPCh: —— $\alpha\Delta f_{УПЧ}\tau_{ПС} = 3$; – – – $\alpha\Delta f_{УПЧ}\tau_{ПС} = 10$.

We shall consider now an example of a more general nature. Let us assume that frequency response of the UPCh has the form of a Gaussian curve of arbitrary width, i.e.,

$$|H(i\omega)|^2 = e^{-\frac{\omega^2}{4\alpha\Delta f_{УПЧ}^2}}, \qquad (8.5.18)$$

and gates also are Gaussian with width $\tau_{ПС}$ and detuning δ, i.e.,

$$v_s(t) = e^{-\frac{\pi(t+\delta)^2}{2\tau_{пс}^2}} - e^{-\frac{\pi(t-\delta)^2}{2\tau_{пс}^2}}. \qquad (8.5.19)$$

Designating $\tau_И/\tau_{ПС} = y$, $\Delta f_{УПЧ}/\Delta f_{СОГЛ} = 2\Delta f_{УПЧ}\tau_И = x$ ($\Delta f_{СОГЛ} = 1/2\tau_И$), $z = \sqrt{2\pi}\delta/\tau_И$ and performing necessary calculations by formulas (8.4.18) and (8.4.19), for equivalent spectral density we obtain the following expression:

$$S_{экв} = \frac{T_r\tau_И^2(1+x^2)[2x^2+y^4(1+x^2)]^2}{\pi q^2\cdot 16z^2x^4y^4\exp\left\{-\frac{z^2x^2y^2}{2x^2+y^2(1+x^2)}\right\}} \times \left\{\frac{1-\exp\left\{-\frac{z^2x^2y^2}{2x^2+y^2}\right\}}{yx^2\sqrt{2x^2+y^2}} + \right.$$

$$\left. + \frac{4q\exp\left\{-\frac{z^2x^2y^2}{2(x^2+y^2+x^2y^2)}\right\}}{x\,y\,(x^2+y^2+x^2y^2)} \times \frac{\left[1-\exp\left\{\frac{-z^2x^2y^4(1+x^2)^2}{2(x^2+y^2+x^2y^2)[y^2+x^2(2+y^2+x^2)]}\right\}\right]}{[y^2+x^2(2+y^2+x^2)]}\right\}. \quad (8.5.20)$$

When $x = 1$ ($\Delta f_{УПЧ} = \Delta f_{СОГЛ}$), $y \to \infty$ ($\tau_{ПС} \to 0$), $z \to 0$ ($\delta \to 0$) this expression turns into $S_{ОПТ}$ with $b = \pi/2\tau_И^2$, reaching its minimum. Investigation of formula (8.5.20) shows that $S_{ЭКВ}$ comparatively weakly depends on detuning z, especially if the passband of the UPCh is close to the matched band. For small z the expression for equivalent spectral density takes the form

$$S_{экв} = \frac{T_r\tau_И^2}{\pi q^2}\cdot\frac{(1+x)^2[2x^2+y^2(1+x^2)]^2}{16x^2y^2}\cdot\left\{\frac{1}{yx(2x^2+y^2)^{3/2}} + \right.$$

$$\left. + \frac{2qy^2(1+x^2)^2}{(y^2+x^2+x^2y^2)^{3/2}(2x^2+y^2+y^2x^2+x^4)^{3/2}}\right\}, \quad (8.5.21)$$

which is considerably more convenient for calculations.

The dependence of $S_{ЭКВ}/S_{ОПТ}$ on q for various x and y is shown in Fig. 8.9. Curves of this figure show that loss in accuracy increases with decrease of the signal-to-noise ratio q and increase of duration of the gate and the difference between $\Delta f_{УПЧ}$ and $\Delta f_{СОГЛ}$. This loss for small q can reach substantial magnitude. Analysis of formulas (8.5.20), (8.5.21), and Fig. 8.9 permits us to draw the following conclusions:

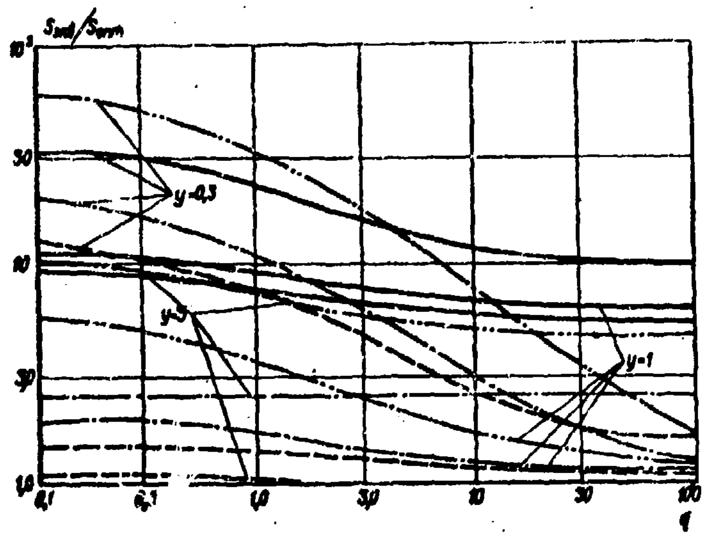

Fig. 8.9. The dependence of ratio $S_{ЭКВ}/S_{ОПТ}$ on q for Gaussian gates and pulse: —— x = 0.3; – – – x = 1; –·– x = 3; –··– x = 10.

1. For $\Delta f_{УПЧ} \leq \Delta f_{СОГЛ}$ the magnitude of $S_{ЭКВ}$ is less, the larger y, i.e., the less the duration of the gate. When $\Delta f_{УПЧ} > \Delta f_{СОГЛ}$ duration of the gate should be increased to a magnitude of the order of $\tau_И$ ($y \sim 1$). The

necessity for this is the faster drop of the gain factor with reduction of the gates as compared to decrease of the spectral density at the discriminator output in the case of an expanded passband of the UPCh.

2. The influence of expansion of the passband, in particular, on the component of $S_{ЭКВ}$ with factor $1/q$ is stronger, the shorter the gates.

3. For large x and y the magnitude of $S_{ЭКВ}$ decreases with increase of detuning to values ensuring maximum slope of the discrimination characteristic, i.e., to $z = \sqrt{(2x^2 + y^2 + x^2y^2)/xy}$. For instance, when $x = 3$ and $y = 10$, selecting z from this condition, we obtain for $S_{ЭКВ}/S_{ОПТ}$ the following values: $S_{ЭКВ}/S_{ОПТ} = 1.51$ for $q = 0.1$; $S_{ЭКВ}/S_{ОПТ} = 1.41$ for $q = 3$; $S_{ЭКВ}/S_{ОПТ} = 1.25$ for $q = 100$, which is considerably smaller than corresponding values for small values of z.

4. In general, selection of parameters of the discriminator should be carried out in such a manner that $x = 1$ to 3, $y = 1$ to 3, and the magnitude of detuning ensures a maximum gain factor. Then, for all signal-to-noise ratios q the magnitude of $S_{ЭКВ}$ will not exceed $S_{ОПТ}$ by more than a factor of two, and, in practice, the range finder will realize its potential accuracy.

Characteristics of a two-channel discriminator with correlation processing of the signal for the considered case were actually already found in Chapter VII. Equivalent spectral density here is determined by formula (8.3.13), where quantities $b_{ЭКВ}$ and $q_{ЭКВ}$, determined by formulas (8.3.14) and (8.3.15), are given for an arbitrary duration of gates of Gaussian form and arbitrary detuning by expressions (7.6.17) and (7.6.18) and the curves of Fig. 7.17. Let us remember that ratio $q_{ЭКВ}/q = h_{ЭКВ}/h$. Therefore, it remains only to repeat what was said in Paragraph 7.6.2 about the given case.

We shall investigate further the influence of the magnitude of detuning on accuracy of a correlation-type discriminator with switching of reference signals. We shall consider that durations of gate pulses coincide with pulse duration. Then

$$S_{ЭКВ} = \frac{T_r \tau_И^2}{\pi q^2} \cdot \frac{1 + 2qe^{-\frac{\pi\delta^2}{2\tau_И^2}}}{\frac{\pi\delta^2}{\tau_И^2} e^{-\frac{\pi\delta^2}{\tau_И^2}}}. \qquad (8.5.22)$$

For both large and small $\delta/\tau_И$ the magnitude of $S_{ЭКВ}$ for all q increases without limit. The minimum of $S_{ЭКВ}$ for small q is reached with detuning δ which ensures a maximum gain factor, i.e., at $\delta = \sqrt{\tau_И^2/\pi}$, and for large q at $\delta = \sqrt{2\tau_И^2/\pi}$. The ratio

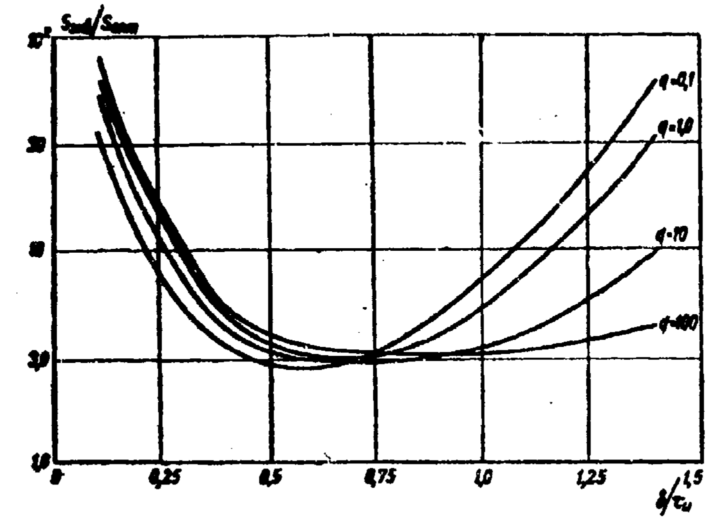

Fig. 8.10. The dependence of $S_{\text{эв}}/S_{\text{ОПТ}}$ on $\delta/\tau_{И}$ for a discriminator with switching of reference signals with a Gaussian form of the pulse.

of $S_{\text{эв}}$ from (8.5.22) to $S_{\text{ОПТ}}$, as a function of detuning δ, is shown for various q in Fig. 8.10. As can be seen from this figure, loss can attain substantial magnitudes; even for optimum selection of detuning ratio $S_{\text{эв}}/S_{\text{ОПТ}}$ is 3. For small detunings the dependence of loss on the signal-to-noise ratio is not very substantial; for large $\delta/\tau_И$ loss is greater, the smaller q.

§ 8.6. Pulse Amplitude Modulation

With incoherent radiation more frequently than with coherent there arises the necessity of expanding the spectrum of the sounding signal by intrapulse modulation. In this and the following paragraphs we shall consider several examples of such modulation and corresponding schemes for construction of discriminators. Furthermore, we shall consider subsequently that in accordance with requirements of practice the width of the spectrum of intrapulse modulation is great as compared to the width of the pulse spectrum.

Let us consider, first, amplitude modulation of a pulse by sinusoidal law with frequency $\omega_m\left(\omega_m >> \frac{1}{\tau_И}\right)$. Obviously, most convenient from the technical point of view for the given case is the scheme of a discriminator with correlation processing and differentiation of the reference signal. Since for large $\omega_m\tau_И$ it is possible to disregard the possibility of additional increase of accuracy of measurement by using the form of the pulse envelope,* which gives a relative contribution to

*We have in mind a pulse envelope without sinusoidal modulation.

accuracy of the order of $1/(\omega_m \tau_И)^2$, then it is sufficient to carry out differentiation of the reference signal only with respect to the additional amplitude modulation, not differentiating the pulse envelope. Then the operation of differentiation is realized by shift of the modulating voltage in phase and heterodyning.

The block diagram of the discriminator allowing for this circumstance is shown in Fig. 8.11. The received signal after transfer to intermediate frequency and preamplification proceeds to two mixers, to which there are fed voltages m cos $\omega_m \times$ $\times (t - \tau)$ and m sin $\omega_m(t - \tau)$ with phase $\omega_m \tau$, controlled by the output quantity of the range finder. The signal from the output of the cosine mixer is added to the input signal of the mixer. Then both signals are gated, are approximately matched with the duration of the signal pulse by gate pulses, are passed through filters, from which there is required sufficient identity of phase responses, and they are mixed in the phase detector, forming output voltage of the discriminator. With respect to the bandwidth of the filters all remarks in §§ 8.2 and 8.3 are valid. However, in this case we assume that bandwidth of the filter does not necessarily satisfy condition $\Delta f_Ф \tau_И \ll 1$, but has arbitrary magnitude.

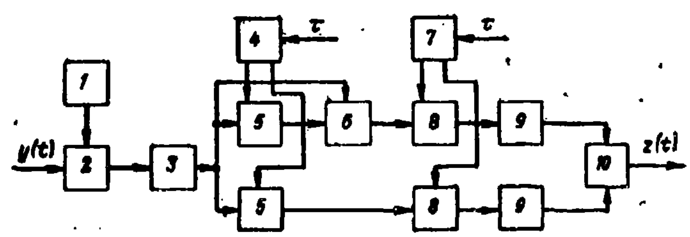

Fig. 8.11. Block diagram of a discriminator for a signal with additional sinusoidal modulation: 1 — local oscillator; 2 — mixer; 3 — preamplifier; 4 — generator of sinusoidal oscillations with controlled phase; 5 — mixer; 6 — adder; 7 — controlled generator of gate pulses; 8 — broad-band gated amplifier; 9 — i-f filter; 10 — phase detector.

If condition $\Delta f_Ф \tau_И \ll 1$ is satisfied, the equivalent spectral density is determined by formula (8.3.19). The law of modulation of the sounding signal can be recorded in this case in the form

$$u_0(t) = u_{н0}(t) \frac{1 + m \cos \omega_m t}{\sqrt{1 + \frac{m^2}{2}}}, \tag{8.6.1}$$

and for reference signals

$$u_1(t) = v(t) \frac{1 + m \cos \omega_m t}{\sqrt{1 + \frac{m^2}{2}}}, \tag{8.6.2}$$

$$u_2(t) = v(t)\,\frac{m\omega_m \sin \omega_m t}{\sqrt{1+\frac{m^2}{2}}}, \tag{8.6.3}$$

where it is assumed that gate pulses in both channels are identical and are described by function v(t), which we for convenience can normalize so that

$$\frac{1}{T_r}\int\limits_{-\infty}^{\infty} v^2(t)\,dt = 1. \tag{8.6.4}$$

Such normalization is valid, too, for the pulse envelope $u_{a0}(t)$.

If durations of pulses $u_{a0}(t)$ and v(t) are great as compared to $1/\omega_m$, the cross-correlation functions $C_{ik}(x)$ will have the form:

$$C_{10}(x) = \frac{1+\frac{m^2}{2}\cos\omega_m x}{1+\frac{m^2}{2}}\cdot\frac{1}{T_r}\int\limits_{-\infty}^{\infty} u_{a0}(t)\,v(t)\,dt, \tag{8.6.5}$$

$$C_{02}(x) = -C_{20}(x) = C'_{10}(x) = -\frac{\omega_m m^2 \sin\omega_m x}{2\left(1+\frac{m^2}{2}\right)}\cdot\frac{1}{T_r}\int\limits_{-\infty}^{\infty} u_{a0}(t)\,v(t)\,dt, \tag{8.6.6}$$

$$c_{12}(x) = -\frac{\omega_m m^2 \sin\omega_m x}{2\left(1+\frac{m^2}{2}\right)}, \tag{8.6.7}$$

$$C_{11}(x) = -\frac{1+\frac{m^2}{2}\cos\omega_m x}{1+\frac{m^2}{2}}, \tag{8.6.8}$$

$$C_{22}(x) = \frac{m^2\omega_m^2\cos\omega_m x}{2\left(1+\frac{m^2}{2}\right)}. \tag{8.6.9}$$

Then, on the basis of formula (8.3.13) for equivalent spectral density we obtain the following expression:

$$S_{\text{экв}} = T_r\,\frac{2\left(1+\frac{m^2}{2}\right)}{m^2\omega_m^2}\cdot\frac{1+q\left(\frac{1}{T_r}\int\limits_{-\infty}^{\infty} u_{a0}(t)\,v(t)\,dt\right)^2}{2q^2\left(\frac{1}{T_r}\int\limits_{-\infty}^{\infty} u_{a0}(t)\,v(t)\,dt\right)^4} = \frac{1+q_{\text{экв}}}{2bq^2_{\text{экв}}}, \tag{8.6.10}$$

where

$$b = \frac{m^2\omega_m^2}{2\left(1+\frac{m^2}{2}\right)} \tag{8.6.11}$$

is the mean square width of the spectrum of modulation for the considered case, and

$$q_{\text{экв}} = q\left(\frac{1}{T_r}\int_{-\infty}^{\infty} u_{a0}(t)\,v(t)\,dt\right)^2 \tag{8.6.12}$$

is the equivalent signal-to-noise ratio, decrease of which is caused by mismatch of durations of the sounding pulse and the gate pulses. With sufficiently large $\tau_c/\tau_и$ ratio $q_{экв}/q$ for any form of pulse and gate pulses is equal to $\tau_и/\tau_c$; for a square pulse and square gate pulses there is exact equality when $\tau_c/\tau_и$, and, conversely, $q_{экв}/q = \tau_c/\tau_и$ when $\tau_c < \tau_и$.

Thus, nonoptimality of processing, consisting of the mismatching of durations of the sounding and reference signals, leads in this case simply to corresponding decrease of the signal-to-noise ratio. If in circuits of this type we cannot achieve identity of phase responses of the two channels, then, just as in corresponding circuits of coherent discriminators, there occurs additional increase of errors of measurement. The equivalent spectral density increases proportionally to $1/\cos^2\varphi$, where φ — phase delay in one channel relative to the other.

Now let us assume that the bandwidth of the filter satisfies requirement $\Delta f_\Phi \gg 1/T_r$, but is not limited from above. Then, using the expression for output voltage of the discriminator and performing the same transformations as in §§ 8.2 and 8.3, for equivalent spectral density we can obtain the following expression:

$$S_{\text{экв}} = \frac{T_r\left(1+\frac{m^2}{2}\right)}{m^2\omega_m^2 q^2}\,\frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}|H(i\omega_1)|^2|H(i\omega_2)|^2\{|V[i(\omega_1+\omega_2)]|^2+ q F(i\omega_1)F(i\omega_2)V^*[i(\omega_1+\omega_2)]\}\,d\omega_1 d\omega_2}{\left[\int\limits_{-\infty}^{\infty}|H(i\omega)|^2\times|F(i\omega)|^2\,d\omega\right]^2}, \tag{8.6.13}$$

where $H(i\omega)$ — frequency response of the filter;

$V(i\omega)$ — Fourier transform of function $v^2(t)$;

$F(i\omega)$ — Fourier transform of function $u_{a0}(t)v(t)$.

If function $|H(i\omega)|$ is narrow as compared to functions $|V(i\omega)|$ and $|F(i\omega)|$, expression (8.6.13) changes into (8.6.10), and for $v(t) = u_{a0}(t)$ into the expression for $S_{\text{опт}}$ with coefficient b, determined by formula (8.6.11). In general, e.g., for a Gaussian form of pulses, of gate pulses and of the frequency-response curve of filters with passband $\Delta f_\Phi = \Delta f_{УПЧ}$ from formula (8.6.13) it follows that

$$S_{\text{экв}}=\frac{T_{r}\left(1+\frac{m^{2}}{2}\right)}{m^{2}\omega_{m}^{2}q^{2}}\left\{\frac{(1+y)^{2}(1+x^{2}+y^{2})}{4y\sqrt{x^{2}+y^{4}}}+q\frac{(1+y)^{2}(1+x^{2}+y^{2})}{\sqrt{(x^{2}+2y^{2}+2)(x^{2}+2y^{2}+2y^{2}x^{2}+2y^{4})}}\right\}, \quad (8.6.14)$$

where all designations are the same as in Paragraph 8.5.3. This formula permits us to investigate a series of limiting cases. For instance, with expansion of the passband of the filter ($x \to \infty$) and $y \ll 1$

$$S_{\text{экв}}=\frac{T_{r}\left(1+\frac{m^{2}}{2}\right)}{m^{2}\omega_{m}^{2}q^{2}}\left(q+\frac{x}{4y}\right)= \frac{T_{r}\left(1+\frac{m^{2}}{2}\right)}{m^{2}\omega_{m}^{2}q^{2}}(q+0{,}5\Delta f_{\text{УПЧ}}\tau_{c}), \quad (8.6.15)$$

which, with the exception of the proportionality factor, depending on the form of modulation, coincides with the expression for equivalent spectral density of a discriminator for a simple pulse signal with a broad-band UPCh.

If duration of the gate pulse considerably exceeds pulse duration $\left(y = \frac{\tau_{\text{г}}}{\tau_{c}} \ll 1\right)$,

$$S_{\text{экв}}=\frac{T_{r}\left(1+\frac{m^{2}}{2}\right)}{m^{2}\omega_{m}^{2}q^{2}}\left(\frac{1+x^{2}}{4yx}+q\frac{1+x^{2}}{x\sqrt{2+x^{2}}}\right), \quad (8.6.16)$$

and, finally, with a filter which is narrow-band as compared to the width of the spectrum of the pulse formula (8.6.10) is valid, where in this case

$$\frac{q_{\text{экв}}}{q}=\frac{2\frac{\tau_{\text{и}}}{\tau_{c}}}{1+\left(\frac{\tau_{\text{и}}}{\tau_{c}}\right)^{2}}. \quad (8.6.17)$$

From formulas (8.6.14) and (8.6.16) it follows that expansion of the filter passband and of duration of the gate pulse for not very small values of x basically affects the noise term in $S_{\text{экв}}$ (i.e., the term with $1/q^{2}$) and, consequently, is especially undesirable with a small signal-to-noise ratio.

The dependence of $S_{\text{экв}}/S_{\text{опт}}$ on q for various x and y, illustrating the influence of basic parameters of the discriminator on accuracy, is shown in Fig. 8.12.

The scheme of a discriminator with correlation processing of the signal and differentiation is, of course, not unique for the given case. For instance, it is also possible to use a discriminator with a "shortening" filter, which is realized in this case by three i-f amplifiers matched with the spectrum of the pulse envelope, tuned to frequencies $\omega_{\text{пр}}$ and $\omega_{\text{пр}} \pm \omega_{m}$, respectively. Gain factors of the UPCh's tuned to frequencies $\omega_{\text{пр}} \pm \omega_{m}$, should be a factor of m/2 less than the gain factor of the basic UPCh. Output voltages of all three UPCh's are added, detected and fed

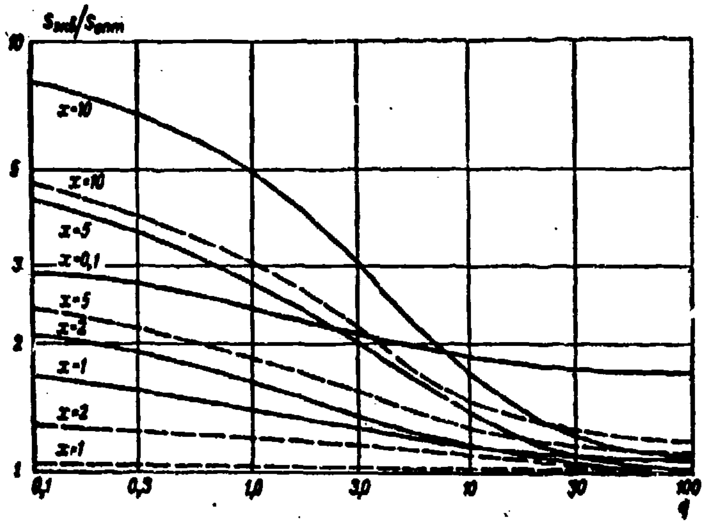

Fig. 8.12. Influence of the passband of the UPCh and duration of the gate pulse on $S_{ЭКВ}$: —— y = 0.3; – – – y = 1.0.

to gating stages. With sufficiently rational selection of parameters the characteristics of this discriminator will be the same as those considered above.

§ 8.7. Frequency Intrapulse Modulation

One of the most studied forms of intrapulse modulation at present is linear frequency modulation [38, 39]. The form of the pulse here, as a rule, is selected symmetric. With the given form of modulation various methods of construction of discriminators may be reasonable from the technical point of view. Let us consider first a discriminator with a shortening filter. The shortening filter for the case of frequency modulation can be made in different ways. The most well-known is use of delay lines with taps [12, 38]. Amplitudes and phases of signals taken from the different taps will be selected in such a manner that as a whole the frequency response of the filter is conjugate with the spectrum of the signal.

Influence of the form of the pulse envelope on accuracy and range resolution capability was already discussed in Chapter VII (see Paragraph 7.8.1); in this case, as follows from general expressions of §§ 8.2, 8.3, and 8.4, in this respect everything remains constant. Therefore, we will be interested basically only in the influence of nonoptimality of processing on accuracy of range finding.

Let us assume first that the shortening filter is optimum. Then for gates which are narrow as compared to the duration of the shortened pulse the equivalent spectral density is determined by expression (8.2.17) for $S_{ОПТ}$. Parameter b in this

formula is determined by the following expressions:

for a square pulse envelope

$$b=\frac{a^2\tau_{\text{и}}^2}{12}=\frac{\omega_m^2}{3}, \qquad (8.7.1)$$

for a Gaussian envelope

$$b=\frac{a^2\tau_{\text{и}}^2}{2\pi}=\frac{2\omega_m^2}{\pi}, \qquad (8.7.2)$$

for a cosinusoidal envelope

$$b=\frac{\pi^2-6}{3\pi^2}a^2\tau_{\text{и}}^2=\frac{4(\pi^2-6)}{3\pi^2}\omega_m^2=0,52\omega_m^2, \qquad (8.7.3)$$

where a — speed of change of frequency within limits of pulse;

$\tau_{\text{и}}$ — effective pulse duration;

$\omega_m = a\tau_{\text{и}}$ — effective frequency deviation for the duration of a pulse.

With finite duration of gates accuracy somewhat worsens. Let us consider, for instance, the case when the pulse envelope has the form of a Gaussian curve. Then, according to (7.3.6) the form of the shortened pulse is also described by a Gaussian curve of form

$$f(t)=e^{-\frac{\omega_m^2 t^2}{\pi}}. \qquad (8.7.4)$$

This means that the considered case with an optimum shortening filter is equivalent to simple pulse modulation by Gaussian pulses with duration

$$\tau_{\text{экв}}=\frac{\pi}{2\omega_m}=\frac{\pi}{a\tau_{\text{и}}} \qquad (8.7.5)$$

and a discriminator with a matched UPCh. Assuming the form of the gates Gaussian (8.5.13), we see that for calculation of $S_{\text{ЭКВ}}$ it is possible to use expression (8.5.20) in which one should set $x = 1$ ($\Delta f_{\text{УПЧ}} = \Delta f_{\text{СТР}}$). Then for the considered discriminator formula (8.5.21) is also valid, corresponding to small detuning between gates; curves of Fig. 8.9 and all conclusions of Paragraph 8.5.3 pertain to the case $x = 1$.

Sometimes in practice one can use a realization of the shortening filter, simpler in certain respects, which ensues directly from the presentation of output voltage of the discriminator in the optimum case. As we already said above, this voltage is determined through the modulus of quantity f_j from (8.2.1), which for any period, when we use optimum shortening filtration, is recorded in the form

$$f(\tau)=\left|\int_{-\infty}^{\tau}u_0(s-\tau)\,y(s)\,e^{j\omega_0 s}\,ds\right|.$$

where moment τ in the upper limit of the integral coincides with delay of the reflected signal, $u_0(t)$ in this case is $u_{a0}(t)e^{iat^2/2}$, a – rate of linear change of frequency.

Then

$$f(\tau)=\left|\int_{-\infty}^{\tau} u_{a0}(s-\tau)\,e^{\frac{ias^2}{2}-ia\tau s+\frac{ia\tau^2}{2}+i\omega_0 s}\,y(s)\,ds\right|=$$

$$=\left|\int_{-\infty}^{\tau} u_{a0}(s-\tau)\,e^{i(\omega_{пр}+a\tau)(\tau-s)}\,y(s)\,e^{i(\omega_0+\omega_{пр})s+i\frac{as^2}{2}}\,ds\right|=$$

$$=\left|\int_{-\infty}^{\tau} h_0(\tau-s)\,e^{i(\omega_{пр}+a\tau)(\tau-s)}\,\eta(s)\,ds\right|. \qquad (8.7.6)$$

This means that the output voltage of the optimum shortening filter at time $t = \tau$ can be formed by heterodyning the received signal $y(t)$ with voltage of a frequency-modulated local oscillator and passing it through a filter with a frequency response matched with the spectrum of the pulse envelope. This filter should be tuned to frequency $\omega_{пр} + a\tau$, depending on the distance interesting us.

In order to ensure full equivalence with the usual shortening filter, forming the integral of $f(\tau)$ for all delays τ_i, we need a unit of identical filters tuned to frequencies $\omega_{пр} + a\tau_i$, and a unit of lagging gate pulses by which there should be carried out tapping of output voltages of filters, each at the corresponding moment τ_i. The set of filters and gate pulses gives at the output a shortened pulse.

Optimality of processing is preserved, of course, only when the pulse response of the filter of the UPCh satisfies the condition of matching $h_1(t) = h_0(t) = u_{a0}(-t)$, and gate pulses are narrow as compared to the duration of the shortened pulse. Such a realization of a shortening filter may be more convenient technically when the radar is intended for simultaneous work in a small range of distances. Then the number of filters can be comparatively small; gate pulses, if the usual condition of small duration of the shortened pulse as compared to the duration of the sounding pulse envelope is satisfied, can be fed simultaneously, and voltage of the local oscillator can also be pulsed. Here it is necessary only that the pulse duration of the local oscillator be sufficient for complete overlapping of the signal pulse. The possibility of using pulse mode in the local oscillator frees us from the necessity of providing too large deviations of its frequency.

For realization of a discriminator sufficient are two channels, tuned to frequencies $\omega_{пр} + a\tau \pm a\delta$, where τ – measured value of delay. For tuning a range

discriminator we can use change of the intermediate frequency, so that $\omega_{\Gamma 1} = \omega_{\Gamma} + a\tau$. From this point of view such a discriminator is like a discriminator of correlation type in which the reference signal is changed in accordance with change of the measured value of delay. In this case the change consists not in shift of the reference signal, but in change of the local oscillator frequency. The block diagram of the discriminator is shown in Fig. 8.13.

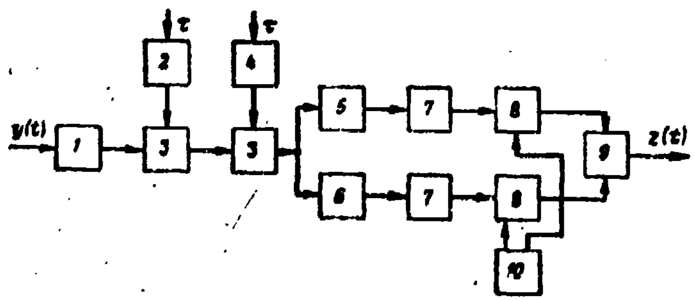

Fig. 8.13. Block diagram of a discriminator for frequency-modulated pulses. 1 — preamplifier; 2 — amplitude- and frequency-modulated local oscillator with controlled delay of amplitude modulation; 3 — mixer; 4 — local oscillator with controlled frequency; 5 — UPCh with frequency of tuning $\omega_{\Pi P} - a\delta$; 6 — UPCh with frequency of tuning $\omega_{\Pi P} + a\delta$; 7 — detector; 8 — gated amplifier; 9 — subtractor; 10 — controlled generator of gate pulses.

To find characteristics of the discriminator we can use general formulas of § 8.4. We shall limit ourselves to the case of narrow gate pulses, but we shall consider the UPCh frequency response $H_1(i\omega)$ arbitrary. From comparison of (8.4.1) with (8.7.6) it follows that the complex pulse response of the filter $h(t)$ in this case should be set equal to

$$h(t) = h_1(t)\, e^{\frac{iat^2}{2}}. \tag{8.7.7}$$

Then, assuming duration of the pulse envelope great as compared to duration of the shortened pulse, we obtain

$$a(\delta) = \frac{1}{T_r} f(\delta) = \frac{1}{T_r}\int_{-\infty}^{\infty} h(t+\delta)\, u_0^*(t)\, dt \approx \frac{1}{T_r}\int_{-\infty}^{\infty} h_1(t)\, u_{a0}(t)\, e^{ia\delta t} dt, \tag{8.7.8}$$

$$b(\delta) = \frac{1}{T_r}\int_{-\infty}^{\infty} h^*(t)\, h(t+\delta)\, dt \approx \int_{-\infty}^{\infty} h_1^2(t)\, e^{ia\delta t} dt \tag{8.7.9}$$

and the expression for equivalent spectral density will take the form

$$S_{\text{экв}}=T_{r}\frac{b^{2}(0)-|b(2\delta)|^{2}+2q\left[b(0)\,|a(\delta)|^{2}-\operatorname{Re}a^{2}(\delta)\,b(2\delta)\right]}{8q^{2}\left[\operatorname{Re}a^{*}(\delta)\,a'(\delta)\right]^{2}}. \tag{8.7.10}$$

In particular, for a Gaussian form of the pulse and of frequency-response curves of the UPCh

$$a(\delta)=\sqrt{\frac{2}{1+4\Delta f^{2}_{\text{УПЧ}}\tau^{2}_{\text{и}}}}\,e^{-\frac{a^{2}\delta^{2}\tau^{2}_{\text{и}}}{2\pi\left(1+4\Delta f^{2}_{\text{УПЧ}}\tau^{2}_{\text{и}}\right)}}=\sqrt{\frac{2}{1+x^{2}}}\,e^{-\frac{2\omega_{m}^{2}\delta^{2}}{\pi(1+x^{2})}}, \tag{8.7.11}$$

$$b(\delta)=\frac{1}{2\Delta f_{\text{УПЧ}}\tau_{\text{и}}}\,e^{-\frac{a^{2}\delta^{2}}{16\pi\Delta f^{2}_{\text{УПЧ}}}}=\frac{1}{x}\,e^{-\frac{\omega_{m}^{2}\delta^{2}}{\pi x^{2}}}, \tag{8.7.12}$$

where, as before, $x = \Delta f_{\text{УПЧ}}/\Delta f_{\text{согл}} = 2\Delta f_{\text{УПЧ}}\,\tau_{\text{и}}$, and $\omega_{m} = a\tau_{\text{и}}/2$.

Then according to (8.7.10) equivalent spectral density is equal to

$$S_{\text{экв}}=\frac{T_{r}\pi}{2\omega_{m}^{2}}\,\frac{(1+x^{2})^{4}\,e^{\frac{4z^{2}}{1+x^{2}}}}{128z^{2}}\times\left\{\frac{1-e^{-\frac{4z^{2}}{x^{2}}}}{x^{2}}+\frac{4qe^{-\frac{2z^{2}}{1+x^{2}}}}{x(1+x^{2})}\left(1-e^{-\frac{2z^{2}}{x^{2}}}\right)\right\}, \tag{8.7.13}$$

where $z = \sqrt{b}\,\delta = \left(\sqrt{2\omega_{m}^{2}/\pi}\right)\delta$ – relative magnitude of detuning.

When $x = 1$ and $z \to 0$ formula (8.7.13) changes into the expression for $S_{\text{опт}}$ with coefficient b (from 8.7.2). As also in other cases, the dependence on z is relatively weak; practically up to values $z/x \sim 1$ in formula (8.7.13) it is possible to consider detuning zero, especially with a sufficiently large signal-to-noise ratio. The exact dependence of $S_{\text{экв}}/S_{\text{опт}}$ on detuning with a matched band ($x = 1$) is shown in Fig. 8.14, from which it is clear that increase of detuning to a value providing a maximum gain factor of the discriminator ($z \approx 0.7$) increases $S_{\text{экв}}$ even when $q \sim 0.1$ by not more than 15%. The influence of mismatching of the band of the UPCh for various q is illustrated in Fig. 8.15. This dependence turns out to be rather strong – expansion of the passband in the considered circuit leads, probably, to more unpleasant consequences than in any other one.

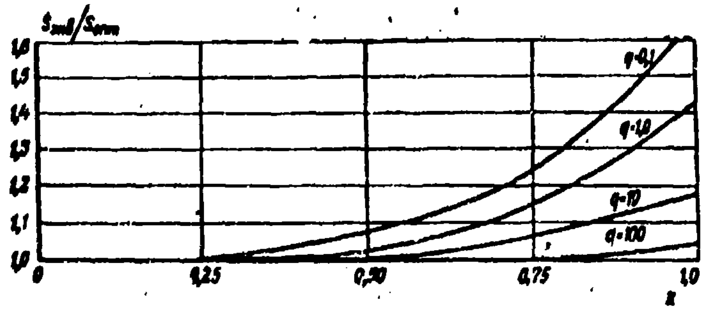

Fig. 8.14. Influence of detuning on equivalent spectral density in an FM range finder.

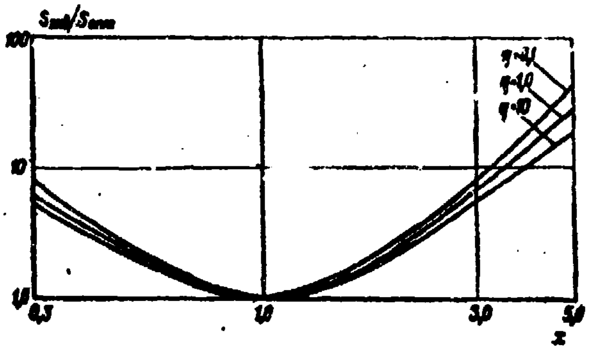

Fig. 8.15. Influence of mismatch of the passband of the UPCh on $S_{\text{ЭКВ}}$ in an FM range finder.

The block diagram of Fig. 8.13 allows for a definite modification, in its essence corresponding to correlation processing of the received signal. With such processing the received signal before being sent to the filters should be multiplied by a gate pulse approximately matched with the signal pulse envelope. This can be attained, for instance, if voltage of the FM local oscillator is modulated in amplitude by a certain pulse v(t) with duration of the order of duration of the sounding pulse.

Filters of the UPCh in this case need not be matched with the width of the spectrum of the pulse envelope; conversely, operation of the discriminator is nearer to optimum, the less the magnitude of $\Delta f_{\text{УПЧ}} \tau_{\text{И}}$. Gating of output voltages of the UPCh also is not necessary; these output voltages should be detected and subtracted. Changes in the block diagram of Fig. 8.13 reduce in this case to modulating the local oscillator in amplitude, and having no gating stages in general. Such a scheme is simpler in that there is no need to carry out control of the narrow gate pulses fed to gating stages; however, in place of this there is required control of the position of the local oscillator pulse.

Thus, in both cases there are required two control circuits — for delay and for frequency — but in the second case requirements on the circuits for control of the reference signal in delay and on accuracy of this control are considerably weaker.

Optimum processing in the given circuit is attained when the form of the pulse envelope of the local oscillator coincides with the form of the sounding pulse envelope and bandwidth of the filters of the UPCh satisfies relationships $1/\tau_{\text{И}} \gg \Delta f_{\text{УПЧ}} \gg 1/T_r$. If the last condition is satisfied, and the shape of the pulse of the local oscillator v(t) differs from $u_{a0}(t)$, the equivalent spectral density, as before, is given by formula (8.7.10), where in expressions (8.7.8) and (8.7.9) for a(δ) and b(δ) one should replace pulse response of the filter $h_1(t)$ by a function describing the form of the local oscillator pulse v(t).

In the more general case, when both conditions of optimality are not satisfied, there is dependence of $S_{\text{ЭКВ}}$ on the form and parameters of the frequency response of the filter of the UPCh. The expression for equivalent spectral density here can be obtained by normal means and has the form

$$S_{экв}=\frac{T_r}{2q^2}\left\{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\{|V(i\omega_1)|^2|H(i\omega_2)|^2\left[\left|H(i(\omega_1-\omega_2))\right|^2-\right.\right.$$
$$\left.-\left|H(i(\omega_1-\omega_2-2a\delta))\right|^2\right]+$$
$$+2qF(i\omega_1)F(i\omega_2)V^*(i(\omega_1+\omega_2))|H(i(\omega_1-a\delta))|\times$$
$$\left.\times\left[\left|H(i(\omega_2+a\delta))\right|^2-\left|H(i(\omega_2-a\delta))\right|^2\right]\}d\omega_1 d\omega_2\right\}:$$
$$:\left[\frac{\partial}{\partial\delta}\int\limits_{-\infty}^{\infty}|F(i\omega)|^2\left|H(i(\omega-a\delta))\right|^2 d\omega\right]^2, \qquad (8.7.14)$$

where, just as in § 8.6, $V(i\omega)$ – Fourier transform of function $v^2(t)$, and $F(i\omega)$ – the transform of function $v(t)u_{a0}(t)$.

Let us give final results for a Gaussian form of the pulse envelope, of the pulse envelope of the local oscillator $v(t)$ and of frequency responses of the UPCh. Substituting the corresponding functions in (8.7.14), we obt

$$S_{экв}=\frac{T_r\pi(1+y^2)(1+y^2+x^2)}{2\omega_m^2\cdot 128z^2q^2}\cdot e^{\frac{4z^2}{1+x^2+y^2}}\times\left\{\frac{1-\exp\left\{-\frac{4z^2}{x^2+y^2}\right\}}{y\sqrt{x^2+y^2}}+\right.$$
$$+\frac{8q\exp\left\{-\frac{4z^2}{x^2+2(1+y^2)}\right\}}{\sqrt{[x^2+2(1+y^2)][x^2+2y^2(1+x^2+y^2)]}}\times$$
$$\left.\times\left[1-\exp\left\{-\frac{8z^2(1+y^2)^2}{[x^2+2(1+y^2)][x^2+2y^2(1+x^2+y^2)]}\right\}\right]\right\}, \qquad (8.7.15)$$

where $x = 2\Delta f_{УПЧ}\tau_И$; $z = \sqrt{2\omega_m^2\delta^2/\pi}$; $y = \tau_И/\tau_{ПС}$.

As also in other cases, this expression weakly depends on detuning z, only slightly increasing with increase of it. Therefore, the magnitude of z should be selected such as to ensure a maximum discriminator gain factor, which corresponds to values of $z \sim 0.6$ to 0.8. In place of formula (8.7.15) here it is possible with error not exceeding 25-30% to use the expression for $S_{ЭКВ}$ when $z = 0$, which has the form

$$S_{экв}=\frac{T_r\pi(1+y^2)(1+y^2+x^2)^2}{64\omega_m^2q^2}\left\{\frac{1}{y(x^2+y^2)^{3/2}}+\right.$$
$$\left.+\frac{16q(1+y^2)^2}{[x^2+2(1+y^2)]^{3/2}[x^2+2y^2(1+x^2+y^2)]^{3/2}}\right\} \qquad (8.7.16)$$

and is considerably more convenient for practical calculation.

When $y = 1$, $z \to 0$, $x \to 0$ the quantity $S_{ЭКВ}$ changes into $S_{ОПТ}$. Mismatch of durations of the sounding pulse and of the local oscillator pulse leads to the same consequences as in a circuit with amplitude modulation. Bandwidth of the UPCh for $\Delta f_{УПЧ} \leq \Delta f_{СОГЛ}$ practically does not affect the magnitude of $S_{ЭКВ}$, and further expansion

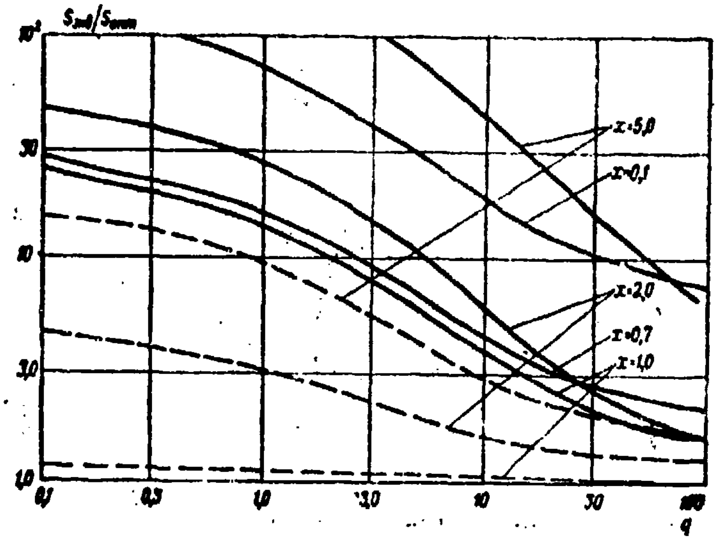

Fig. 8.16. Dependence of $S_{ЭКВ}/S_{ОПТ}$ on q for a range finder with frequency modulation: —— y = = 0.3; – – – y = 1.

of the band leads to increase of $S_{ЭКВ}$ basically due to the noise term. Curves of the dependence of $S_{ЭКВ}/S_{ОПТ}$ on q for various values of x and y, illustrating the influence of parameters on accuracy of measurement, are shown in Fig. 8.16.

§ 8.8. Phase-Code Intrapulse Manipulation

With incoherent radiation there can also be used phase-code manipulation inside the pulse. Questions of selection of codes and general properties of the autocorrelation function of the phase-code-manipulated pulse signal were already discussed in Chapter I, and measurement of distance by the delay of such signals in the coherent case was considered in Chapter VII. Here it remains to add only very little to what has already been said above.

The optimum discriminator in this case can be realized both by correlation processing, and also by a shortening filter [12]. One possible block diagram of a discriminator with a shortening filter is shown in Fig. 8.17. The received signal after conversion of frequency and transmission through a UPCh, matched in bandwidth with duration of the code interval τ_k is fed to a delay line with taps, corresponding to delay $k\tau_k$ (k = 1, 2, ..., n). Delayed signals are fed to an adder with a plus or minus sign depending upon what value (0 or π) the phase of the signal in the corresponding code interval has. At the output of the adder there will form a pulse which is then processed just as in a discriminator for a normal pulse signal. Such a circuit is completely equivalent to the usual pulse range finder (see § 8.5),

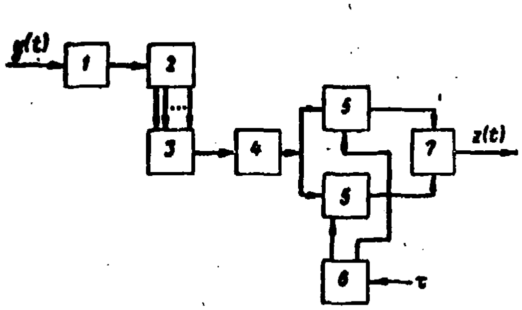

Fig. 8.17. Block diagram of a discriminator with PCM and a shortening filter: 1 — filter matched with τ_k; 2 — delay line with taps; 3 — adder; 4 — detector; 5 — gating stage; 6 — controlled generator of gate pulses; 7 — subtractor.

working on a square pulse of duration τ_k. Therefore, for calculation of characteristics of the range finder it is possible to use all the findings of Paragraph 8.5.2.

Characteristics of discriminators with correlation processing, for instance, with two detuned channels or with switching of reference signals, also actually were already found above. It is necessary only to use formulas relating to the case of a square pulse everywhere replacing its duration by the duration of the code interval τ_k. Here one should consider that at present in examining possibilities of using phase-code manipulation with a pulse signal they almost exclusively consider a pulse with a square envelope.

This is explained, first, by the simplicity of technical realization of circuits for processing a signal with a square pulse and, second, by the fact that for an assigned peak power of the signal selection of a square envelope ensures maximum pulse energy. At the same time with a square envelope in discriminators of correlation type it is possible without special difficulties to ensure coincidence of the reference and sounding signals. Therefore it is possible to consider only the case of coinciding modulations of the sounding and reference signals. Here in the formulas determining the discriminator characteristics $C_{1k}(x) = C_{00}(x) = C(x)$, i.e., the cross-correlation functions can be assumed equal to the autocorrelation functions of the sounding signal. This circumstance simplifies calculations.

The above investigation of discriminators of incoherent radar range finders shows that, as for a coherent signal, accuracy of range finding is determined by characteristics of modulation of the sounding signal and the magnitude of the signal-to-noise ratio. For a sufficiently large signal-to-noise ratio and other conditions being equal accuracy of range finding is not worse than in the case of a coherent signal. For small signal-to-noise ratios there is obtained a loss, precisely depending on relationship $\Delta f_c / f_r = \Delta f_c T_r$. The dependence of accuracy on characteristics of modulation of the sounding signal, on the form of the reference signals and the magnitude of detuning in discriminators using correlation processing also is identical in both cases. Mismatch of characteristics of "shortening" filters with the spectrum of the sounding signal in incoherent discriminators leads to the same

consequences as noncoincidence of modulations of reference and sounding signals in correlation discriminators.

Discriminator characteristic found above can be directly used for analysis of tracking range finders as a whole with any smoothing circuits.

§ 8.9. Analysis of Accuracy of Incoherent Range Finders

As was shown above, the problem of analysis of the accuracy of tracking meters is solved by a single method regardless of what magnitude is measured and how the discriminator of the meter is constructed. If characteristics of the discriminator are found, it remains only to use general formulas of Chapter VI, substituting in them the corresponding expressions for characteristics of the discriminator. According to this the whole problem of analysis of accuracy of incoherent range finders reduces to simple replacement in formulas of § 7.10, characterizing error of measurement, of the expressions for $K_{Д}$ and $S_{ЭКВ}$ found for coherent discriminators by the corresponding expressions for $K_{Д}$ and $S_{ЭКВ}$ which were found in the preceding paragraphs for incoherent discriminators. This means that all laws governing accuracy of range finding which were investigated in § 7.10 are completely preserved and do not need repeated discussion. Therefore, we shall limit ourselves in this chapter only to consideration of a series of illustrative examples.

We shall discuss the influence of Automatic Gain Control (AGC) on accuracy of real meters. In incoherent range finders, in general, there are the same regularities caused by the normalizing action of the AGC system; however, the concrete form of the dependence of discriminator gain on the signal-to-noise ratio is somewhat different. In discriminators with correlation processing of the signal the AGC system should be closed from the output of the integrating filter. Then the average power of output voltage of a receiver with AGC is

$$K_1^2(P_0+2N_0\Delta f_\Phi)$$

and discriminator gain varies as

$$K_{Д}=K_{Д0}\frac{1}{1+\frac{2N_0\Delta f_\Phi}{P_0}}=K_{Д0}\frac{1}{1+\frac{\Delta f_\Phi T_r}{q}}, \qquad (8.9.1)$$

where, as before $K_{Д0}$ — the gain factor of the discriminator in the absence of noises.

In discriminators with a "shortening" filter the AGC system is closed from the output of the detector, which follows after the "shortening" filter, where to decrease noise components in the control voltage of the system of AGC output voltage of the detector is gated by a pulse travelling after the pulse of the signal. In

practice gating is frequently carried out directly in the "shortening" filter — in normal pulse receivers, as a rule, the UPCh is gated. Duration of the gate pulse here is several times greater than the duration of the "shortened" pulse. Gating leads to decrease of intensity of noise with respect to τ_c/T_r, and thanks to this in such discriminators

$$K_{д}=K_{д0}\frac{1}{1+\frac{\Delta f_{упч}\tau_c}{q}}, \tag{8.9.2}$$

where τ_c — duration of the gating pulse;

$\Delta f_{УПЧ}$ — bandwidth of the "shortening" filter (UPCh without intrapulse modulation).

If in the correlation discriminator the passband of the filter $\Delta f_{ф}$ is great, to decrease noise components of the control voltage in this case, too, it is possible to use gating of the output voltage of the filter. Formulas (8.9.1) and (8.9.2), obviously, can be reduced to the form of (7.10.4) where quantity y is equal to $\Delta f_{ф}/\Delta f_c$ and $\Delta f_{УПЧ}\tau_c/\Delta f_c T_r$, respectively; however, in the given case it is more convenient to consider the dependence of $K_Д$ on q. Both formulas (8.9.1) and (8.9.2) can be recorded in a single form

$$K_{д}=K_{д0}\frac{q}{q+y_1}, \tag{8.9.3}$$

where $y_1 = \Delta f_{ф} T_r$ or, accordingly, $y_1 = \Delta f_{УПЧ}\tau_c$.

Thanks to this the dependence of the effective bandwidth of closed-loop tracking systems with incoherent discriminators and smoothing filters with constant parameters on the signal-to-noise ratio q coincides with the dependence of effective bandwidth on signal-to-noise ratio h in coherent range finders (see Paragraph 7.10.2).

Let us consider a range finder with a smoothing circuit in the form of a single integrator with gain factor $K_И$. We consider that measurement of distance is carried out by a pulse signal without additional modulation. Let us assume that the discriminator of the range finder has a UPCh, matched with pulse duration $\tau_И$, and gates are located end-to-end and have duration $\tau_{ПС} = \tau_И$. Then, according to the formula (8.5.10), equivalent spectral density is

$$S_{экв}=T_r\tau_и^2\left(\frac{0,09}{q}+\frac{0,21}{q^2}\right). \tag{8.9.4}$$

The dimensional gain factor of the open loop of a tracking meter is

$$K=K_дK_и=K_{д0}K_и\frac{q}{q+y_1}=K_0\frac{q}{q+y_1}, \tag{8.9.5}$$

where K_0 – nominal value of the gain factor in the absence of noises, and y_1 in this case is $\Delta f_{\text{упч}} \tau_0$.

Since the effective passband of the closed-loop tracking system $\Delta f_{\text{эф}}$ is equal to $K/4$, fluctuation error of measurement is determined by expression

$$\sigma^2_{\phi д} = 2\Delta f_{эф} S_{экв} = K_0 T_r \tau_и^2 \left(\frac{2,25\cdot 10^{-2}}{q+y_1} + \frac{5,25\cdot 10^{-2}}{q(q+y_1)}\right) = \\ = \Delta f_{эф0} T_r \tau_и^2 \cdot 10^{-2} \left(\frac{9}{q+y_1} + \frac{21}{q(q+y_1)}\right), \qquad (8.9.6)$$

where $\Delta f_{\text{эф}\ 0} = K_0/4$ – nominal value of the effective passband of the closed-loop tracking system.

The dependence of the relative magnitude of fluctuation error $\sigma_{\phi д}/\tau_и \sqrt{\Delta f_{эф\ 0} T_r}$ on q for various y_1 is shown in Fig. 8.13. With an assigned value of $\Delta f_{эф\ 0}$ fluctuation error decreases with increase of y_1; this corresponds to the fact that with increase of y_1 decrease of q leads to still more considerable narrowing of the effective bandwidth. The influence of y_1 is more essential, the less the magnitude of the signal-to-noise ratio. For large q change of y_1 does not lead to noticeable change of fluctuation error. The magnitude of product $\Delta f_{эф\ 0} T_r$ is usually 10^{-3} to 10^{-1}; then error of measurement may comprise tenths and hundredths of the pulse duration.

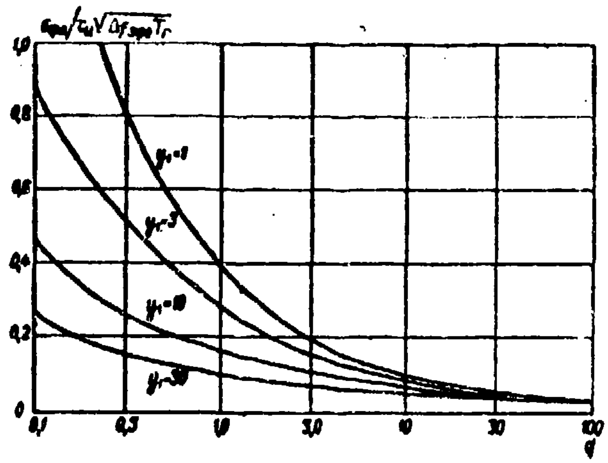

Fig. 8.13. Fluctuation error of range finding in a system with one integrator.

Dynamic error, corresponding to a nonstatistical approach (see § 7.10), in the considered case in accordance with (7.10.29) is equal to

$$\sigma_д = \frac{a_1}{K} = \frac{a_1}{K_0}\left(1 + \frac{y_1}{q}\right) = \frac{a_1}{4\Delta f_{эф0}}\left(1 + \frac{y_1}{q}\right), \qquad (8.9.7)$$

where a_1 – speed of the target.

Dynamic error increases with decrease of the signal-to-noise ratio, where this increase is stronger, the larger y_1.

Let us consider now the example of a range finder with a smoothing filter in the form of a double integrator with correction, whose transfer function is determined by expression (7.10.11). Let us assume that measurement of distance is produced with the help of a pulse signal with a Gaussian envelope and intrapulse

frequency modulation with frequency deviation $2\omega_m$. We consider that the discriminator is close to optimum, so that

$$S_{экв} = S_{опт} = \frac{T_r \pi (1+q)}{4\omega_m^2 q^2}. \qquad (8.9.8)$$

Assuming in the circuit of the discriminator the presence of an AGC system, for the effective bandwidth we obtain the following expression:

$$\Delta f_{эф} = \frac{1 + K_0 T_к^2 \frac{q}{q+y_1}}{4T_к}, \qquad (8.9.9)$$

where K_0 — as before, the gain factor of the open system in the absence of noises ($K_0 = K_{ч} K_{д\,0}$ has dimensionality of $1/\sec^2$).

We saw in Paragraph 7.10.3 that the minimum of errors of measurement which is obtained in an optimum system with double integrator is attained under condition $KT_к^2 = 2\{K = K_0[q/(q + y_1)]\}$. Consider that this condition is satisfied for a certain value of q_0, i.e.,

$$K_0 T_к^2 \frac{q_0}{q_0 + y_1} = 2.$$

Then

$$\Delta f_{эф} = \frac{1}{4T_к}\left[1 + 2\frac{1 + \frac{y_1}{q_0}}{1 + \frac{y_1}{q}}\right]. \qquad (8.9.10)$$

Using formula (8.9.8) and (8.9.10), for variance of fluctuation error we obtain the following expression:

$$\sigma^2_{фл} = \frac{T_r \pi}{8\omega_m^2 T_к} \cdot \frac{1+q}{q^2}\left[1 + 2\frac{1 + \frac{y_1}{q_0}}{1 + \frac{y_1}{q}}\right].$$

The dependence of the relative magnitude of error $\sigma_{фл}/\sqrt{T_r \pi / 8\omega_m^2 T_к}$ on the signal-to-noise ratio q is shown in Fig. 8.19 for different y_1 and q_0. Investigation of this dependence shows that fluctuation error of measurement with a fixed magnitude of $T_к$ rather essentially depends on selection of q_0 and y_1. It turns out that for fixed q_0 error increases with increase of y_1 if $q > q_0$, and decreases if $q < q_0$.

On the whole fluctuation error for any q and y_1 decreases with increase of q_0; first this decrease is rather fast (with change of q_0 from 0.1 to 1, $\sigma_{фл}$ decreases for various q and y_1 by a factor of 1.5-3), and then slows, so that increase of q_0 above values $q_0 \sim 10$ no longer practically leads to change of error for values of y_1 considered in the example. More exactly, dependence on q_0 disappears if $y_1 \ll q_0$.

Decrease of $\sigma^2_{\text{фл}}$ with increase of q for values of q_0 which are not very small is somewhat slower than analogous decrease of equivalent spectral density. In order to obtain an idea about absolute values of fluctuation error, we consider the example when frequency deviation is $2\omega_m = 2\pi \cdot 10^6$ rad/sec, frequency of repetition of pulses is $f_r = 1/T_r = 100$ cps, and the time constant of the correcting circuit T_K = 0.1 s. Then the scale factor $\frac{c}{2}\sqrt{\pi T_r / 8\omega_m^2 T_K}$ is equal to 9.4 m, and fluctuation error, corresponding to Fig. 8.19, will be in the range 170 to 1.7 m.

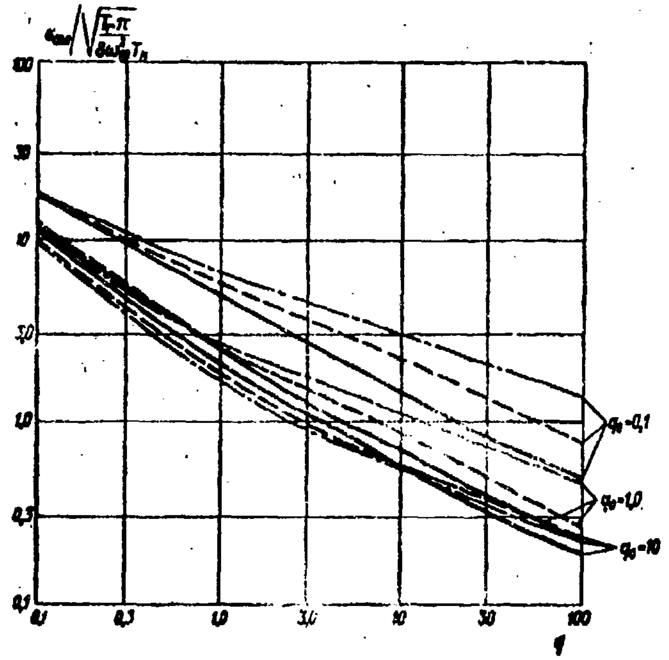

Fig. 8.19. Fluctuation error of range finding in a system with a double integrator: —— y_1 = 1; – – – – y_1 = 3; —·—·— y_1 = 10.

Let us consider now the dependence of dynamic error on the signal-to-noise ratio. With a smoothing filter with two integrators error in speed in steady-state operating conditions is absent, and stationary error is

$$\sigma_{\text{д}} = \frac{a_2}{K} = \frac{a_2 (q + y_1)}{K_0 q} = a_2 T^2_{\kappa} \frac{q_0 (q + y_1)}{q (q_0 + y_1)}. \qquad (8.9.11)$$

In formula (8.9.11) a_2 — acceleration of the target, and it is assumed that condition KT^2_K [sic] is satisfied when $q = q_0$.

The dependence of the relative magnitude of dynamic error $\sigma_{\text{Д}} / a_2 T^2_K$ on q for various q_0 and y_1 is shown in Fig. 8.20. For all q_0 and y_1 dynamic error decreases with increase of q, where this decrease is rather sharp.

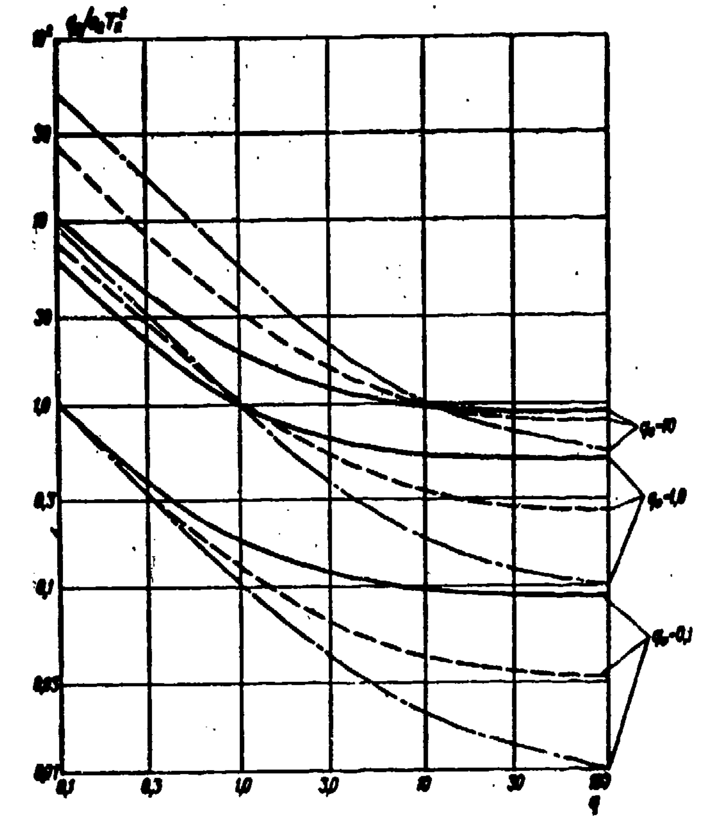

Fig. 8.20. Dynamic error of range finding in a system with a double integrator: —— $y_1 = 1$; – – – $y_1 = 3$; —·— $y_1 = 10$.

Dependence of $\sigma_Д$ on q_0 and y_1 has a sharper character than in the case of $\sigma_{фл}$. For all y_1 and q quantity $\sigma_Д$ decreases with decrease of q_0. The dependence on y_1 is such that for $q < q_0$ quantity $\sigma_Д$ increases with increase of y_1, and when $q > q_0$ it decreases with increase of y_1. Since for large q dynamic error is thus small, for its decrease for small q we must select sufficiently small values of y_1, the more so since the quantity y_1 comparatively weakly affects the magnitude of fluctuating error.

Selection of small values of q_0 is impermissible from the point of view of quantity $\sigma_{фл}$; obviously, an acceptable compromise in many cases will be selection of a magnitude of q_0 of the order of 1, and y_1 of the order of 2-5. If, as above, $T_К$ = 0.1 sec, at a_2 = 100 m/sec^2 product $a_2 T_К$ is 1 m, and dynamic error in accordance with Fig. 8.20 will not exceed 50 m in the most unfavorable case.

Let us consider one more example, when the measured distance varies as a polynomial of degree n, i.e.,

$$d(t)=\sum_{k=0}^{n}\mu_k t^k, \tag{8.9.12}$$

and in the range finder there is used an optimum smoothing filter, the structure of which is determined by expression (7.10.44), i.e., a filter with pulse response

$$G(t,\ \tau)=\sum_{i,\ k=0}^{n} A_{ik}(\tau)\, t^i \tau^k, \tag{8.9.13}$$

where $A_{ik}(\tau)$ – matrix elements;

$$\mathbf{A}(\tau)=\|A_{ik}(\tau)\|=[\mathbf{M}^{-1}+\mathbf{U}(\tau)]^{-1}; \tag{8.9.14}$$

$\mathbf{M} = \|\overline{\mu_i\mu_k}\|$ – matrix of second moments for the coefficients of the polynomial $(\overline{\mu_k} = 0)$;

$$\mathbf{U}(\tau)=\|u_{ik}(\tau)\|=\left\|\int_0^{\tau}\frac{t^{i+k}dt}{S_{\text{вхв}}(q)}\right\|=\frac{1}{S_{\text{вхв}}(q)}\left\|\frac{\tau^{i+k+1}}{i+k+1}\right\|. \tag{8.9.15}$$

For a sufficiently large time of measurement

$$\mathbf{A}(\tau)\approx\mathbf{U}^{-1}(\tau), \tag{8.9.16}$$

where this equality is fulfilled faster, the greater the uncertainty in knowledge of coefficients of the polynomial; the condition of validity of this approximate equality is

$$\sum_{i,\ k=0}^{n} M_{ik}\tau^{i+k}=\sum_{i,\ k=0}^{n}\overline{\mu_i\mu_k}\,\tau^{i+k}\gg\frac{S_{\text{вхв}}}{\tau}. \tag{8.9.17}$$

Then, from (8.9.15) it follows that

$$A_{ik}(\tau)=S_{\text{вхв}}(q)\frac{a_{ik}}{\tau^{i+k+1}}, \tag{8.9.18}$$

where a_{ik} – elements of a matrix which is not time-dependent and is the reciprocal of matrix

$$\mathbf{a}^{-1}=\left\|\frac{1}{i+k+1}\right\|. \tag{8.9.19}$$

As it was shown in Chapter VI, error of measurement is equal to $G(t,\ t)$; therefore in this case for large t we have the following equality:

$$\sigma^2_{\text{вых}}(t)=\sum_{i,\ k=0}^{n} A_{ik}(t)\, t^{i+k}\approx\frac{S_{\text{вхв}}(q)}{t}\sum_{i,\ k=0}^{n} a_{ik}. \tag{8.9.20}$$

It is possible to show that for matrix **a**, defined by relationship (8.9.19), the sum of all its elements $\sum_{i,k=0}^{n} a_{ik}$ is equal to $(n + 1)^2$; therefore the final expression for variance of error of range finding has the form

$$\sigma^2_{\text{вых}}(t) = \frac{(n+1)^2}{t} S_{\text{экв}}(q), \tag{8.9.21}$$

so that $\sigma^2_{\text{ВЫХ}}$ with an accuracy of coefficient $(n + 1)^2/t$ coincides with the equivalent spectral density. Error decreases without limit with time, and for any fixed t it decreases more, the higher the degree of the polynomial. For small t expression (8.9.21) gives an overstated value of error.

We shall touch on nonlinear phenomena in incoherent tracking range finders. It is obvious that in the framework of the theory of these phenomena developed in Chapter VI incoherence introduces nothing new. Actually, with the idealizations used in Chapters VI and VII characteristics of breakoff of tracking and variance of error, taking into account nonlinearity of the discriminator, are completely determined by the form of the discrimination characteristic and the magnitude of fluctuation error in a linearized system, where the dependence on the detailed form of the discrimination characteristic is not very essential.

In incoherent range finders, sufficiently close in their structure to optimum, the form of the discrimination characteristic, as before, is determined by the autocorrelation function of the sounding signal, i.e., remains the same as in the case of a coherent range finder. Therefore, from this point of view nothing changes.

Variance of fluctuation error of a linearized system is completely determined by the equivalent spectral density. This means that the whole difference reduces to corresponding replacement of some expressions for $S_{\text{ЭКВ}}$ by others. Taking this circumstance into account, for determination of the average time to breakoff of tracking and of variance of fluctuation error of range finding we can use formulas (7.15.3)-(7.15.5), (7.15.7)-(7.15.10).

Most interesting is the question of the critical magnitude of the signal-to-noise ratio, at which still it is possible to ignore nonlinearity of the discriminator.

As it was shown in § 7.15 the critical magnitude of the signal-to-noise ratio is determined by the critical value of parameter μ^2, equal to the product of variance of fluctuation error of a linearized system $\sigma^2_{\text{Л}}$ and the mean square width of the spectrum of modulation of the sounding signal, i.e., $\mu^2 = b\sigma^2_{\text{Л}}$. The critical value

of μ varies somewhat, depending upon determination of the limit of applicability of the linearized consideration — by average time to the first breakoff of tracking or by the magnitude of variance of fluctuation error taking into account nonlinearity. However, as shown in § 7.15, this difference leads only to slight change of the critical value of the signal-to-noise ratio. Oriented to the worst case, it is possible to consider (see § 7.15) that $\mu_{кр} \approx 0.12$ to 0.15. Then with a discriminator sufficiently close from the point of view of the magnitude of equivalent spectral density to optimum the critical value of the signal-to-noise ratio will be determined from equation

$$2S_{экв}(q_{кр})\,\Delta f_{эф} b = \frac{T_r \Delta f_{эф}(1+q_{кр})}{q_{кр}^2} = \mu_{кр}^2 \approx 0,02, \qquad (8.9.22)$$

which coincides with the corresponding equation for $h_{кр}$ with a square spectrum of fluctuations and replacement of Δf_c by $1/T_r$. Solving equation (8.9.22), we obtain

$$q_{кр} = 25\left[T_r \Delta f_{эф} + \sqrt{(T_r \Delta f_{эф})^2 + 0,02 T_r \Delta f_{эф}}\right]. \qquad (8.9.23)$$

This dependence is shown in Fig. 8.21. From this figure it follows that linear conditions in a tracking meter are preserved for magnitudes of signal-to-noise ratio q exceeding 0.03-10, depending on inertia of the tracking system and the pulse repetition period.

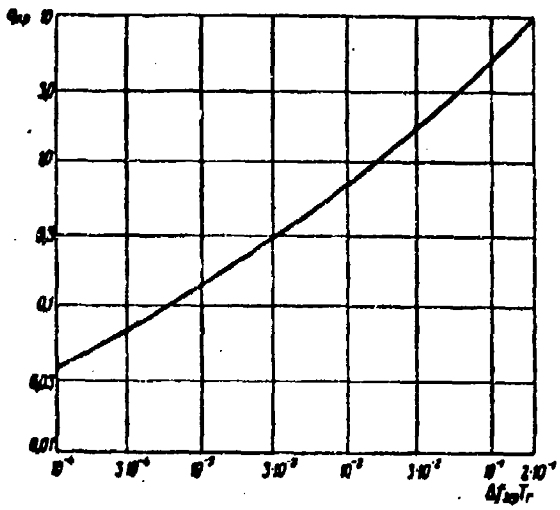

Fig. 8.21. Dependence of q_{cr} on $\Delta f_c\ T_r$.

§ 8.10. Influence of Interferences on Incoherent Range Finders

The influence of interferences on incoherent tracking range finders [69-70] leads to results which in many respects are similar to the case of coherent range finders. This especially pertains to active interferences. The character of the influence of passive interferences in this case is somewhat different, and noise immunity of incoherent range finders with respect to it is lower. We shall consider these questions in greater detail.

8.10.1. Active Noise Interference

As also in the case of coherent radars, the influence of active noise interference, due to its broadband nature as compared to the width of the spectrum of

the sounding signal, is equivalent to the influence of natural noises of the receiver. Therefore, the presence of noise interference can be accounted for by introduction of a new signal-to-noise ratio $q_{\text{пш}}$, equal to the ratio of the energy of the signal received for a period to the sum of spectral densities of natural noise and interference at the input of the receiver of the radar. This quantity is connected with the signal-to-noise ratio q in the absence of interferences by an expression similar to (7.14.1) and (7.14.3), i.e.,

$$q_{\text{пш}} = q\frac{1}{1+\frac{N_{\text{п}}}{N_0}} = \frac{q}{1+\frac{P_{\text{п}}G_{\text{ап}}}{2N_0\Delta f_{\text{п}}}\frac{G_{\text{пр}}\lambda^2}{(4\pi)^2 d_{\text{п}}^2}}, \tag{8.10.1}$$

where all designations are the same as in Paragraph 7.14.1.

From comparison with (7.14.3) it follows that ratio $q_{\text{пш}}/q$ coincides with ratio $h_{\text{пш}}/h$ for a coherent radar; therefore $q_{\text{пш}}/q$ in various situations, as before, is given by the curves of Fig. 7.51, and all conclusions made in Paragraph 7.14.1 are preserved.

Thus, the relative influence of active noise interference in both cases is equal, but, in general, accuracy of range finding by an incoherent range finder under the influence of interference is lowered more than in a coherent one. For low intensities of interference this difference may be immaterial; however, if the intensity of interference is great, even for q >> 1 and identical accuracy of coherent and incoherent range finders in the absence and in the presence of interferences quantity $q_{\text{пш}}$ may be less than 1. Here, along with lowering of accuracy of range finding due to decrease of the signal-to-noise ratio there appears loss of accuracy due to incoherence (see § 8.2).

Thereby, an incoherent range finder, providing identical accuracy with a coherent one in the absence of interferences, will give greater errors of measurement in the presence of interferences, in spite of the fact that in both cases the signal-to-noise ratio is lowered by an identical factor.

8.10.2. Pulse Chaotic Interference

In this case there also is an analogy with a coherent range finder, and the influence of interference, true, with a somewhat worse approximation than during noise interference, can be described as the influence of equivalent white noise. This analogy is evident for incoherent range finders with correlation processing of the signal, when the passband of the integrating filter is small as compared to the width of the spectrum of the pulse envelope of the signal.

Actually, for pulses of interference which are not very long there is no qualitative difference between coherent and incoherent range finders — in both cases pulses of interference after multiplication by the reference signal are passed through narrowband filters and are expanded. As a result they form a random process equivalent in statistical characteristics to a random process at the output of this filter upon feeding its input white noise with spectral density N_{Π}, which is determined by formulas (7.14.6)-(7.14.9).

Then the influence of interference, as before, can be quantitatively described by introduction of a new signal-to-noise ratio $q_{\Pi Ш}$ and use of the former formulas. Ratio $q_{\Pi Ш}/q$, as with noise interference, coincides with $h_{\Pi Ш}/h$ and is given by formula (7.14.10) and curves of Fig. 7.51. It is obvious that here the finding of the preceding paragraph about the worse noise immunity of incoherent range finders remains in effect, since quantitative characteristics of pulse chaotic interference are such that $q_{\Pi Ш}$ in real conditions may be less than one.

In incoherent discriminators with "shortening" filters, in general, there are less grounds for replacing pulse chaotic interference with equivalent white noise, especially if the pulse duration of interference is small as compared to the duration of the pulse envelope. However, even in this case pulse chaotic interference can be replaced by a continuous random process with the same level of spectral density at the maximum, but now of finite spectrum width. Therefore, all results are approximately valid for such discriminators, but the formula for $q_{\Pi Ш}$ gives a somewhat understated value, i.e., estimation of noise immunity by $q_{\Pi Ш}$ is obtained with a certain safety margin.

8.10.3. Return Interference

The character and results of the influence of return interference on incoherent range finders completely coincide with the case of coherent range finders. Therefore, here there remain all the same problems which we discussed in Paragraph 7.14.3.

8.10.4. Passive Interference

Well-known is the fact that noise immunity of incoherent radars without special means of protection from passive interferences is very low. This has a clear physical foundation and is illustrated in detail in Chapter V in examining incoherent detection systems. Analogously, the influence of passive interferences on incoherent range finders leads to sharp increase of errors of measurement, and with sufficient intensity of interference, to breakoff of tracking and cessation of the regime of

tracking. Let us consider as confirmation of this one example of the influence of interference on an incoherent range finder.

In order to be sure that low noise immunity of incoherent range finders to passive interferences has a fundamental character, we consider a range finder with an optimum discriminator. For definitiveness we consider that the discriminator is realized by a "shortening" filter ideally matched with the sounding signal, a square-law detector and two gates, narrow as compared to the duration of the "shortened" pulse, with detuning 2δ, which we now will consider finite, and in final results we pass to the limit $\delta \to 0$, i.e., we turn to the case of an optimum discriminator. Modulation of the signal we consider arbitrary. For narrow triangular gate pulses the process at the output of the discriminator can be considered a discrete random process with a period of repetition of the signal T_r.

If the value of this process in the j-th period is equal to $v_j(\Delta)$, where Δ is mismatch, it has mean value $\overline{v_j(\Delta)}$ and correlation function

$$R_{\text{д}jk}(0)=\overline{[v_j(0)-\overline{v_j(0)}]\,[v_k(0)-\overline{v_k(0)}]},$$

the gain factor and the equivalent spectral density of the equivalent continuous process are determined by relationships

$$K_{\text{д}}=\left.\frac{\partial\overline{v_j(\Delta)}}{\partial\Delta}\right|_{\Delta=0}, \tag{8.10.2}$$

$$S_{\text{экв}}=T_r\sum_{j-k}\frac{R_{\text{д}j-k}}{K_{\text{д}}^2}. \tag{8.10.3}$$

Output voltage of the discriminator $v_j(\Delta)$ can be recorded through the square of the envelope of output voltage of the "shortening" filter in the j-th period $z_j(t)$ in the following way:

$$v_j(\Delta)=z_j(\tau_0+\delta-\Delta)-z_j(\tau_0-\delta-\Delta). \tag{8.10.4}$$

Here τ_0 — true value of delay; $\tau_0 - \Delta = \tau$ — its measured value,

$$z_j(t)=\left|\int_{-\infty}^{t}u_0(s-t)\,y_j(s)\,e^{i\omega_0 s}ds\right|^2 \tag{8.10.5}$$

— the square of the envelope of output voltage of the "shortening" filter with pulse response $h(t) = u_0(-t)$ ($u_0(t)$ describes modulation in one period); $y_j(t)$ — the signal received in the j-th period.

Then

$$K_{\text{д}}=2\,\frac{\partial\overline{z_j(\tau_0+\delta)}}{\partial\delta}, \tag{8.10.6}$$

$$R_{\text{Д}jk} = 2R_{zjk}(\tau_0+\delta,\ \tau_0+\delta) - 2R_{zjk}(\tau_0+\delta,\ \tau_0-\delta), \tag{8.10.7}$$

where $R_{zjk}(t_1 t_2) = \overline{[z_j(t_1) - \overline{z_j(t_1)}][z_k(t_2) - \overline{z_k(t_2)}]}$ – correlation function of values of the square of the envelope in the j-th and k-th periods.

Let us find the mean value of the square of the envelope. It is obvious that

$$\overline{z_j(t)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(s_1-t)\,u^*_0(s_2-t)\,e^{i\omega_0(s_1-s_2)} R_{yjk}(s_1,\ s_2)\,ds_1 ds_2, \tag{8.10.8}$$

where $R_{yjk}(t_1,\ t_2)$ – correlation function of the received signal, consisting of a mixture of the signal reflected from the target, internal noise and the signal from passive interference. As it was shown in Chapter I, this correlation function is equal to

$$R_{yjk}(t_1,\ t_2) = P_c \operatorname{Re} u_0(t_1-\tau_0)\,u^*_0(t_2-\tau_0)\,\rho_{jk} e^{i\omega_0(t_1-t_2)} +$$
$$+ N_0\delta(t_1-t_2)\,\delta_{jk} + \operatorname{Re}\int_{-\infty}^{\infty} \sigma(\tau)\,u_0(t_1-\tau)\,u^*_0(t_2-\tau) \times d\tau\, r_{jk} e^{i\omega_0(t_1-t_2)} e^{i\theta_{\text{Д}}(j-k)}, \tag{8.10.9}$$

where

ρ_{jk} – coefficient of interperiod correlation of signal;

r_{jk} – coefficient of interperiod correlation of interference;

$\theta_{\text{Д}} = \Delta\omega_{\text{Д}} T_r - [\Delta\omega_{\text{Д}} T_r/2\pi]2\pi$ – phase shift for a period, caused by difference of Doppler frequencies of the signal and interference;

$[\Delta\omega_{\text{Д}} T_r/2\pi]$ – integral part of ratio $\Delta\omega_{\text{Д}} T_r/2\pi$;

$\Delta\omega_{\text{Д}}$ – difference of Doppler frequencies of signal and interference.

Phase shift $\theta_{\text{Д}}$ is conveniently presented in the form

$$\theta_{\text{Д}} = \Delta\omega'_{\text{Д}} T_r,$$

where $\Delta\omega'_{\text{Д}} = \Delta\omega_{\text{Д}} - [\Delta\omega_{\text{Д}} T_r/2\pi]2\pi/T_r$ -- difference of the difference Doppler frequency and the nearest frequency to it, which is a multiple of the frequency of repetition of the signal.

Function $\sigma(\tau)$ in formula (8.10.9) is the distribution density of power with respect to range reflected from passive interference, expressed, for instance, in w/μsec. This function is connected with the distribution density of range reflectors by the simple relationship

$$\sigma(\tau) = P_c \frac{\sigma_0}{\sigma_{\text{Ц}}} \frac{c}{2} n\left(\frac{c\tau}{2}\right), \tag{8.10.10}$$

where σ_0 – effective reflecting surface of one reflector;

$\sigma_{\text{Ц}}$ – effective reflecting surface of target;

$n(r)$ – distribution density of range reflectors (number of reflectors in a layer of unit thickness).

Substituting expression (8.10.9) in formula (8.10.8) and integrating, we obtain

$$\overline{z_j(t)} = \frac{T_r^2}{2}\left[P_c\,|C(\tau_0 - t)|^2 + \int_{-\infty}^{\infty} \sigma(\tau)\,|C(\tau - t)|^2\,d\tau + \frac{2N_0}{T_r}\right], \qquad (8.10.11)$$

where C(x) – autocorrelation function of one period of modulation.

If density of reflectors $\sigma(\tau)$ is constant within limits of the interval of range resolution, the component caused by the presence of passive interference is

$$\int_{-\infty}^{\infty} \sigma(\tau)\,|C(\tau - t)|^2\,d\tau \approx \sigma(t)\int_{-\infty}^{\infty} |C(\tau - t)|^2\,d\tau \approx \sigma(\tau_0)\int_{-\infty}^{\infty} |C(x)|^2\,dx, \qquad (8.10.12)$$

i.e., does not depend on t. Therefore, the presence of passive interference, as also the presence of internal noises, does not affect the gain factor of the discriminator, and magnitude of $K_{\text{Д}}$ remains the same as before (§ 8.2).

Note that in a nonoptimal discriminator, having in the absence of interference null shift of the discrimination characteristic due to noises, in the presence of passive interference due to imbalance of channels there will appear additional systematic error, the magnitude of which may be very significant due to the high level of interference.

Likewise, calculating the correlation function, as we have done many times for the case when there are only a reflected signal and noise, we can obtain the following expression:

$$\begin{aligned} R_{\text{Д}jk}(t_1, t_2) &= \frac{\dot{P}_c^2}{4} T_r^1 \rho_{jk}^2 \,|C(\tau_0 - t_1)\,C(\tau_0 - t_2)|^2 + \\ &+ N_0^2 T_r^2 |C(t_1 - t_2)|^2 \delta_{jk} + \mathrm{Re}\Big\{N_0 T_r^3 P_c C(t_1 - t_2)\,C^*(\tau_0 - t_1)\,C(\tau_0 - t_2) + \\ &+ N_0 T_r^3 \sigma(\tau_0)\int_{-\infty}^{\infty} C^*(x)\,C(x + t_1 - t_2)\,dx\Big\}\delta_{jk} + \frac{T_r^4}{4} r_{jk}^2 \sigma^2(\tau_0)\left|\int_{-\infty}^{\infty} C(x - t_1)\,C^*(x - t_2)\,dx\right|^2 + \\ &+ \mathrm{Re}\left\{\frac{P_c}{2} T_r^1 \rho_{jk} r_{jk} e^{i\theta_{\text{Д}}(j-k)} \sigma(\tau_0)\,C(\tau_0 - t_1) \times C^*(\tau_0 - t_2)\int_{-\infty}^{\infty} C^*(x)\,C(x + t_1 - t_2)\,dx\right\}, \end{aligned} \qquad (8.10.13)$$

where we again assumed that the density of reflectors within limits of the range resolution interval near point τ_0 is constant and equal to $\sigma(\tau_0)$, i.e., its value at the point of location of the target. This assumption sufficiently well is realized in practice already for comparatively low range resolution capability of the radar.

Substituting the found expression for the correlation function in (8.10.7), summing in accordance with formula (8.10.3) and passing in the obtained expression to limit $\delta \to 0$, for equivalent spectral density we find the following expression:

$$S_{\text{экв}} = \frac{T_r}{2bq^2}\left\{1 + q + \frac{\sigma(\tau_0)T_r}{2N_0}\int_{-\infty}^{\infty}\left[|C(x)|^2 + \frac{1}{b}|C'(x)|^2\right]dx + \right.$$

$$\left. + \frac{\sigma^2(\tau_0)T_r^2}{4N_0^2 b}\int_{-\infty}^{\infty}|C(x)|^2 dx\int_{-\infty}^{\infty}|C'(x)|^2 dx + \frac{q\sigma(\tau_0)T_r}{2N_0 b}\int_{-\infty}^{\infty}|C'(x)|^2 dx f\left(\Delta f_{\text{п}}T_r, \frac{\Delta\omega'_{\text{д}}}{\Delta f_{\text{п}}}\right)\right\}, \quad (8.10.14)$$

where it is assumed that the width of spectral band of passive interference $\Delta f_{\text{п}}$ is great as compared to the width of the spectrum of fluctuations of the signal $\Delta f_{\text{с}}$, and function $f(\Delta f_{\text{п}} T_r, \Delta\omega'_{\text{д}}/\Delta f_{\text{п}})$ is equal to

$$f\left(\Delta f_{\text{п}}T_r, \frac{\Delta\omega'_{\text{д}}}{\Delta f_{\text{п}}}\right) = \sum_{j-k} r_{jk} e^{i\theta_{\text{д}}(j-k)}. \quad (8.10.15)$$

For great width of the spectrum of interference as compared to the frequency of repetition, when $r_{jk} = \delta_{jk}$, this function is equal to one, and in the opposite case, when $\Delta f_{\text{п}} T_r \ll 1$, summation in formula (8.10.15) can be replaced by integration, and then

$$f\left(\Delta f_{\text{п}}T_r, \frac{\Delta\omega'_{\text{д}}}{\Delta f_{\text{п}}}\right) = \frac{1}{\Delta f_{\text{п}}T_r} S_{\text{п}}(\Delta\omega'_{\text{д}}), \quad (8.10.16)$$

where $S_{\text{п}}(\omega)$ – spectral density of interference, normalized so that $S_{\text{п}}(0) = 1$.

In the particular case of exponential correlation of interference, when $r_{jk} = e^{-2\Delta f_{\text{п}} T_r |i-k|}$, the exact expression for $f(\Delta f_{\text{п}} T_r, \Delta\omega'_{\text{д}}/\Delta f_{\text{п}})$ has form

$$f\left(\Delta f_{\text{п}}T_r, \frac{\Delta\omega'_{\text{д}}}{\Delta f_{\text{п}}}\right) = 1 + \frac{1}{\Delta f_{\text{п}}T_r\left[1 + \left(\frac{\Delta\omega'_{\text{д}}}{2\Delta f_{\text{п}}}\right)^2\right]}. \quad (8.10.17)$$

Considering in greater detail the expression for $S_{\text{экв}}$, we see that the first two terms are caused by the presence of internal noises and remain in the absence of interference, characterizing accuracy of measurement with internal noises alone.

The last term is caused by the interaction of the signal with interference and depends, at least for a frequency of repetition high as compared to $\Delta f_{\text{п}}$, on the difference of Doppler frequencies of the signal and interference. This term coincides with the corresponding component of $S_{\text{экв}}$, caused by the presence of passive interference; in coherent systems and with selection of a high frequency of repetition it is determined only by the magnitude of the difference Doppler frequency. Therefore, for all "nonblind" speeds (see Paragraph 7.14.4) this term does not lead to

increased errors of measurement.

The third term in formula (8.10.14) is caused by interaction of interference with noise and has an order of 1/q (let us remember that in accordance with (8.10.10) $\sigma(\tau_0)$ has an order of P_c). This term has a specific incoherent form – it does not depend on Δf_Π T_r and $\Delta\omega'_Д$. In coherent systems, thanks to the presence of narrow-band filtration of noise at the signal frequency, this term also depends on the difference Doppler frequency and for sufficiently great detuning turns out to be small as compared to q.

An analogous state of affairs occurs with respect to the fourth term, which is caused by the interaction of interference with interference. This term gives the main component of spectral density – it does not depend on the signal-to-noise ratio q. In coherent systems, thanks to the fact that to the discriminator output there pass only lateral components of the spectrum of interference, this term also depends on detuning.

Integrals in formula (8.10.14) have the order of magnitude of the effective range resolution interval $\Delta\tau_{эф} \sim 1\sqrt{b}$, i.e.,

$$\int_{-\infty}^{\infty} |C(x)|^2 dx = \Delta\tau_{эф} \approx \frac{1}{\sqrt{b}}, \tag{8.10.18}$$

$$\frac{1}{b}\int_{-\infty}^{\infty} |C'(x)|^2 dx \approx \Delta\tau_{эф}. \tag{8.10.19}$$

Then

$$\int_{-\infty}^{\infty} |C(x)|^2 dx \cdot \sigma(\tau_0) = P_c \frac{\sigma_0}{\sigma_ш} \frac{c\Delta\tau_{эф}}{2} n\left(\frac{c\tau_0}{2}\right) = P_c \frac{\sigma_п}{\sigma_ш}, \tag{8.10.20}$$

where $\sigma_\Pi = \sigma_0 \frac{c\Delta\tau_{эф}}{2} n\left(\frac{c\tau_0}{2}\right)$ – total reflecting surface of interference in the resolving volume of the radar.

With an accuracy of a numerical coefficient of the order of unity, equal to this quantity, too, is the product

$$\sigma(\tau_0)\frac{1}{b}\int_{-\infty}^{\infty} |C'(x)|^2 dx \simeq P_c \frac{\sigma_п}{\sigma_ш}. \tag{8.10.21}$$

More exactly, integrals (8.10.18) and (8.10.19), for instance, for the Gaussian autocorrelation function

$$C(x) = \exp\left(-\frac{bx^2}{2}\right)$$

are equal to

$$\Delta\tau_{эф} = \int_{-\infty}^{\infty} |C(x)|^2 dx = \sqrt{\frac{\pi}{b}}, \tag{8.10.22}$$

$$\frac{1}{b}\int_{-\infty}^{\infty} |C'(x)|^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{b}} = \frac{1}{2}\Delta\tau_{эф}. \tag{8.10.23}$$

Thus, taking into account (8.10.20) and (8.10.21), we can give equivalent spectral density the form

$$S_{экв} = \frac{T_r}{2b}\left\{\frac{1}{q^2}+\frac{1}{q}+\frac{1}{q}\frac{\sigma_п}{\sigma_ц}(1+a)+\left(\frac{\sigma_п}{\sigma_ц}\right)^2 a + a\frac{\sigma_п}{\sigma_ц} f\left(\Delta f_п T_r \frac{\Delta\omega'_д}{\Delta f_п}\right)\right\}, \tag{8.10.24}$$

where α — numerical coefficient of the order of unity (for Gaussian $C(x)$ it is equal to 1/2).

Consequently, even for a large signal-to-noise ratio q, a narrow spectrum of interference, and great detuning in frequency ($\Delta\omega'_Д/\Delta f_П \gg 1$), the equivalent spectral density may be very great. Its limiting value is determined by $(\sigma_П/\sigma_Ц)^2$. If it is considered that in the resolved volume there are reflectors with total reflecting surface $\sigma_П \sim 50\ m^2$, then even for the largest air targets with a reflecting surface of the order of $\sigma_Ц = 10$ to $20\ m^2$ [26] the spectral density of $S_{ЭКВ}$ is (6 to 13)T_r/b, i.e., is the same as in the absence of interferences and for a signal-to-noise ratio $q \sim 0.4$ to 0.2. Here, as follows from results in § 8.9, in most cases there occurs disturbance of linear conditions in the tracking system, errors of measurement essentially increase, and breakoff of tracking becomes possible. In many cases ratio $\sigma_П/\sigma_Ц$ can reach considerably larger values. Therefore, the influence of passive interference on an incoherent range finder leads, actually, to the impossibility of range finding.

The only specifically incoherent means of combatting passive interferences known at present is incoherent alternating-period compensation [1, 34]. It consists of the following: "shortened" pulses after detection are delayed a period and are subtracted from the undelayed sequence (see Chapter IV). Then, the difference signal is fed to gating stages for formation of the signal of range mismatch.

If the width of the spectrum of interference is sufficiently small as compared to the frequency of repetition, the components of output voltages, caused by the presence of interference, due to their correlatedness are partially compensated, and the signal-to-interference ratio is improved. With incoherent compensation there occurs, basically, decrease of the basic component of equivalent spectral density, which is caused by interaction of interference with itself (the term in

formula (8.10.24) which is proportional to $(\sigma_{\Pi} / \sigma_{II})^2$). The magnitude of suppression essentially depends on the degree of interperiod correlation of interference and the form of its spectral density. The relative magnitude of suppression, i.e., the degree of increase of the signal-to-interference ratio, depends also on the speed of the target.

On the basis of estimates given in Chapter IV one may assume that suppression of interference has an order of $1/\Delta f_{\Pi} T_r$. For normally used frequencies of repetition and width of the spectrum of interference of several tens of cycles per second, which is characteristic of fixed, e.g., ground radars, this suppression can attain 10-20 db. In the case of moving airborne radars, due to great width of the spectrum of passive interference suppression turns out to be essentially smaller or may, in general, be absent.

Thus, incoherent compensation of passive interference is not a radical means of protecting radars and in many cases cannot ensure reliable range finding. Therefore, the only truly effective means of protecting pulse range finders from passive interferences is application of coherent techniques.

Most frequently this is attained by introduction of coherent alternating-period compensation (Chapter IV), which for sufficiently high multiplicity ensures approximately the same results as the coherent range finders considered in the preceding chapter. Theoretically, it turns out that all terms in formula (8.10.24) depend on the difference Doppler frequency and for sufficiently great detuning are small. In practice, due to the presence of blind speeds and nonoptimality of processing, efficiency of a system with alternating-period subtraction is lower than efficiency of a radar with a coherent range finder (see Chapter IV); however, in many important cases suppression of interference is sufficient to guarantee normal efficiency of the range finder.

§ 8.11. Conclusion

The conducted investigation of incoherent tracking radar range finders and comparison with results pertaining to coherent meters shows that from many points of view differences in the form of sounding radiation of the radar and in methods of processing the received signal do not lead to any essential difference in characteristics of accuracy of measurement.

In particular, with external interferences and a sufficiently large signal-to-noise ratio potential accuracy of range finding is identical for coherent and incoherent signals. Just as in coherent systems, different approximate methods of

realization of optimum operations considered in this chapter ensure accuracy close to the potential, if only we reasonably select parameters of the corresponding circuits. The dependence of accuracy of measurement on the form of modulation in incoherent discriminators is the same as in coherent. The influence of various kinds of nonidealnesses of processing also, basically, remains the same.

Essential difference in the characteristics of accuracy of coherent and incoherent radar range finders can take place with external interferences, especially passive ones. In this respect incoherent systems give considerable loss, and in a number of cases the presence of passive interferences leads to impossibility of their application. Another peculiarity of the investigated incoherent range finders is worse accuracy for low signal levels and, connected with this, the necessity of allowing for nonlinearity of discriminators and phenomena of breakoff of tracking.

Regarding questions requiring further investigation and more detailed working out, here it is possible to repeat everything said in § 7.16 with respect to problems concerning coherent range finders. Actually, problems enumerated there have no less urgency and interest in this case. This list can be supplemented by a series of specifically "incoherent" problems. From the practical point of view one of the most interesting such problems is more detailed study of interference resistance of incoherent range finders with respect to passive interferences. We consider correct the investigation of the system of incoherent alternating-period compensation, seeking and studying other possible means of protection from passive interferences, a combined system using for protection from passive interferences coherent compensation with external and internal coherence, etc.

Another interesting question is more precise definition of solution of the problem of synthesis of an optimum incoherent discriminator. Here we should consider the presence of interperiod correlation of the received signal, find operators by which there is replaced accumulation of squares of the envelope, and analyze their influence on accuracy of measurement. Although physical considerations and consideration of the limiting cases force us to think that the influence of interperiod correlation is hardly essential, nonetheless it would be interesting to obtain strict analytic confirmation of this circumstance.

One more important question is investigation of nontracking incoherent range finders. Practically, most interesting are meters using a unit of detection channels. This problem obtains special urgency in connection with application of incoherent signals with high range resolution capability, provided by intrapulse

modulation. In the given question one should seek optimum operations of processing the signal from the output of the unit of channels, and study characteristics of accuracy for these and other possible operations.

In incoherent range finders we can use quantization of the output signal by level. (This quantization may be, in principle, applied in coherent range finders, too.) Investigation of systems with quantization is also interesting for practice. Here the most important question is the question of the influence of the magnitude of quantization on accuracy of measurement and of finding conditions when quantization can be disregarded.

However, in spite of the fact that there are a number of unsolved questions, results of the present chapter, in general, permit us with sufficient knowledge of the matter to approach the problem of construction of incoherent radar range finders with arbitrary intrapulse modulation and of investigation of their accuracy in various conditions. These results give the possibility of correctly selecting the functional circuit of the discriminator of a range finder, to estimate the influence of various kinds of deviations from optimality of processing, to calculate characteristics of discriminators taking into account these deviations, and to analyze accuracy of range finding with different laws of motion of the target and different smoothing filters. The formulas for accuracy of measurement obtained above permit us to correctly select the required magnitudes of width of the spectrum of modulation and of the energy of the received signal.

CHAPTER IX

MEASUREMENT OF SPEED

§ 9.1. Introductory Remarks

Requirements on contemporary radars frequently lead to the necessity of the measurement not only of current coordinates of targets, but also rates of their change. In certain radar systems it is desirable to measure both radial, and also tangential components of velocity and to find as a final result the velocity vector of the target with respect to the radar. Moreover, there exist technical problems leading to the necessity of measurement of the second derivative of coordinates of targets for discerning the latter and for exact prolongation of their trajectories.

However, of greatest interest is measurement of the radial component of velocity. It is needed, for instance, for navigational systems, where there are used Doppler groundspeed meters (dead reckoners). Measurement of radial velocity in the variant of automatic tracking of frequency of reflected signal is necessary during construction of any coherent radars, since they contain narrow-band filters (see Chapters IV and VII), which it is necessary to tune upon change of frequency of the signal.

Thus, from the principle of construction of coherent radars there ensues the necessity of measurement of Doppler shift of frequency of the signal

$$f_{д} = \frac{2V}{\lambda},$$

where V — radial velocity;

λ — wavelength of the generated oscillations.

Namely for this of greatest interest is investigation of Doppler speed meters, which is the basic content of this chapter. Besides direct measurement of radial velocity using the Doppler effect there are applied other methods of finding it,

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based on use of range-finding systems, since voltages at certain points of filters of range finders are proportional to radial velocity. Such a method is essentially the only one for incoherent pulse radars with short pulse duration, since attempts to measure speed by Doppler shift of frequency, determined during a pulse duration, usually do not stand up to criticism in connection with low accuracy of measurement. Measurement of tangent components of velocity can be performed using goniometrical systems of radars. Questions of measurement of speed with the help of range finders and goniometers also will be seen in this chapter.

As also in preceding chapters, we will be interested first of all in finding optimum methods of construction of speed meters (their discriminators and smoothing circuits), allowing us to obtain minimum errors of measurement in the presence of noises and with allowance for fluctuations of the signal reflected from the target. Quantitative estimation of these errors, just as of errors of measurement of speed with various applied methods of construction of discriminators and smoothing circuits, is the next problem of this chapter. Furthermore, we analyze nonlinear phenomena in speed meters, taking place with intense noises and leading to breakoff of tracking of automatic tracking meters, and also questions of the influence on them of certain forms of active and passive interferences.

As also during investigation of range systems, we will widely use results of Chapter VI in accordance with which during synthesis of optimum speed meters which are tracking systems of definite form, we shall be basically interested in synthesis of the optimum discriminator. Smoothing circuits are synthesized for any meters in Chapter VI. Correspondingly, being interested in Doppler speed meters, we shall first determine operations of an optimum frequency discriminator, judge different technical methods of construction of the discriminator, giving results close to optimum, and make certain recommendations ensuing from comparison of circuits of discriminators. Smoothing circuits will be considered from the point of view of application of results of Chapter VI to measurement of speed giving physical treatments and examples, illustrating accuracy of the obtained meters as a whole. Special attention will be paid to the question of deviations from the optimum method of construction of meters and their influence on accuracy of measurement of speed.

Estimates of the influence of strong noises and interferences on Doppler tracking speed meters are given on the basis of a consideration of the phenomenon of breakoff of tracking, which is conducted from tenets of the two criteria of breakoff considered in Chapter VI.

In a number of cases they apply nontracking speed meters, in particular, meters containing a unit of filters tuned to various Doppler frequencies. Analysis of potentialities of such meters is also given in this chapter.

Investigating properties of meters of speed, determined by differentiation of a coordinate (range, angle), we shall find optimum circuits of meters for certain particular forms of statistics of change of speed, and also we shall analyze accuracy of measurement during application of usual forms of linear filters with constant parameters.

All these investigations give us the possibility to estimate performance of speed meters not only in the presence of internal noises of systems, but also under the influence of certain wide-spread forms of interferences, whose effect, as will be shown, is equivalent to the effect of noise.

§ 9.2. Synthesis of an Optimum Frequency Discriminator

We turn first of all to measurement of radial velocity on the basis of the Doppler effect. The measured parameter of the received radar signal here is Doppler shift of frequency $\omega_Д = \frac{4\pi V}{\lambda}$ which varies, in general, randomly in time. The function of the discriminator of a tracking radar meter is singling out voltage proportional to current mismatch between the true magnitude of $\omega_Д(t)$ and its measured value, which is the output variable of the tracking system. Here, as follows from Chapter VI, an optimum discriminator, ensuring minimum spectral density of the random component of voltage at its output, should form the derivative $\partial|Q(t, \omega_Д)|^2/\partial\omega_Д$ for $\omega_Д$, equal to the estimated value (output variable of the tracking meter). Here $Q(t, \omega_Д)$ is determined by the presentation of the logarithm of the functional of the probability density of the received signal y(t) for an assigned value of parameter $\omega_Д$ in the form

$$\int_0^T |Q(t, \omega_Д)|^2 dt + C,$$

where C does not depend on $\omega_Д$.

To find the functional of probability density $P[y(t)/\omega_Д(t)]$ necessary for further calculations it is necessary to assign the form of signal y(t). In accordance with the description of the signal, given in Chapter I, the reflected signal can often be considered a normal random process. We shall be interested in its reception in white Gaussian noise of spectral density N_0. This corresponds to cases of the presence of internal noises of the receiver, of broad-band active interferences, and, with certain assumptions, stated in § 9.9, passive interferences.

Then the received signal (mixture of reflected signal and interference) $y(t)$ is a normal random process whose correlation function, according to (1.4.3), is determined by expression

$$R(t_1, t_2) = P_c u_a(t_1-\tau) u_a(t_2-\tau) \rho(t_1-t_2) \times \\ \times \cos[\omega_0(t_1-t_2) + \omega_Д(t_1)(t_1-t_2) + \psi(t_1-\tau) - \psi(t_2-\tau)] + \\ + N_0 \delta(t_1-t_2), \quad (9.2.1)$$

where, as before, $u_a(t)$ and $\psi(t)$ — laws of amplitude and phase modulation of the signal, respectively; τ — delay of the reflected signal; P_C — its mean power; $\rho(t)$ — correlation function of fluctuations of the signal; $\omega_Д(t)$ — a function, representing Doppler shift of the frequency of the signal,* slow as compared to $\rho(t)$.

Calculation of the functional of probability density of a normal random signal with correlation function of form (9.2.1) already is given in Chapter IV during synthesis of an optimum detector of a coherent signal. According to (4.3.7) the logarithm of the likelihood function (functional of probability density of signal $y(t)$) is presented in the form just now mentioned:

$$L(\omega_Д) = C + \frac{1}{N_0}\int_0^T |Q(t, \omega_Д)|^2 dt, \quad (9.2.2)$$

*Strictly speaking, with a random signal and radial velocity of the target varying in time one should talk about random changes of phase of the signal, connected with radial velocity. These changes can be presented in the form

$$\varphi(t) = \int_{t_0}^{t} \omega_Д(s)\, ds.$$

In the expression for the correlation function of the signal there is the difference

$$\Delta\varphi = \varphi(t_1) - \varphi(t_2) = \int_{t_2}^{t_1} \omega_Д(s)\, ds.$$

In connection with the assumption of the quickness of change of components of the random signal as compared to changes of speed of the target for intervals $t_1 - t_2$, corresponding to the interval of correlation of the shown components, this approximate equality is valid:

$$\Delta\varphi \approx \omega_Д(t_1)(t_1 - t_2).$$

It directly leads to formula (9.2.1) for the correlation function and to the above-indicated interpretation of the function of the discriminator.

which gives the possibility of direct interpretation of operations of an optimum discriminator. Here $Q(t, \omega_Д)$ is the result of passage of the product of the observed realization of signal y(t) and a reference signal, which in complex form is expressed as

$$u_a(t-\tau)\, e^{i(\omega_0+\omega_Д)t+\psi(t-\tau)},$$

through a filter, the square of the modulus of whose frequency response, according to (4.3.8), is equal to

$$|H_0(i\omega)|^2 = \frac{hS_0(\omega)}{1+hS_0(\omega)}. \qquad (9.2.3)$$

Quantity h is the signal-to-noise ratio and is defined as

$$h = \frac{P_c}{2N_0 \Delta f_c},$$

Δf_c — effective band of fluctuations of the signal;
$S_0(\omega)$ — normalized spectral density of fluctuations.

Correspondingly,

$$|Q(t, \omega_Д)|^2 = \Big|\int_{-\infty}^{t} y(s)\, u_a(s-\tau)\, e^{i[(\omega_0+\omega_Д)s+\psi(s-\tau)]} h_0(t-s)\, ds\Big|^2, \qquad (9.2.4)$$

where $h_0(t)$ — pulse response of a filter, whose frequency response is $H_0(i\omega)$. This filter is a low-frequency equivalent of the optimum filter of a coherent receiver, encountered already in problems of detection and range finding. The obtained result is valid in the case of rapid fluctuations of the signal.

Output voltage of the optimum frequency discriminator is defined as

$$z(t) = \frac{1}{N_0}\frac{\partial |Q(t, \omega_Д)|^2}{\partial \omega_Д}\Big|_{\omega_Д=\hat{\omega}_Д} = \frac{1}{N_0}\frac{\partial}{\partial \omega}\Big\{\Big|\int_{-\infty}^{t} h_0(t-s)\times$$
$$\times e^{i\omega_{пр}(t-s)} y(s)\, u_a(s-\tau) e^{i[\omega_r s+\psi(s-\tau)]}\, ds\Big|^2\Big\}, \qquad (9.2.5)$$

where $\omega_{пр}$ — fixed frequency ($h_0(t) \cos \omega_{пр} t$ — pulse response of the optimum filter);

$$\omega_r = \omega_0 + \hat{\omega}_Д + \omega_{пр};$$

$\hat{\omega}_Д(t)$ — estimated value of Doppler frequency, obtained from the output of the meter.

Recording of output voltage of the discriminator z(t) in the form of (9.2.5) permits us to easily give the following physical interpretation of operations of the optimum discriminator. The expression in braces is the square of the envelope of voltage, formed by mixing the received signal y(t) with reference signal

$u_a(t - \tau)\cos[\omega_\Gamma\ t + \psi\ (t - \tau)]$ and passing this signal through an optimum filter, tuned to intermediate frequency $\omega_{\Pi\Gamma}$. Here, delay of modulation of the reference signal τ should correspond to the true distance to the target; it is determined practically by output voltage of the range system. The frequency of the heterodyne oscillator should be tuned to $\omega_0 + \hat{\omega}_Д(t) + \omega_{\Pi\Gamma}$, which is attained with the help of the voltage from the output of the synthesized speed meter. Mixing of the recieved signal with the shown reference signal ensures convolution of phase modulation, as a result of which there is obtained a signal unmodulated in phase and the best separation of the amplitude-modulated signal from noises.

Naturally, convolution of phase modulation during mixing with heterodyne voltage and multiplication by function $u_a(t - \tau)$, expressing the law of amplitude modulation (gating during pulse modulation of the signal) by no means must necessarily be produced by giving to heterodyne voltage the corresponding law of amplitude modulation. These two functions can also be realized separately.

Formation of the square of the envelope of voltage at the output of the filter can be carried out by a square-law detector. In order to produce voltage, proportional to the derivative with respect to $\omega_Д$, it is possible to realize two channels of the described form, filters of which are detuned relative to $\omega_{\Pi\Gamma}$ by $\pm\frac{\Delta\omega}{2}$, and to subtract the output voltages of these channels. As a result we replace differentiation by calculation of the difference and as $\Delta\omega \rightarrow 0$ output voltage of the resulting circuit seeks $z(t)$. Optimum speed meter is seen in Fig. 9.1.

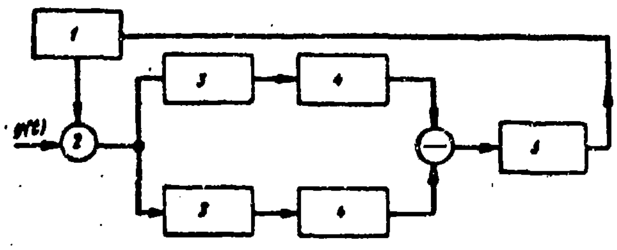

Fig. 9.1. Optimum speed meter: 1 — controlled local oscillator; 2 — mixer; 3 — filter; 4 — square-law detector; 5 — smoothing circuit.

Comparison of the obtained form of optimum processing of a signal with results of Chapter IV shows that an optimum frequency discriminator consists of two channels of optimum detection, detuned with respect to the measured parameter (frequency), which facilitates construction of radars optimum both in conditions of detection and in conditions of measurements.

We emphasize that the optimum circuit is synthesized for an arbitrary form of radiation, both continuous and also pulse. Operations of processing the signal, starting with narrow-band filtration, do not depend on the method of radiation and form of modulation. This will facilitate our further consideration of different

deviations from the optimum method of construction of the frequency discriminator, allowing us not to turn to concrete forms of modulation.

§ 9.3. Characteristics of an Optimum Frequency Discriminator

The found frequency discriminator is optimum from the point of view of obtaining maximum accuracy of measurement of speed. Therefore, its basic characteristics are magnitudes characterizing accuracy, i.e., in accordance with Chapter VI, equivalent spectral density $S_{\text{опт}}$ and a coefficient determining the intensity of parametric fluctuations $S_{\text{пар}}$.

Equaivalent spectral density with an optimum discriminator can be found according to (6.7.33) by the formula

$$S_{\text{опт}}^{-1} = -\frac{1}{2T}\int_0^T\int_0^T \frac{\partial W(t_1, t_2, \omega_{\text{д}})}{\partial\omega_{\text{д}}} \frac{\partial R(t_1, t_2, \omega_{\text{д}})}{\partial\omega_{\text{д}}} dt_1 dt_2, \tag{9.3.1}$$

where $R(t_1, t_2, \omega_{\text{д}})$ – correlation function of the received signal, determined by formula (9.2.1);

$W(t_1, t_2, \omega_{\text{д}})$ – function, found from solution of equation (6.7.20), equal to

$$\begin{aligned} W(t_1, t_2, \omega_{\text{д}}) = & -u_a(t_1-\tau)\,u_a(t_2-\tau)\cos[(\omega_0+\omega_{\text{д}})(t_1-t_2)+ \\ & +\psi(t_1-\tau)-\psi(t_2-\tau)]\,\frac{1}{\pi N_0}\int_{-\infty}^{\infty}\frac{hS_0(\omega)}{1+hS_0(\omega)}e^{i\omega\tau}d\omega + \\ & +\frac{1}{N_0}\delta(t_1-t_2). \end{aligned} \tag{9.3.2}$$

Substituting (9.2.1) and (9.3.2) in (9.3.1) and performing calculations, we obtain

$$S_{\text{опт}}^{-1} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{hS'_0(\omega)}{1+hS_0(\omega)}\right]^2 d\omega, \tag{9.3.3}$$

where the stroke is the sign of differentiation.

The same result can be obtained by another method, if we consider that for an optimum discriminator

$$S_{\text{опт}} = \frac{1}{K_{\text{опт}}}, \tag{9.3.4}$$

where $K_{\text{опт}}$ – the gain factor of the discriminator, defined as

$$K_{\text{опт}} = \left|\overline{\frac{\partial z(t, \omega_{\text{д}})}{\partial\omega_{\text{д}}}}\right|_{\omega_{\text{д}}=\hat{\omega}_{\text{д}}}. \tag{9.3.5}$$

Averaging in set (9.2.5), we easily obtain for the discrimination characteristic of an optimum frequency discriminator

$$a(\delta)=\overline{z(t,\delta)}=-\frac{h}{2\pi}\int_{-\infty}^{\infty}|H_0(i\omega)|^2 S'_0(\omega+\delta)\,d\omega, \qquad (9.3.6)$$

where $\delta = \omega_Д - \hat{\omega}_Д$, and $|H_0(i\omega)|^2$ is determined by expression (9.2.3).

From (9.3.5) and (9.3.6) we can obtain formula (9.3.3) for equivalent spectral density. Moreover, we incidentally obtain the expression for the discrimination characteristic of the optimum discriminator, needed during the analysis of processes occurring with strong noises and interferences.

For finding the coefficient of parametric fluctuations $S_{пар}$ we calculate the equivalent fluctuation characteristic of the discriminator

$$S_{экв}(\delta)=\frac{1}{K^2_{опт}}\int_{-\infty}^{\infty}[\overline{z(t,\delta)z(t+\tau,\delta)}-\overline{z(t,\delta)}\,\overline{z(t+\tau,\delta)}]\,d\tau. \qquad (9.3.7)$$

Performing calculations, we obtain

$$S_{экв}(\delta)=\frac{1}{2\pi K^2_{опт}}\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\omega}|H_0(i(\omega+\delta))|^2\right]^2[hS_0(\omega)+1]^2 d\omega. \qquad (9.3.8)$$

Coefficient $S_{пар}$ is now calculated as

$$S_{пар}=\frac{1}{2}\left.\frac{d^2S_{экв}(\delta)}{d\delta^2}\right|_{\delta=0}. \qquad (9.3.9)$$

From the obtained expressions it follows that both equivalent spectral density $S_{опт}$, and coefficient $S_{пар}$ do not depend on the form of modulation of the sounding signal but are determined only by the signal-to-noise ratio h and the form of the spectrum of fluctuations of the reflected signal $S_0(\omega)$.

Of basic importance for determining error of the speed meter is the equivalent spectral density $S_{опт}$. Let us find it for certain characteristic forms of $S_0(\omega)$.

If spectral density of the signal corresponds to white noise passing through an RC-filter, then

$$S_0(\omega)=\frac{1}{1+\left(\frac{\omega}{2\Delta f_c}\right)^2}, \qquad (9.3.10)$$

where Δf_c — effective band of fluctuations of the signal.

Substituting (9.3.10) in (9.3.3), we obtain

$$S_{опт}=2\Delta f_c\,\frac{\sqrt{1+h}\,(1+\sqrt{1+h})^2}{h^2}. \qquad (9.3.11)$$

When the spectral density of the signal corresponds to white noise passing through two series-coupled RC-filters

$$S_0(\omega)=\frac{1}{\left[1+\left(\frac{\omega}{4\Delta f_c}\right)^2\right]^2}. \tag{9.3.12}$$

From (9.3.3) we obtain

$$S_{\text{опт}}=\Delta f_c\left[1-3\sqrt{\frac{\sqrt{1+h}-1}{2h}}+\right.$$
$$\left.+\frac{1}{2}\sqrt{\frac{h}{2(1+h)(\sqrt{1+h}-1)}}\right]^{-1}. \tag{9.3.13}$$

If, however, the spectrum of the signal has a Gaussian form,

$$S_0(\omega)=\exp\left[-\frac{1}{4\pi}\left(\frac{\omega}{\Delta f_c}\right)^2\right], \tag{9.3.14}$$

it is possible to calculate $S_{\text{опт}}$ approximately, obtaining here

$$S_{\text{опт}}\approx\begin{cases}4\sqrt{2}\pi\frac{\Delta f_c}{h^2} & h\ll 1,\\ \frac{\Delta f_c}{(\ln h)^{3/2}-\frac{3}{2}(\ln h)^{1/2}} & h\gg 1.\end{cases} \tag{9.3.15}$$

Dependences of $S_{\text{опт}}/\Delta f_c$ on the signal-to-noise ratio h for the three considered cases are shown in Fig. 9.2.

These three cases do not exhaust, of course, all possibilities. However, they belong to one family of functions $S_0(\omega, n)$, depending on integral parameter n,

$$S_0(\omega, n)=\frac{1}{[1+(\omega\tau_n)^2]^n}, \tag{9.3.16}$$

where

$$\tau_n=\frac{1}{\Delta f_c}\frac{(2n-2)!}{2^{2n-1}[(n-1)!]^2}.$$

Generalizing the obtained results for an arbitrary n, we can obtain limiting relationships for $S_{\text{опт}}$ for small and large signal-to-noise ratios:

$$\left.\begin{aligned}S_{\text{опт}}(n)&=\frac{\Delta f_c}{h^2}\frac{2^{4n-1}(2n+1)[(n-1)!]^2[(2n)!]^2}{n^2(4n)!(2n-2)!} && h\to 0,\\ S_{\text{опт}}(n)&=\Delta f_c\frac{2^{2n-1}[(n-1)!]^2}{n^2(2n-2)!} && h\to\infty.\end{aligned}\right\} \tag{9.3.17}$$

From these relationships and from Fig. 9.2 it follows that for small h the magnitude of $S_{\text{опт}}$ depends little on the form of the spectrum, being proportional

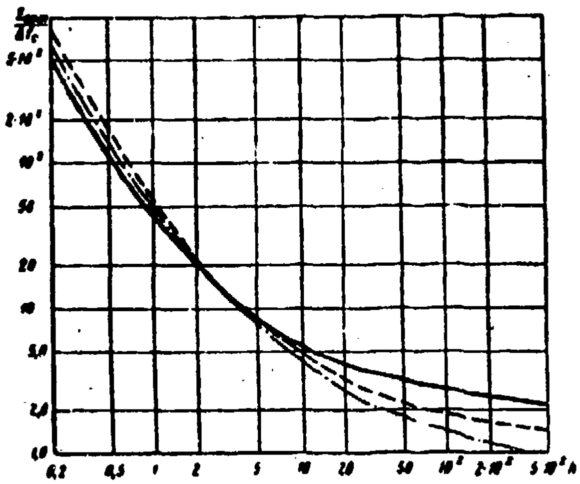

Fig. 9.2. Dependence of equivalent spectral density of an optimum frequency discriminator $S_{\text{опт}}$ on the signal-to-noise ratio h:
—— $S_0(\omega)$ is expressed by formula (9.3.10);
--- $S_0(\omega)$ is expressed by formula (9.3.12);
-.- $S_0(\omega)$ is expressed by formula (9.3.14);

to $1/h^2$. For large $h(h > 100)$ the magnitude of $S_{\text{опт}}$ essentially depends on the form of the spectrum, varying as $h \to \infty$ from $2\Delta f_c$ ($n = 1$) to 0 ($n \to \infty$). In Table 9.1 is given the dependence of limiting values of $S_{\text{опт}}$ on the approximation of the spectrum for family (9.3.16).

For very large signal-to-noise ratios the question of optimization of a meter usually does not arise. Therefore, working signal-to-noise ratios should be considered magnitudes of h = 0.1 to 100. In this range of h the dependence of $S_{\text{опт}}$ on the form of the spectrum of fluctuations of the signal is not great and it often can be ignored, considering for simplicity approximation (9.3.10).

It should be noted that one should not especially trust the obtained values of $S_{\text{опт}}$ for very large h, since the calculating formulas used here become insufficiently exact.

Table 9.1

n	1	2	3	4	5	. . .	∞
$\lim\limits_{h \to 0} \frac{h^2 S_{\text{опт}}}{\Delta f_c}$	16	18,3	18,35	18,32	18,3	. . .	17,7
$\lim\limits_{h \to \infty} \frac{S_{\text{опт}}}{\Delta f_c}$	2	1	0,59	0,4	0,29	. . .	0

Besides consideration of the dependence of $S_{\text{опт}}$ on h one should emphasize the proportionality of $S_{\text{опт}}$, and consequently, and of variance of error of measurement of speed, to the effective width of the spectrum of fluctuations of the signal Δf_c. Here there exists a fundamental difference between results obtained for Doppler speed meters and meters for range and coordinates, and using range finders

and goniometers. It consists in the fact that there exists a limiting error of measurement of speed for any size signal-to-noise ratio, while in range finders and goniometers error decreases without limit with increase of this ratio. Physically this is explained by the fact that in a speed meter subjected to measurement is the derivative of the phase of the signal, which for the accepted idealizations contains, due to fluctuations of the signal, an additive random addition, not depending on the signal-to-noise ratio.

Thus, the method of encoding the measured parameter (speed) in the signal in principle does not permit us to compensate the influence of fluctuations of the signal, in distinction from range finders and goniometers where such compensation is possible.

Assigning the simplest form of spectral density of fluctuations of the signal (9.3.10), by formulas (9.3.8) and (9.3.9) we find $S_{пар}$. Results of rather cumbersome calculations are presented in the form of the curve of Fig. 9.3. As $h \to \infty$ quantity $S_{пар}$ reaches its maximum and is equal to $1/2\Delta f_c$. We do not give values of $S_{пар}$ for small h, since here conditions of optimality of the synthesized circuit are no longer realized, and the model of the tracking meter accepted for analysis of accuracy loses its meaning, which leads during formal application of the expression for $S_{пар}$ to physically inexplicable results. We shall postpone estimation of the increase of fluctuation error due to parametric fluctuation until our consideration of smoothing filters and of errors of the speed meter as a whole.

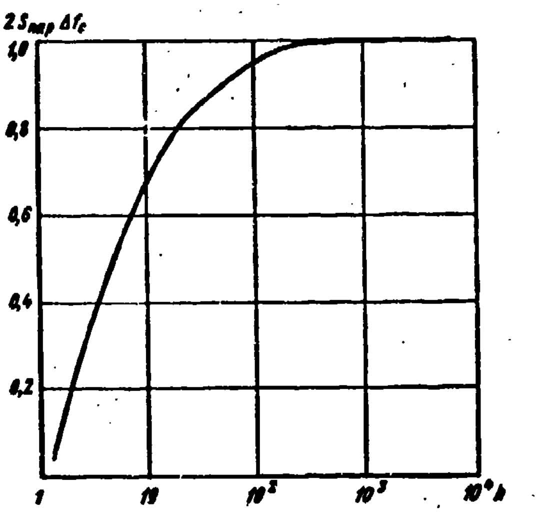

Fig. 9.3. Dependence of the coefficient of parametric fluctuations $S_{пар}$ of an optimum frequency discriminator on the signal-to-noise ratio h.

§ 9.4. Analysis of Certain Circuits of Frequency Discriminators

Let us turn to analysis of frequency discriminators, whose circuits differ from the above-considered optimum circuit. Analyzing properties of these discriminators and characterizing them by parameters $S_{экв}$ and $S_{пар}$, we have the possibility of comparing different discriminators with one another and with an optimum discriminator. As a result we shall find conditions in which certain of the circuits are close in

performance to optimum ones, i.e., accuracy of measurement of speed provided by them is close to potential accuracy belonging to an optimum meter.

For convenience of consideration we introduce certain nonessential idealizations. During the analysis of different circuits we assume that convolution of the spectrum of an arbitrarily modulated received signal is realized ideally. This assumption means that the arbitrarily modulated signal will be converted into a signal, containing only one spectral component of width Δf_c, by which we measure the Doppler frequency, without change of the signal-to-noise ratio with respect to the input signal of the receiver. As a result characteristics of the considered circuits will not depend on the form of modulation similarly to how this took place for the optimum circuit. Subsequently, we shall take into account those changes to which imperfectness of convolution of the spectrum of the signal leads. It turns out that this is possible to do in sufficiently general form without turning to consideration of the concrete forms of modulation.

We shall subsequently assume that the receiver in the considered discriminators, has an automatic gain control [AGC] system, which reduces the average level of voltage at the output of the amplifier covered by it to a constant level. Thus, assumptions about the function executed by the AGC system remain the same as in Chapter VII. With application of square-law detectors or multipliers in circuits of discriminators this leads, naturally, to the same dependence of the discriminator gain factor $K_Д$ on the signal-to-noise ratio h

$$K_Д = K_{Д0}\left[1 + \frac{1}{h}\left(\frac{\Delta f_y}{\Delta f_c}\right)\right]^{-1}, \qquad (9.4.1)$$

where $K_{Д0}$ — nominal value of the gain factor in the absence of noises;

Δf_y — effective bandwith of the amplifier covered by the AGC loop ($\Delta f_y \gg \Delta f_c$).

When there is no AGC system, or it does not work due to smallness of the amplitude of the signal, the dependence of $K_Д$ on h changes. Here, the gain factor of the discriminator is proportional to h. There may be other dependences, determined by suppression of the signal by noise due to various nonlinearities in the radio channel. However, we henceforth take dependence (9.4.1).

Let us consider in the above-described plan some of the most widely used circuits of frequency discriminators.

9.4.1. Circuit with a Tuned Loop and a Phase Shifter

The diagram of a frequency discriminator is shown in Fig. 9.4. Input signal y(t) is fed to a mixer. Heterodyne voltage, proceeding to the same mixer, is modulated in phase in such a way that there is ensured phase demodulation of the signal. Heterodyne frequency is controlled by the output voltage of the speed meter. After heterodyning the signal is subjected to amplification, where the gain factor changes in accordance with the law of amplitude modulation of the received signal.

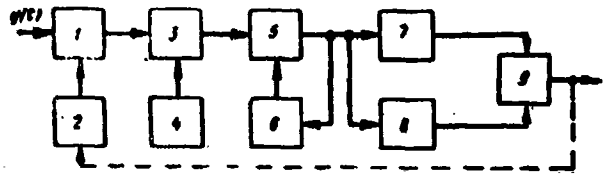

Fig. 9.4. Diagram of a frequency discriminator with a tuned loop and a phase shifter: 1 — mixer; 2 — controlled local oscillator; 3 — amplifier; 4 — modulator; 5 — amplifier with controlled gain; 6 — AGC circuit; 7 — tuned loop; 8 — phase shifter; 9 — phase detector.

In the case of a pulse signal the shown operations simply signify gating of the amplifier with the proper form of gate pulse. Delay of phase modulation of the local oscillator and amplitude modulation in the amplifier is controlled from the output of the range finder. Then the signal is passed through an amplifier with band $\Delta f_y \gg \Delta f_c$, which is subject to automatic gain control. Note that in practice preceding stages (gated stages) may also have gain control.

We assume that band Δf_y is sufficiently small to ensure suppression of sidebands of amplitude modulation of the signal. This assumption is made for convenience of further analysis, although it is obvious that results are not changed if it is not realized. The described part of the circuit, ensuring demodulation of the signal and reduction of it to a level corresponding to work of subsequent stages in a linear regime, is common both for the considered, and also for the following circuits of discriminators.

After demodulation the signal proceeds to the tuned loop of the discriminator with passband Δf_K and from its output it is fed to a phase detector; simultaneously to the phase detector there is fed voltage from the output of the amplifier, shifted $\pi/2$ by a phase shifter. Output voltage of the phase detector, which multiplies these voltages, is the output variable of the discriminator, which after amplification and smoothing controls the frequency of the local oscillator. The principle of action of the circuit is based on the fact that the phase-frequency response of a loop tuned to a certain frequency is an odd function of detuning relative to this frequency, approximately linear with small detuning. As a result during detuning of middle frequencies of the filter and of the signal beats between

components of the signal at the phase detector input, shifted correspondingly by the phase shifter and filter, form a constant component, approximately proportional to detuning.

The mixture of the signal and noise at the input of the discriminator loop tuned to frequency $\omega_{\text{пр}}$ can be presented in the form

$$u_1(t) = A(t)\cos\omega_{\text{пр}}t + B(t)\sin\omega_{\text{пр}}t, \tag{9.4.2}$$

where

$$\left.\begin{aligned} A(t) &= a(t)\cos\delta t + b(t)\sin\delta t + \xi(t), \\ B(t) &= -a(t)\sin\delta t + b(t)\cos\delta t + \eta(t), \end{aligned}\right\} \tag{9.4.3}$$

δ – detuning between frequencies of the local oscillator and the signal;

$a(t)$ and $b(t)$ – independent normal random processes, characterizing fluctuations of the received signal, with spectral density $S_0(\omega)$;

$\xi(t)$ and $\eta(t)$ – noises with spectral density, uniform in the passband of the loop and equal to $1/h$.

Strictly speaking, voltage at the output of the loop is not equal, but proportional to $u_1(t)$ with a proportionality factor which depends on amplification of the channel and the power of the signal at the input. However, subsequently this proportionality factor has no significance, since we introduced characteristics of the discriminator which do not depend on constant proportionality factors.

Output voltage of the discriminator $u_{\text{Д}}(t, \delta)$ is obtained by multiplying the result of passage of $u_1(t)$ through a filter with pulse response $h(t)\cos\omega_{\text{пр}}t$ and the result of shift $u_1(t)$ in phase by $\pi/2$. Performing the shown operations, we have, with an accuracy of a constant coefficient,

$$u_{\text{Д}}(t, \delta) = \frac{1}{2}\int_{-\infty}^{t} h(t-\tau)\,[A(\tau)B(t) - A(t)B(\tau)]\,d\tau. \tag{9.4.4}$$

Determining by usual method the gain factor of the discriminator $K_{\text{Д}}$, we have

$$K_{\text{Д}} = \left|\frac{\partial \overline{u_{\text{Д}}(t, \delta)}}{\partial\delta}\right|_{\delta=0} = \frac{i}{2\pi}\int_{-\infty}^{\infty} H^*(i\omega)\,S'_0(\omega)\,d\omega, \tag{9.4.5}$$

where $H(i\omega)$ – Fourier transform of $h(t)$; * – designates a complex conjugate magnitude; and the stroke – sign of differentiation.

Equivalent spectral density $S_{\text{экв}}$ determined by the value of the fluctuation characteristic for $\delta = 0$, is equal to

$$S_{\text{экв}} = \frac{1}{K_{\text{Д}}^2}\frac{1}{4\pi}\int_{-\infty}^{\infty}[|H(i\omega)|^2 - H^{*2}(i\omega)]\left[S_0(\omega) + \frac{1}{h}\right]^2 d\omega. \tag{9.4.6}$$

For the simplest form of spectral density (9.3.10) and a loop whose

low-frequency equivalent has frequency response

$$H(i\omega)=\frac{1}{1+\frac{i\omega}{2\Delta f_{\kappa}}}, \tag{9.4.7}$$

expressions (9.4.5) and (9.4.6) take form

$$K_{\text{Д}}=k\frac{\Delta f_{c}\Delta f_{\kappa}}{2(\Delta f_{\kappa}+\Delta f_{c})^{2}}=k\frac{\alpha}{2(1+\alpha)^{2}}, \tag{9.4.8}$$

$$\begin{aligned}S_{\text{экв}}&=2\Delta f_{c}\left(1+\frac{\Delta f_{c}}{\Delta f_{\kappa}}\right)\left[1+\frac{2}{h}\left(1+\frac{\Delta f_{\kappa}}{\Delta f_{c}}\right)+\right.\\&\left.+\frac{1}{h^{2}}\left(1+\frac{\Delta f_{\kappa}}{\Delta f_{c}}\right)^{2}\right]=2\Delta f_{c}\left(1+\frac{1}{\alpha}\right)\left[1+\frac{2}{h}(1+\alpha)+\right.\\&\left.+\frac{1}{h^{2}}(1+\alpha)^{2}\right].\end{aligned} \tag{9.4.9}$$

Here $\alpha = \Delta f_{K}/\Delta f_{C}$; k – proportionality factor, not affecting the magnitude of $S_{\text{ЭКВ}}$, but, in general, depending on the signal-to-noise ratio due to the infleunce of the AGC system, described above. As a result the dependence of $K_{\text{Д}}$ on h is determined by formula (9.4.1). Formula (9.4.8) determines the dependence of $K_{\text{Д}}$ on the pass-band of the loop Δf_{K}. From (9.4.8) it follows that maximum $K_{\text{Д}}$ is attained at $\alpha = \Delta f_{K}/\Delta f_{C} = 1$, but as in other meters this still does not minimize $S_{\text{ЭКВ}}$.

Quantity $S_{\text{ЭКВ}}$ consists of three components, caused by beats of components of the signal at inputs of the phase detector (component not depending on h), beats of the signal with noise (component containing 1/h) and beats of noise with noise (component with $1/h^2$). For low-level noises the first component predominates; for high levels, the third. For any value of h quantity $S_{\text{ЭКВ}}$ has a minimum at a certain value of the ratio of bandwidth of the loop and signal α. This testifies to presence, as in the optimum circuit, of an optimum value of bandwidth Δf_{K}. Minimum $S_{\text{ЭКВ}}$ is attained at

$$\alpha=\frac{\Delta f_{\kappa}}{\Delta f_{c}}\approx\begin{cases}\frac{1}{3}+\frac{h}{4} & h<1,\\ \sqrt{\frac{h}{2}} & h>1;\end{cases} \tag{9.4.10}$$

in the optimum circuit, according to results obtained above, the filter should have the same form of frequency response with the relationship of bandwidths

$$\alpha_{\text{опт}}=\sqrt{1+h}. \tag{9.4.11}$$

Thus, in a circuit with a tuned loop and a phase shifter the best bandwidth of the loop is approximately 1/3 as broad as in the optimum circuit for small h and $\frac{1}{\sqrt{2}}$ as broad as in the optimum for large h. This is explained by the difference in

the character of processing of the signal. It is useful to estimate qualities of the considered circuit, comparing it with the optimum circuit. Therefore, of interest is comparison of $S_{ЭКВ}$, det[illegible] by formula (9.4.9), with $S_{ОПТ}$, expressed by (9.3.11), for various values of [illegible] α.

Results of such comparison are presented in Fig. 9.5. For low-level noises and large α from formula (9.4.9) and from the given curves it is clear that the magnitude of $S_{ЭКВ} \to 2\Delta f_C$, i.e., the circuit in its properties does not differ from the optimum one. For high-level noises and $\alpha \gg$ [illegible] the magnitude of $S_{ЭКВ}$ is approximately proportional to α^2. This is explained both by lowering of the gain factor of the discriminator, and also by increase of the power of noises at the output of the loop without essential increase of signal power. At the same time narrowing of Δf_K with respect to the optimum value of the loop passband also leads to growth of $S_{ЭКВ}$ due to decrease of the gain factor. From the given curves we see that in the range of signal-to-noise ratios

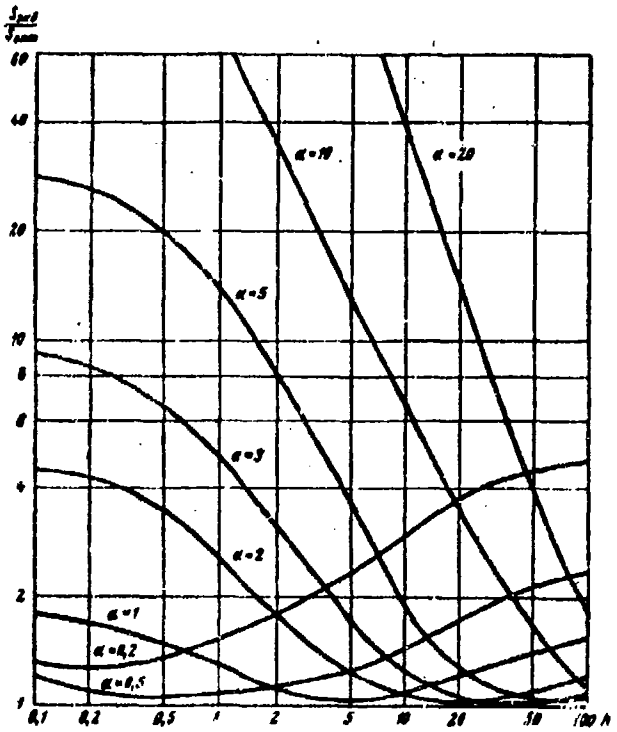

Fig. 9.5. Dependence of $S_{ЭКВ}/S_{ОПТ}$ on the signal-to-noise ratio h for discriminators with a tuned loop and a phase shifter and with mixers and differentiation for various α.

h = 0.1 to 100 loss in the considered circuit as compared to the optimum can be reduced by selection of the passband of the loop for small h to 20% or less, and for large h practically to zero.

During the analysis of nonlinear phenomena, occuring for high-level noises, it may be useful to know the form of the discrimination characteristic of the considered circuit. By averaging (9.4.4) we obtain for it the expression

$$a(\delta)=\overline{u_{д}(t,\,\delta)}=\frac{1}{4\pi i}\int_{-\infty}^{\infty} H(i\omega)\,[S_0(\omega-\delta)-S_0(\omega+\delta)]\,d\omega. \qquad (9.4.12)$$

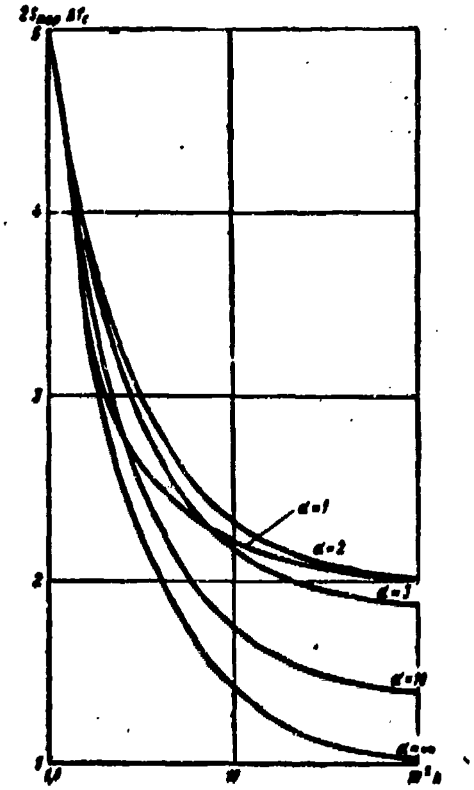

Fig. 9.6. Dependence of $S_{\text{пар}}$ on h for a discriminator with a tuned loop and a phase shifter for different α.

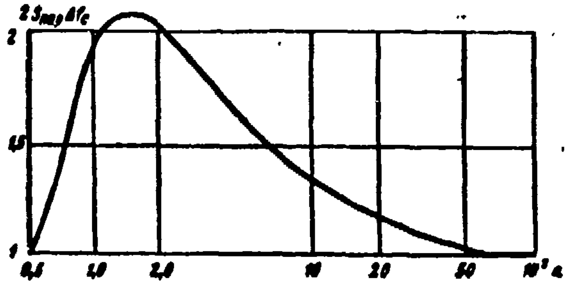

Fig. 9.7. Dependence of $S_{\text{пар}}$ on α for discriminators with a tuned loop and a phase shifter and with mixers and differentiation for signal-to-noise ratio $h \to \infty$.

For spectral density of signal (9.3.10) and frequency response (9.4.7) this expression, after calculations, gives

$$a(\delta) = k_1 \frac{\alpha}{1+\alpha}\frac{x}{1+x^2}, \qquad (9.4.13)$$

where

$$x = \frac{\delta}{2(\Delta f_c + \Delta f_n)}, \qquad (9.4.14)$$

and k_1 – proportionality factor.

Calculation of the fluctuation characteristic of the considered discriminator on the same assumptions with respect to the form of the spectrum of fluctuations and frequency response of the loop leads to expression

$$S_{\text{экв}}(x) = 2\Delta f_c \frac{1+\alpha}{\alpha}\left\{\frac{1+[(1+\alpha)^2+3\alpha]x^2+\alpha(1+\alpha)x^4}{(1+x^2)^3} + \frac{2(1+\alpha)}{h}\,\frac{1+(1+2\alpha)x^2}{(1+x^2)^2} + \frac{(1+\alpha)^2}{h^2}\right\}. \qquad (9.4.15)$$

Coefficient $S_{\text{пар}}$ is found from this as

$$S_{\text{пар}} = \frac{1}{2}\,\frac{S''_{\text{экв}}(0)}{[2(\Delta f_c + \Delta f_n)]^2} = \frac{1}{2}\,\frac{1}{\Delta f_c}\,\frac{1}{1+\alpha}\left[5+\alpha-\frac{2}{\alpha} + \frac{2(1+\alpha)(2\alpha-1)}{\alpha h}\right], \qquad (9.4.16)$$

where the two strokes designate the second derivative of $S_{\text{экв}}$ with respect to δ.

In Fig. 9.6 are dependences of $S_{\text{пар}}$ on the signal-to-noise ratio h for various values of the ratio of bandwidths of the loop and signal. As $h \to \infty$ the magnitude of $S_{\text{пар}}$ seeks a limiting value, the dependence of which on α is presented in Fig. 9.7. With decrease of h the magnitude of $S_{\text{пар}}$ grows inversely proportionally to h.

9.4.2. Circuit with Mixers and Differentiation

During presentation of circuits of frequency discriminators here and subsequently we shall not depict circuits of demodulation of the signal and of preliminary processing, which remain the same as in the circuit of Fig. 9.4. The part of the circuit determining the principle of action of the considered discriminator is shown in Fig. 9.8. Output voltage of amplifier enters two mixers to which as the reference voltages there are fed the voltage of a local oscillator tuned from the output of the meter and the same voltage, shifted in phase $\pi/2$ (with the help of a phase shifter). Output voltage of one of the mixers is fed to a low-frequency filter, and from the other to a differentiating circuit with the same time constant. Output voltages of the filter and the differentiating circuit are multiplied in a phase detector, forming the output voltage of the discriminator.

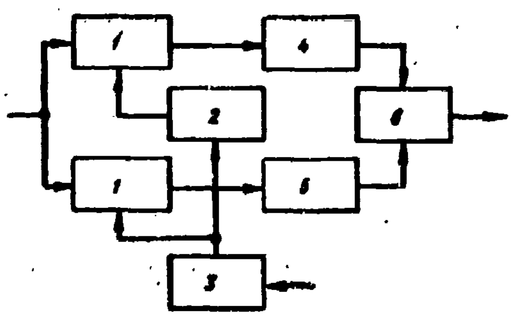

Fig. 9.8. Circuit of a frequency discriminator with mixers and differentiation: 1 — mixer; 2 — phase shifter; 3 — controlled local oscillator; 4 — low-frequency filter; 5 — differentiator; 6 — phase detector.

The principle of action of this discriminator is based on the fact that with detuning δ between frequencies of the local oscillator and the signal at the output of mixers there occur variable components of frequency δ, shifted in phase $\pi/2$. The differentiator removes this phase shift, forming voltage whose amplitude is determined by detuning. As a result, after multiplication in the phase detector there is obtained a quantity which is proportional to detuning δ.

In connection with the fact that the local oscillator is tuned from the frequency of the input signal of the mixers, voltage at the output of one of them is equal to $A(t)$, and at the output of the other is $B(t)$ (9.4.3). At the discriminator output there will be formed

$$u_{\text{д}}(t, \delta)=\int\limits_{-\infty}^{t}\int\limits_{-\infty}^{t} h(t-t_1)\, h_1(t-t_2)\, A(t_1)\, B(t_2)\, dt_1 dt_2, \tag{9.4.17}$$

where $h(t)$ — pulse response of a low-frequency filter;

$h_1(t)$ — pulse response of the differentiating circuit, whose frequency response is related to the frequency response of the low-frequency filter $H(i\omega)$ by relationship

$$H_1(i\omega)=\frac{i\omega}{2\Delta f_{\text{ш}}} H(i\omega). \tag{9.4.18}$$

Using formula (9.4.17) and performing proper averaging, we obtain the gain factor of the discriminator

$$K_{\text{д}}=\left|\frac{\partial \overline{u_{\text{д}}(t,\delta)}}{\partial\delta}\right|_{\delta=0}=-\frac{i}{2\pi}\int_{-\infty}^{\infty}H(i\omega)H^{*}_{1}(i\omega)S'_{0}(\omega)\,d\omega \qquad (9.4.19)$$

and equivalent spectral density

$$\begin{gathered}S_{\text{экв}}=\frac{1}{K_{\text{д}}^{2}}\int_{-\infty}^{\infty}[\overline{u_{\text{д}}(t,0)\,u_{\text{д}}(t+\tau,0)}-\\ -\overline{u_{\text{д}}(t,0)}\;\overline{u_{\text{д}}(t+\tau,0)}]\,d\tau\doteq\\ =\frac{1}{K_{\text{д}}^{2}}\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(i\omega)|^{2}|H_{1}(i\omega)|^{2}\left[S_{0}(\omega)+\frac{1}{h}\right]^{2}d\omega.\end{gathered} \qquad (9.4.20)$$

In order to have the possibility of comparing properties of the considered discriminator with properties of the preceding circuits, we consider that the low-frequency filter in this case has frequency response H(iω), determined by (9.4.7), i.e., is the l-f equivalent of the bandpass filter of the preceding circuit. Then, as it is easy to show, the discrimination and fluctuation characteristics of the considered discriminator and of the discriminator with a tuned loop and a phase shifter coincide. In particular, expressions for $K_{\text{Д}}$ and $S_{\text{ЭКВ}}$ obtained from (9.4.19) and (9.4.20) coincide with formulas (9.4.8) and (9.4.9), respectively. The coefficient of parametric fluctuations is determined by formula (9.4.16). This means that although in the method of formation of the signal of error the circuits of Figs. 9.4 and 9.8 differ from each other, their characteristics completely coincide (with satisfaction of the imposed conditions), and all dependences obtained above remain in force for the given discriminator.

Thus, these two discriminators are equivalent. The basis for application of the circuit of Fig. 9.8 instead of the preceding one, in spite of greater complexity, is the possibility of using low-frequency filters instead of bandpass filters, creation of which causes technical difficulties in cases when the signal has a very narrow band. In these cases there is required high stability of the bandpass filters, obtained by application of crystal filters, whereas during creation of low-frequency filters no difficulties arise.

9.4.3. Circuit with Detuned Loops

The considered circuit without networks intended for convolution of the spectrum of the signal is presented in Fig. 9.9. The signal from the output of the

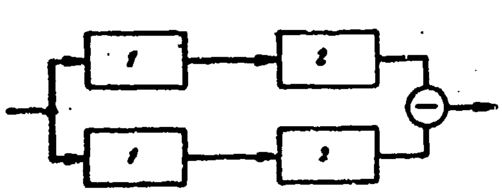

Fig. 9.9. Circuit of a frequency discriminator with detuned loops: 1 — loop; 2 — detector.

amplifier proceeds to two mutually detuned (by $\Delta\omega$) loops. Output voltages of the loops are detected and substracted, forming the output quantity of the discriminator. It is clear that as $\Delta\omega$ approaches zero and with matching of frequency responses of the loops with the spectrum of the signal the circuit exactly executes optimum operations. However, here the transmission factor of the discriminator seeks zero, which should be compensated by a high-gain amplifier, coupled in series with the depicted part of the discriminator circuit. In order to avoid considerable loss in amplification, in the real circuit detuning of loops is produced by a finite quantity $\Delta\omega$, comparable with the width of the spectrum of the signal. Furthermore, there is practically selected some transmission band of the loops $\Delta f_{к}$, which does not change with change of the signal-to-noise ratio h. Frequently, due to technical conditions it is impossible to select this band in such a way as to ensure matching of the loops with the spectrum of the signal.

Considering the shown peculiarities, we shall analyze qualities of the circuit with detuned loops. Voltage to the input of the analyzed part of the circuit is determined, as before, by expression (9.4.2). This voltage is passed through filters with pulse responses $h(t)\cos(\omega_{пр} - \Delta\omega/2)t$ and $h(t)\cos(\omega_{пр} + \Delta\omega/2)t$. Squares of the moduli of voltages obtained at the output of the filters, formed by square-law detectors, are subtracted. Considering these operations, for the output of the discriminator, with an accuracy of a constant coefficient, we obtain

$$u_{д}(t, \delta) = \int_{-\infty}^{t}\int_{-\infty}^{t} h(t-t_1)\, h(t-t_2) \sin\frac{\Delta\omega}{2}(t_2 - t_1) \times \\ \times [A(t_2)B(t_1) - A(t_1)B(t_2)]\, dt_1 dt_2. \tag{9.4.21}$$

Calculating by the usual method the slope of the discrimination characteristic and equivalent spectral density, we have

$$K_{д} = -\frac{1}{\pi}\int_{-\infty}^{\infty} |H(i\omega)|^2 S'_0\left(\omega + \frac{\Delta\omega}{2}\right) d\omega \tag{9.4.22}$$

and

$$S_{экв} = \frac{1}{K_{д}^2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\left[\left|H\left(i\left(\omega + \frac{\Delta\omega}{2}\right)\right)\right|^2 - \right.\right. \\ \left.\left. - \left|H\left(i\left(\omega - \frac{\Delta\omega}{2}\right)\right)\right|^2\right]\left[S_0(\omega) + \frac{1}{h}\right]\right\}^2 d\omega. \tag{9.4.23}$$

The discrimination characteristic of the given form of a frequency discriminator is expressed as

$$a(\delta)=\overline{u_{д}(t,\delta)}=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}|H(i\omega)|^2\left[S_0\left(\omega+\frac{\Delta\omega}{2}-\delta\right)-\right.$$
$$\left.-S_0\left(\omega+\frac{\Delta\omega}{2}+\delta\right)\right]d\omega, \qquad (9.4.24)$$

whereas its equivalent fluctuation characteristic is found by formula

$$S_{экв}(\delta)=\frac{1}{K_{д}^2}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[\left|H\left(i\left(\omega+\frac{\Delta\omega}{2}\right)\right)\right|^2-\right.$$
$$\left.-\left|H\left(i\left(\omega-\frac{\Delta\omega}{2}\right)\right)\right|^2\right]^2\left[S_0(\omega-\delta)+\frac{1}{h}\right]^2d\omega. \qquad (9.4.25)$$

The obtained characteristics determine all properties of the considered discriminator interesting us. As also for the preceding circuits, we calculate them with spectral density $S_0(\omega)$ having the form of (9.3.10) and frequency response $H(i\omega)$ of form (9.4.7). Then the transmission factor of the discriminator is expressed by formula

$$K_{д}=k\frac{2\alpha(1+\alpha)\mu^2}{[\mu^2+(1+\alpha)^2]^2}, \qquad (9.4.26)$$

where k, as before, is a coefficient which depends on h due to the influence of automatic gain control;

$$\alpha=\frac{\Delta f_{к}}{\Delta f_c};\ \mu=\frac{\Delta\omega}{4\Delta f_c}.$$

Equivalent spectral density takes the form

$$S_{экв}=\frac{\Delta f_c}{4\alpha}\frac{\mu^2+(1+\alpha)^2}{(\mu^2+\alpha^2)(1+\alpha)^2}\left\{[\mu^2+(1+\alpha)(1+5\alpha+8\alpha^2)]+\right.$$
$$+\frac{2}{h}[\mu^2+(1+\alpha)^2](\mu^2+1+4\alpha+5\alpha^2)+$$
$$\left.+\frac{1}{h^2}[\mu^2+(1+\alpha)^2]^2\right\}. \qquad (9.4.27)$$

Quantity $S_{ЭКВ}$, as also for the preceding circuits, has three terms, variously depending on the signal-to-noise ratio h and explained by beats of different components of the mixture of the signal with noise. The dependence of $S_{ЭКВ}$ on h, α, and μ can be investigated most vividly by graph. In Fig. 9.10 there is shown the dependence of $S_{ЭКВ}/S_{ОПТ}$ on h for different values of α and μ. From these curves we see that the dependence of $S_{ЭКВ}/S_{ОПТ}$ on the ratio of bandwidths $\alpha = \Delta f_К/\Delta f_С$ is not monotonic, and as for the preceding circuits there exists an optimum value of the bandwith of the loops $\Delta f_К$, minimizing $S_{ЭКВ}$. For small μ this optimum band

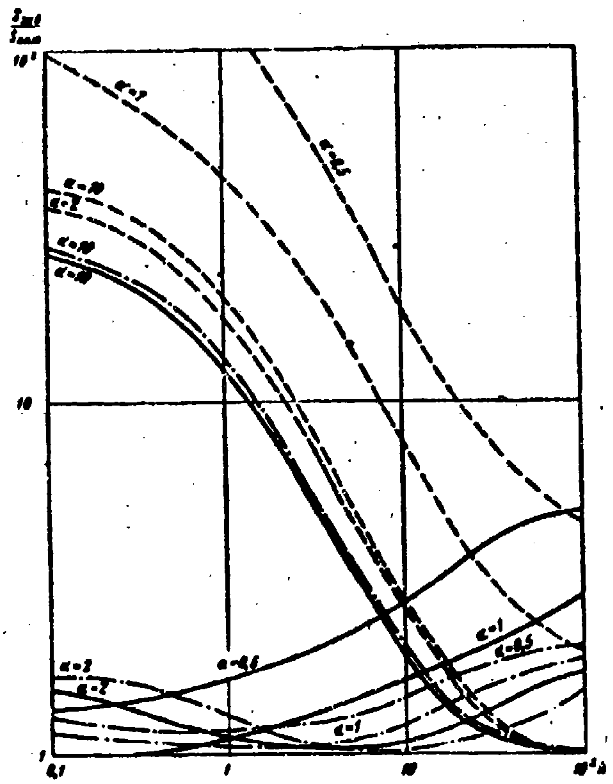

Fig. 9.10. Dependence of $S_{\text{экв}}/S_{\text{опт}}$ on the signal-to-noise ratio h for a discriminator with detuned loops: —— $\mu = 0.3$; --- $\mu = 1$; -.- $\mu = 5$.

is close to $\Delta f_c \sqrt{1 + h}$, as one should have expected, since as $\mu \rightarrow 0$ and with such selection of the loop bandwidth the circuit approaches its optimum form. For large μ the value of the optimum bandwidth of the loops changes.

If, fixing band of the loops and the signal-to-noise ratio, we investigate the dependence of the magnitude of $S_{\text{экв}}$ on detuning of loops μ, it is easy to see that for large α it increases with growth of detuning μ, while for small α in a certain range of variation of h there is the minimum spectral density for a certain optimal value of detuning μ.

Thus, in this circuit, just as in the others, deviation from optimum value by one of the parameters (bandwidth of loops) leads to change of the optimum value of the other parameter (detuning). From Fig. 9.10 it follows that with some, at first glance, fully reasonable selection of parameters of the circuit its accuracy may be an order or two lower than the potential. This testifies to the necessity of careful selection of bands of the loops and their detuning in the considered circuit, and also the great sensitivity of this form of discriminator to its parameters.

For a more complete judgement of accuracies which are provided by the given circuit it is also necessary to calculate the coefficient of parametric fluctuations $S_{\text{пар}}$. It, as earlier, is calculated from the fluctuation characteristic and is presented graphically in Fig. 9.11. Quantity $S_{\text{пар}}$ contains a component not depending on h, owing its origin to beats of signal components. With decrease of h coefficient $S_{\text{пар}}$ grows inversely proportionally to h. For very small h there is no sense in using the obtained dependences, since error of the system of measurement no longer is expressed correctly through $S_{\text{пар}}$.

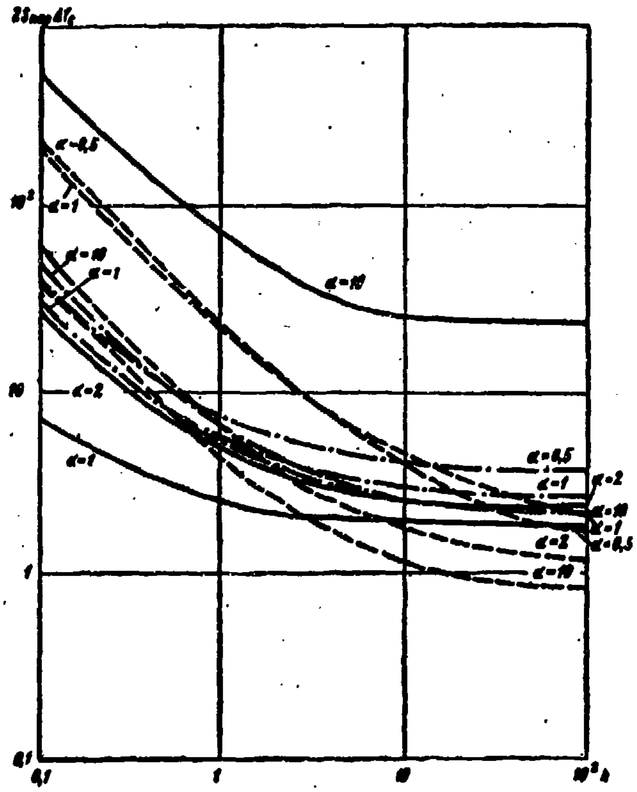

Fig. 9.11. Dependence of $S_{пар}$ on h for a discriminator with detuned loops: —— μ = 0.3; --- μ = 1; -.- μ = 5.

9.4.4. Circuit with Frequency Variation

Part of the considered discriminator circuit, determining the principle of its action, without circuits of preliminary processing of the signal, is presented in Fig. 9.12. The input signal is mixed with output voltage of a local oscillator controlled from the output of the meter. The output of the meter determines the mean value of the frequency of the local oscillator, periodically varying (near this mean value) according to a law assigned by the generator of reference voltage. From the output of the mixer the signal proceeds to a bandpass filter (tuned loop), is detected by a square-law amplitude detector and enters a stage which multiplies it by the reference voltage. Output of this stage is the output of the discriminator.

The control voltage will be formed by frequency variation of the signal at the output of the mixer near the frequency of tuning of the loop. The reference voltage, varying the frequency of the local oscillator, and consequently also of the signal at the input of the loop, may have different laws of change. Most frequently this voltage has a square form, so that it constitutes a sequence of positive and negative pulses of identical amplitude, of

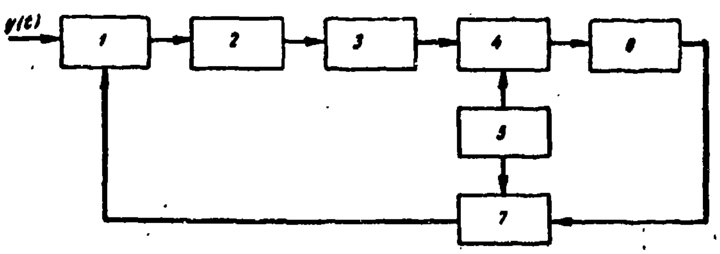

Fig. 9.12. Circuit of a frequency discriminator with frequency variation: 1 — mixer; 2 — tuned loop; 3 — square-law detector; 4 — stage of multiplication; 5 — generator of reference voltage; 6 — filter; 7 — controlled local oscillator.

duration $T_{\Pi}/2$, with period T_{Π}. As a result frequencies of the local oscillator and the signal at the input of the loop deviate from face values in successive half-periods by $\pm\frac{\Delta\omega}{2}$, respectively. With such form of reference voltage the element multiplying output voltage of the detector by the reference signal is simply a commutation switch.

To ensure normal work of the circuit the frequency of pulses of reference voltage should be great as compared to the effective bandwidth of the tracking speed meter. At the same time it should be small as compared to bandwidth of the loop of the discriminator.

In this case output voltage of the discriminator $u_{Д}(t, \delta)$ can be found if we present the signal at the input of the depicted circuit in the form

$$u_1(t)=E(t)\cos[(\omega_Г+\omega_0)t+\varphi(t)], \tag{9.4.28}$$

where $E(t)$ and $\varphi(t)$ – random functions, where

$E(t)\cos\varphi(t)$ and $E(t)\sin\varphi(t)$ – independent normal random processes with spectral density $S_0(\omega) + 1/h$;

$\omega_Г$ – center frequency of tuning of the heterodyning oscillator;

$\omega_0 = \omega_{ПР} + \delta$ – frequency of tuning of the loop.

Designating the pulse response of the loop $h(t)\cos\omega_{ПР}t$, for output voltage of the discriminator we can obtain the following expression;

$$\begin{aligned}u_Д(t, \delta)=&f(t)\int_{-\infty}^{t}\int_{-\infty}^{t}h(t-\tau_1)h(t-\tau_2)E(\tau_1)E(\tau_2)\times\\ &\times\cos\left[\left(\frac{\Delta\omega}{2}+\delta\right)(\tau_1-\tau_2)+\varphi(\tau_1)-\varphi(\tau_2)\right]d\tau_1d\tau_2-\\ &-f\left(t-\frac{T_п}{2}\right)\int_{-\infty}^{t}\int_{-\infty}^{t}h(t-\tau_1)h(t-\tau_2)E(\tau_1)E(\tau_2)\times\\ &\times\cos\left[\left(\frac{\Delta\omega}{2}-\delta\right)(\tau_1-\tau_2)-\varphi(\tau_1)+\varphi(\tau_2)\right]d\tau_1d\tau_2,\end{aligned} \tag{9.4.29}$$

where $f(t)$ – periodic function with period $T_П$, determined as

$$f(t)=\begin{cases}1 & nT_п<t<\left(n+\frac{1}{2}\right)T_п,\\ 0 & \left(n+\frac{1}{2}\right)T_п<t<(n+1)T_п,\end{cases} \tag{9.4.30}$$

n – integers.

Considering the inertia of subsequent smoothing circuits, the characteristic of the considered circuit interesting us can be found by averaging in period $T_П$ the results of averagings of the set, connected with use of (9.4.29). Then for

the gain factor and equivalent spectral density of the discriminator it is possible to write

$$K_Д = \frac{1}{T_и}\int_0^{T_и} \overline{\frac{\partial u_Д(t,\delta)}{\partial\delta}}\Big|_{\delta=0} dt =$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} |H(i\omega)|^2 \left[S_0'\left(\omega+\frac{\Delta\omega}{2}\right) - S_0'\left(\omega-\frac{\Delta\omega}{2}\right)\right] d\omega, \quad (9.4.31)$$

$$S_{экв} = \frac{1}{T_и}\int_0^{T_и}\frac{1}{K_Д^2}\int_{-\infty}^{\infty}[\overline{u_Д(t,0)\,u_Д(t+\tau,0)} -$$

$$- \overline{u_Д(t,0)}\;\overline{u_Д(t+\tau,0)}]\,d\tau dt =$$

$$= \frac{1}{K_Д^2}\,\frac{2}{\pi}\int_{-\infty}^{\infty}\left\{|H(i\omega)|^2\left[S_0\left(\omega-\frac{\Delta\omega}{2}\right)+\frac{1}{h}\right]\right\}^2 d\omega, \quad (9.4.32)$$

where as before $H(i\omega)$ – Fourier transform of $h(t)$.

For a spectral density of the signal $S_0(\omega)$ of form (9.3.10) and frequency response $H(i\omega)$ of form (9.4.7) calculation by formulas (9.4.31) and (9.4.32) gives

$$K_Д = k\frac{\alpha(1+\alpha)\mu}{[\mu^2+(1+\alpha)^2]^2} \quad (9.4.33)$$

and

$$S_{экв} = \frac{2\Delta f_c[\mu^2+(1+\alpha)^2]}{\alpha\mu^2(1+\alpha)^2}\Big\{(1+\alpha)[\alpha+\mu^2(1-\alpha)^2+$$

$$+(1+\alpha)^2(\alpha+(1+\alpha)^2)]+\frac{2}{h}[\mu^2+(1+\alpha)^2][\mu^2+$$

$$+(1+\alpha)^2(1+2\alpha)]+\frac{1}{h^2}[\mu^2+(1+\alpha)^2]^2\Big\}, \quad (9.4.34)$$

where $\alpha = \Delta f_К/\Delta f_С$; $\mu = \Delta\omega/4\Delta f_С$.

The transmission factor of the discriminator here depends on parameters of the circuit just as in the circuit with detuned loops; however, the equivalent spectral density has another magnitude. Investigation of formula (9.4.34) and the graphic presentation of the dependence of $S_{ЭКВ}/S_{ОПТ}$ on h for different α and μ, shown in Fig. 9.13, show that the spectral density of fluctuations at the output of the given form of discriminator considerably exceeds that for the optimum discriminator. This occurs for any combinations of parameters $\Delta\omega$; $\Delta f_К$; $\Delta f_С$ and h.

In distinction from the circuit with detuned loops, as $\Delta\omega \to 0$ (with simultaneous increase of amplification in the circuit) the magnitude of $S_{ЭКВ}$ for a circuit with frequency variation grows proportionally to $1/\mu^2$. Increase of frequency deviation $\Delta\omega$ also leads to growth of $S_{ЭКВ}$, proportional to $(\Delta\omega)^6$. With increase of h, if

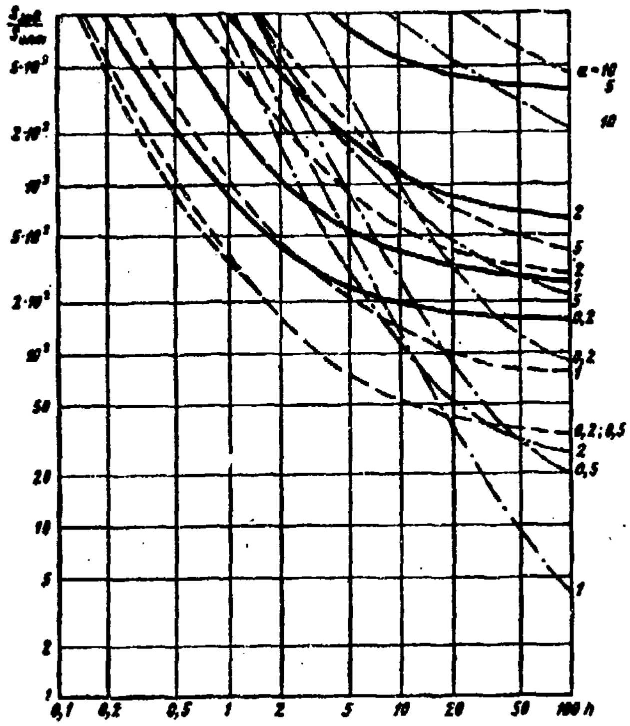

Fig. 9.13. Dependence of $S_{вкв}/S_{опт}$ on the signal-to-noise ratio h for a discriminator with frequency variation: —— μ = 0.3; --- μ = 1; -.- μ = 5.

$\Delta f_K/\Delta f_C$ and $\Delta f_K/\Delta\omega$ are great, this equality is satisfied:

$$\lim_{h\to\infty} S_{вкв} = \frac{32\Delta f_K^4}{\Delta f_c \Delta\omega^2}, \qquad (9.4.35)$$

i.e., for large signal-to-noise ratios the spectral density of fluctuations at the output of a discriminator with frequency variation will be in $16(\Delta f_K/\Delta f_C)^2 \times$ $\times (\Delta f_K/\Delta\omega)^2$ greater than in an optimum discriminator and all discriminators previously considered.

Thus, the considered circuit gives sharp worsening of results as compared to the preceding circuits. The physical explanation of nonoptimality of the discriminator with frequency variation is the following. Signals in various half-periods $T_п/2$ are uncorrelated, since for normal work of the circuit there should be satisfied the relationship $T_п/2 \gg$ $\gg 1/\Delta f_C$. Therefore, any averaging of results of sequential amplitude detection of signals in various half-periods cannot compensate fluctuations, whereas in the previous circuits such compensation occurred.

Considering nonoptimality of the considered circuit, we shall not investigate it in more detail. In particular, we shall not calculate the coefficient of parametric fluctuations $S_{пвг}$.

It is necessary to note that a circuit with frequency variation is sensitive to interferences modulated by the frequency of switching or its odd harmonics. In this respect the given discriminator is analogous to goniometric discriminators with conical or square scanning, which will be considered in detail in subsequent chapters. Therefore, we will not discuss this question in detail here.

9.4.5. Inaccuracy of Reproduction of Modulation of the Signal During Reception

During the analysis of different frequency discriminators we assumed that in

them there is performed exact demodulation in phase (frequency) and exact mutiplication by a function expressing the law of amplitude modulation of signal, taking into account the true value of its delay (matched gating in the case of a pulse signal). In real radar receivers exact fulfillment of these functions is impossible. Besides technical tolerances on laws of gating and change of frequency of heterodyne oscillators, here more essential circumstances play a role. Thus, delay of the function expressing the law of modulation of the signal reproduced during reception never coincides exactly with true delay of the reflected signal due to the presence of errors of range finders; during gating we often specially apply gate pulses which are longer than the pulse of the signal, proceeding from considerations connected with breakoff of tracking during intense noises, etc.

We shall show that mismatch of laws of modulation of the reflected signal and of signals used for processing the received signal in the receiver leads simply to decrease of the signal-to-noise ratio. Actually, if in the receiver as laws of amplitude and phase modulation of the signal we use functions $u_{a1}(t - \tau - \Delta)$ and $\psi_1(t - \tau - \Delta)$ instead of functions $u_a(t - \tau)$ and $\psi(t - \tau)$, the signals at the input of the narrow-band filters of the discriminators have the form

$$v(t)=\frac{1}{2}\sqrt{P_c}E(t)u_a(t-\tau)u_{a1}(t-\tau-\Delta)\cos[\omega_{\text{пр}}t+ \\ +\psi(t-\tau)-\psi_1(t-\tau-\Delta)+\varphi(t)]+ \\ +u_{a1}(t-\tau-\Delta)n(t)\cos[\omega_r t-\psi_1(t-\tau-\Delta)], \quad (9.4.36)$$

where Δ – time shift caused by inaccuracy of range tracking;

τ – delay of the function describing the law of modulation of the reflected signal;

$E(t)$ and $\varphi(t)$ – random functions representing fluctuations of the reflected signal;

$n(t)$ – white noise with spectral density N_0;

P_c – mean signal power.

We assume the following normalization of signals:

$$\overline{E^2(t)}=1,\ \frac{1}{T_r}\int_0^{T_r}u_a^2(t-\tau)\,dt=1,$$

where T_r – period of functions $u_a(t)$, $\psi(t)$, $u_{a1}(t)$, $\psi_1(t)$.

Due to narrow-bandedness of filters of the discriminators through them there only pass components of the signal spectrum which are concentrated near frequency $\omega_{\text{пр}}$. Therefore, according to the character of their influence on the discriminator, signals with periodic modulation are equivalent to continuous unmodulated signals. As it was shown in Chapter IV, the signal-to-noise ratio for such an equivalent

unmodulated signal with ideal matching of $u_{a1}(t)$ with $u_a(t)$ and $\psi_1(t)$ with $\psi(t)$ is equal to $h = P_c/2\Delta f_c N_0$. With imperfect matching of these functions at the input of the narrow-band filter there acts v(t), determined by (9.4.36). Averaging the result of passage of v(t) through the filter and calculating the power of the signal and of the components, we find the new signal-to-noise ratio

$$h_1 = h \frac{\left|\frac{1}{T_r}\int_0^{T_r} u_a(t-\tau)\, u_{a1}(t-\tau-\Delta)\, e^{i[\psi(t-\tau)-\psi_1(t-\tau-\Delta)]}\, dt\right|^2}{\frac{1}{T_r}\int_0^{T_r} u_{a1}^2(t-\tau-\Delta)\, dt}. \tag{9.4.37}$$

All formulas obtained for the above-considered discriminators remain valid upon replacement of h_1 for h. It should be noted that in such an approach we consider assigned both the form of functions $u_{a1}(t)$ and $\psi_1(t)$, and also quantity Δ. By virtue of the random nature of the output of the range finder, Δ, in general, randomly varies in time. It is possible, however, to fix some, for instance, the worst value of Δ and, by calculating the corresponding quantity h_1, to find how much accuracy of measurement of speed will decrease.

For illustration of magnitudes of change of the signal-to-noise ratio we shall consider two examples.

1. Let us assume that the signal is modulated only in amplitude by square pulses of duration $\tau_И$, and that the gate pulse also has square form and duration $\tau_c \geq \tau_И$. From (9.4.37) we obtain

$$h_1 = \begin{cases} h\frac{\tau_И}{\tau_c} & \text{at } \Delta < \frac{\tau_c - \tau_И}{2} \\ h\frac{(\tau_И + \tau_c - 2\Delta)^2}{4\tau_И\tau_c} & \text{at } \frac{\tau_c - \tau_И}{2} \leq \Delta < \frac{\tau_c + \tau_И}{2}. \end{cases} \tag{9.4.38}$$

This relationship shows the power loss connected with increase of duration of the gate as compared to $\tau_И$. Corresponding loss in spectral density $S_{ЭКВ}$, and consequently, in accuracy of measurement of speed can be estimated from the preceding formulas and graphs.

2. Let us assume that the signal is modulated in frequency by a sinusoidal law with frequency Ω and deviation ω_m, and heterodyne voltage has different frequency deviation ω_{m1}. Then, with zero mismatch Δ calculations by (9.4.37) give

$$h_1 = hJ_0^2\left(\frac{\omega_m - \omega_{m1}}{\Omega}\right), \tag{9.4.39}$$

where $J_0(x)$ — Bessel function of zero order.

For sufficiently large values of $\omega_m - \omega_{m1}/\Omega$ the signal-to-noise ratio rapidly drops, and for certain values of $\omega_m - \omega_{m1}/\Omega$ the signal of the mixer, in general, is absent. From relationship (9.4.39) there ensues the expediency of selection of a low frequency of modulation Ω at which a comparatively large difference in deviations ω_m and ω_{m1} does not lead to substantial increase of errors of measurement.

9.4.6. Comparison of Circuits of Discriminators

As follows from the conducted analysis, the first three of the four considered forms of discriminators with proper selection of their parameters give results very close to an optimum discriminator. The fourth circuit, with frequency variation of the heterodyne oscillator, is the farthest from optimum. It gives substantially less accuracy of measurements, is sensitive to easily created interferences with periodic modulation and, furthermore, on it there must be stringent requirements of identity of the positive and negative half-periods of the reference voltage. Nonfulfillment of these requirements and also instability of the frequency of tuning of filters lead to the appearance of systematic errors of measurement of speed. By virtue of these circumstances it is doubtful whether one should recommend application of this scheme of a frequency discriminator, if, of course, requirements on accuracy of measurement are rather high.

The circuit with detuned loops ensures accuracy of measurements close to the potential ($S_{экв}$ is close to $S_{опт}$), even if we select detuning of loops proceeding from the condition of a maximum gain factor of the discriminator $K_Д$. However, with small variation of parameters of the circuit its errors can increase rather considerably, i.e., among deficiencies of the circuit are criticality with respect to the parameters. One should note especially that with discrimination at sufficiently high frequencies it is necessary to have two filters, tuned to close, but, in principle, different frequencies. Instability of these frequencies can lead here to appearance of systematic error of measurement, which is characteristic for other circuits, to substantial change of the fluctuation error of measurement and even to cessation of work of the discriminator if the frequencies of tuning become identical or, conversely, go too far apart. Therefore, although in principle the circuit with detuned loop is also close to optimum, it is not always possible to recommend its use.

The most acceptable of those considered for practical application are circuits of

discriminators with a tuned loop and a phase shifter and those with mixers and differentiation. These circuits are identical in their characteristics, and with correct selection of parameters, in particular the bandwidth of the loops, they give results sufficiently close to an optimum discriminator. In particular, equivalent spectral density for them does not exceed $S_{\text{опт}}$ by more than 20%. The influence of instability of parameters for such circuits is considerably less than for the ones discussed above. Choice between these two circuits should be made from considerations of technical convenience. The circuit with a tuned loop and a phase shifter in principle is less bulky, but requires creation of a narrow-band (in most cases, quartz) filter. The circuit with mixers and differentiation does not possess this deficiency; during its realization it is not necessary to think about stability of tuning of the loop, but it is sensitive to inaccuracy of the phase shifter of the heterodyne voltage. Furthermore, this circuit contains two filters instead of the one in the circuit with a tuned loop.

We can, of course, use other schemes of frequency discriminators. Their qualities should be compared with qualities of the optimum circuit to find the expedienc, f their application.

Let us discuss briefly selection of the transmission band of filters in the two best circuits. Usually meters on which there are requirements of high accuracy work at a high signal-to-noise ratio h. Here, the selected transmission band should be changed with change of h and depending upon the width of the spectrum of the signal. However, in practice neither the signal-to-noise ratio nor the width of the spectrum of the signal are ever known exactly. Therefore, it is necessary to select bandwidth of the filters proceeding from the worst case, considering that $S_{\text{экв}}$ monotonically decreases with growth of h and with decrease of width of the spectrum of the signal. Considering these circumstances, it is necessary to select filter bandwidth from the highest possible width of the spectrum of the signal and from a minimum signal-to-noise ratio, occurring at the maximum range, determined by conditions of lock-on and transition to automatic tracking. With such selection of bandwidth of the filters with decrease of distance and with narrowing of the width of the spectrum of the signal the system of measurement of speed will be further from the optimum than under the assumed conditions; however, its error will always be within permissible limits.

Often from technical considerations it is necessary to expand the bandwidth of the filters with respect to its optimum value, since the signal spectrum has a

width of the order of units of cycles per second or even fractions of cycles per second, and stabilities of frequencies obtainable relatively simple are such that the bandwidth of filters cannot be made smaller than ten cycles per second. Usually this does not lead to essential impairment of qualities of the discriminator.

§ 9.5. Smoothing Circuits and Accuracy of Measurement of Speed

Studying properties of discriminators of Doppler speed meters, we turn to estimation of accuracy of these meters. For this it is necessary to assign the second component part of the tracking meter — the smoothing circuits. Very often there are applied linear smoothing circuits with constant parameters of certain very simple forms. Let us consider, first of all, error of speed meters with such smoothing circuits. This can be done by applying formulas of § 6.2, determining fluctuation, dynamic and systematic errors of tracking meters. We shall specially discuss cases in which optimum smoothing circuits have constants parameters, and we shall estimate accordingly accuracies of optimum speed meters. However, in other cases optimum smoothing circuits, as it was shown in Chapter VI, have variable parameters. Therefore, we shall also investigate accuracy of speed meters with variable parameters of their smoothing circuits.

9.5.1. Smoothing Circuits with Constant Parameters

In speed meters there are applied smoothing circuits of the same forms as in range finders. Therefore, many formulas and results of § 7.10 (Paragraph 7.10.2) remain in force for speed meters.

If we disregard parametric fluctuations, fluctuation error of the meter is expressed as before by formula

$$\sigma^{2}_{\text{фл}}=2S_{\text{экв}}\Delta f_{\text{эф}}, \tag{9.5.1}$$

in which $S_{\text{экв}}$ is determined by the above-mentioned analysis of frequency discriminators, and $\Delta f_{\text{эф}}$ — effective bandwidth of the closed tracking speed meter.

As it was shown in § 7.10, for a smoothing circuit in the form of an integrator

$$\Delta f_{\text{эф}}=\frac{K_{\text{и}}K_{\text{д}}}{4}, \tag{9.5.2}$$

where $K_{\text{и}}$ — gain factor (dimensional) of the integrator;

$K_{\text{д}}$ — discriminator gain factor.

For an RC-filter

$$\Delta f_{\text{эф}}=\frac{KK_{\text{д}}}{4T_1}. \tag{9.5.3}$$

where $T_1 = RC$, and K — gain factor.

For a smoothing circuit in the form of a double integrator with correction

$$\Delta f_{э\phi} = \frac{1 + KK_д T_к^2}{4T_к}, \tag{9.5.4}$$

where K — gain factor of the two integrators;

$T_к$ — time constant of the correcting network.

The value of $\Delta f_{э\phi}$ reaches a minimum, equal to $\sqrt{KK_д}/2$, at $T_к = 1/\sqrt{KK_д}$.

Finally, if the smoothing filter consists of two RC-circuits (inertial links) with correction

$$\Delta f_{э\phi} \approx \frac{KK_д T_к}{T_1 T_2}, \tag{9.5.5}$$

where T_1 and T_2 — time constants of the inertial links.

In the last case the minimum of $\Delta f_{э\phi}$ is reached at $T_к = \sqrt{T_1 T_2 / 2KK_д}$ and is

$$\Delta f_{э\phi} = \frac{3\sqrt{2KK_д}}{8\sqrt{T_1 T_2}}. \tag{9.5.6}$$

If we select the mirimum effective bandwidth in the absence of noise, the dependences of the effective bandwidth on the signal-to-noise ratio h, determined by formulas (7.10.19), (7.10.14) and (7.10.10), also remain in force. Then with use of an optimum discriminator [with the simplest form of spectral density of fluctuations (9.3.10)] for smoothing circuits of the first order we have

$$\sigma^2_{\phi п} = 4\Delta f_с \Delta f_{э\phi 0} \frac{\sqrt{1+h}\,(1+\sqrt{1+h})^2}{h} \frac{1}{h+y}, \tag{9.5.7}$$

where $\Delta f_{э\phi\ 0}$ — value of the effective bandwidth of a tracking speed meter in the absence of noises, and

$$y = \frac{\Delta f_у}{\Delta f_с} \tag{9.5.8}$$

is the ratio of bandwidth of the amplifier covered by the AGC loop to the signal bandwidth. In the case of application of a smoothing circuit in the form of a double integrator with correction

$$\sigma^2_{\phi п} = 2\Delta f_с \Delta f_{э\phi 0} \frac{\sqrt{1+h}\,(1+\sqrt{1+h})^2}{h^2}\left(1 + \frac{h}{h+y}\right). \tag{9.5.9}$$

Curves of the dependence of variance of fluctuation error on the signal-to-noise ratio h for various y are shown in Fig. 9.14. Consideration of them leads to the conclusion that with the selection of an identical effective bandwidth in the absence of noise $\Delta f_{э\phi\ 0}$ flucuation error with noise in the case of application

of filters of the 2nd order is greater than with application of filters of the 1st order. This is explained by the varying dependence of the effective bandwidth of the system on the gain factor of the discriminator in the two considered cases.

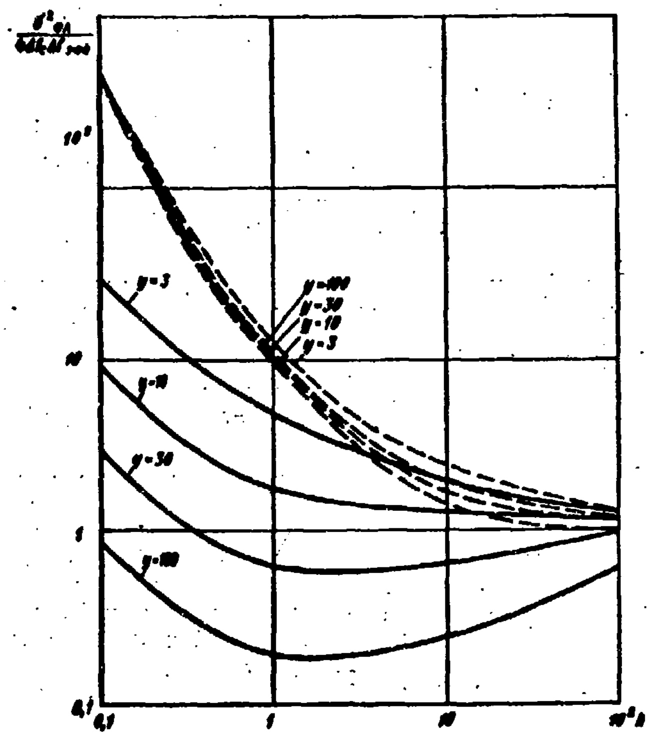

Fig. 9.14. Dependence of variance of fluctuation error of a speed meter $\sigma^2_{\text{фл}}$ on the signal-to-noise ratio h during application of an optimum discriminator and smoothing circuits with constant parameters: —— for circuits of the 1st order; --- for circuits of the 2nd order.

For smoothing circuits of the 1st order there is noticeable dependence of fluctuation error on selection of amplifier bandwidth Δf_y (or y). With broadening of this band error decreases due to increase of intensity of noise at the output of the amplifier, suppression of the gain factor, and consequently also of the effective bandwidth of the system. For smoothing circuits of the 2nd order the dependence of fluctuation error on Δf_y is immaterial.

Increase of fluctuation errors with application of non-optimal discriminators is easily found by multiplication of the obtained values of $\sigma^2_{\text{фл}}$ by ratio $S_{\text{экв}}/S_{\text{опт}}$, found for various discriminators in the preceding paragraph.

It should be noted that from consideration of Fig. 9.14 one should not make conclusions about the advantages of filters of the 1st order, since it is also necessary to consider other components of errors of measurement (dynamic and systematic errors), which turn out to be larger for smoothing circuits of the 1st order.

Passing to consideration of dynamic and systematic errors, let us note that all results obtained in Chapter VII for systems of range finding remain valid for speed meters. In particular, considering that speed of the target has a component which varies randomly, and assuming that this component is a stationary random process with spectral density $S(\omega)$, for dynamic error, according to (7.10.20), we have formula

$$\sigma^2_{\text{дин}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{S(\omega)\, d\omega}{|1 + K_{\text{д}} H(i\omega)|^2}, \qquad (9.5.10)$$

where $K_{\text{Д}}$ – transmission factor of the discriminator;

$H(i\omega)$ – frequency response of the smoothing circuit.

With a very simple form of spectral density of speed

$$S(\omega) = \frac{2\sigma_0^2 T}{1 + \omega^2 T^2} \qquad (9.5.11)$$

(where σ_0^2 – variance of the random component of speed, and T – time of correlation of the process of its change) formulas (7.10.22) and (7.10.23) for $\sigma^2_{\text{ДИН}}$ turn out to be valid, from which, in particular, it follows that dynamic error increases with decrease of the effective bandwidth of the tracking meter, taking place with decrease of the signal-to-noise ratio h.

If the measured speed varies as a linear combination of assigned functions with unknown coefficients

$$V(t) = \sum_{i=1}^{m} \mu_i f_i(t) + \overline{V(t)}, \qquad (9.5.12)$$

where $\overline{\mu}_1 = 0$, $\overline{\mu_1 \mu_k} = M_{1k}$, $\overline{V(t)}$ – mathematical expectation of speed, according to (7.10.25) we have

$$\sigma^2_{\text{дин}} = \sum_{i,k=1}^{m} M_{ik} \varepsilon_i(t) \varepsilon_k(t). \qquad (9.5.13)$$

Here

$$\varepsilon_i(t) = \int_{t_0}^{t} \vartheta(t, \tau) f_i(\tau)\, d\tau, \qquad (9.5.14)$$

$\vartheta(t, \tau)$ is determined by equation (6.2.6), and by virtue of constancy of parameters of smoothing circuits the Fourier transform of $\vartheta(t - \tau)$ is found as

$$\Theta(i\omega) = \frac{1}{1 + K_{\text{д}} H(i\omega)}. \qquad (9.5.15)$$

With a nonstatistical approach to the question of changes of speed dynamic error can be obtained if in (9.5.13) we set $M_{1k} = 1$. Then, using as the smoothing circuit an integrator and with change of speed according to the law $V(t) = v_0 + v_1 t$ dynamic error as $t \to \infty$ we determine as

$$\varepsilon_{\text{дин}} = \frac{v_1}{K_{\text{д}} K_{\text{и}}}. \qquad (9.5.16)$$

Analogously, with smoothing circuits with two integrators and $V(t) = v_0 +$ $+ v_1 t + v_2 t^2$,

$$\varepsilon_{\text{дин}} = \frac{v_1}{K_{\text{д}} K}. \tag{9.5.17}$$

Dynamic error is inversely proportional to the gain factor of the open circuit of the tracking speed meter and increases with decrease of the signal-to-noise ratio.

Dynamic errors in the nonstatistical approach to measurement of speed are determined in the same manner as systematic errors, so that it is possible to consider we have simultaneously discussed the method of finding the latter.

By virtue of the different dependences of fluctuati and dynamic errors on the gain factor, and thus on the signal-to-noise ratio h, as, too, for range finders, a compromise selection of the gain factor of the open loop is reasonable. Let us consider two examples of such selection.

1. Let us assume that the smoothing circuit is an integrator, and that the speed of the target varies as $V(t) = v_0 + v_1 t$. Then, the square of total error of measurement of speed in accordance with (9.5.1) and (9.5.16) is

$$\sigma^2 = \sigma^2_{\text{фл}} + \sigma^2_{\text{дин}} = \frac{K_{\text{и}} K_{\text{д}}}{2} S_{\text{экв}} + \frac{v_1^2}{(K_{\text{и}} K_{\text{д}})^2}. \tag{9.5.18}$$

In the statistical approach to dynamic error v_1^2 must be replaced by $\overline{v_1^2}$. Minimizing σ^2 by selection of $K_{\text{И}}K_{\text{Д}}$, we obtain the optimum value of the gain factor of the open loop

$$(K_{\text{и}} K_{\text{д}})_{\text{опт}} = \sqrt[3]{\frac{4v_1^2}{S_{\text{экв}}}}, \tag{9.5.19}$$

at which total error is

$$\sigma^2 = 1{,}19\,(v_1 S_{\text{экв}})^{2/3}. \tag{9.5.20}$$

Substituting in (9.5.19) the dependence of $S_{\text{ЭКВ}}$ on h, it is easy to find how the gain factor of the open loop should be changed with change of the signal-to-noise ratio. In particular, for an optimum frequency discriminator and spectral density of fluctuations of the signal of form (9.3.10)

$$(K_{\text{и}} K_{\text{д}})_{\text{опт}} = K_0 \sqrt[3]{\frac{h^2}{\sqrt{1+h}\,(1+\sqrt{1+h})^2}}, \tag{9.5.21}$$

where K_0 — the value of the gain factor when $h \to \infty$, i.e., in the absence of noise.

The dependence of the optimum gain factor on h is shown in Fig. 9.15.

2. Let us consider a smoothing circuit in the form of a double integrator with correction and the case of a square-law change of speed. The square of total error in this case is determined by expression (7.10.33) and

$$(KK_{\text{д}})_{\text{опт}} = \left(\frac{4v_2^2}{S_{\text{экв}}}\right)^{1/5}. \qquad (9.5.22)$$

so that with optimum gain

$$e^2 = 1{,}68 v_2^{2/5} S_{\text{экв}}^{4/5}. \qquad (9.5.23)$$

In the statistical approach to dynamic error in the formulas instead of v_2^2 there enters $\overline{v}_2^2$. With the same dependence of $S_{\text{экв}}$ on h as in the preceding case the optimum gain factor should vary with change of the signal-to-noise ratio as

$$(KK_{\text{д}})_{\text{опт}} = K_0\left[\frac{h^4}{\sqrt{1+h}\,(1+\sqrt{1+h})^4}\right]^{1/5}. \qquad (9.5.24)$$

This change is also presented in Fig. 9 15, where it is easy to see that dependences of the gain factor on the signa -to-noise ratio for smoothing circuits in the form of one integrator and of two egrators with correction are very close.

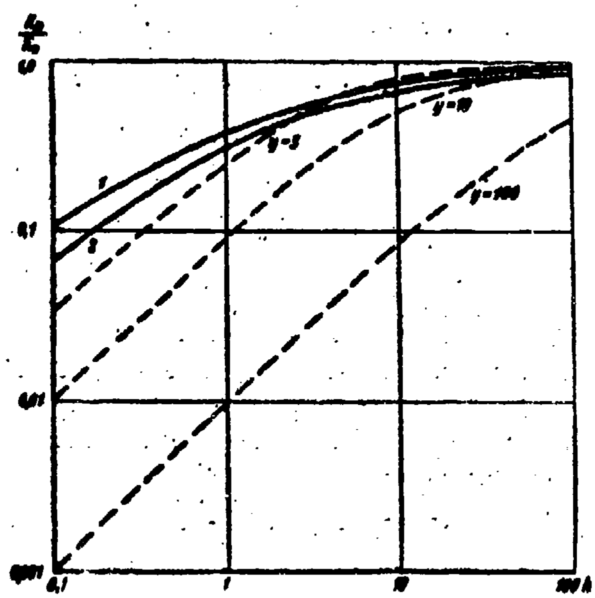

Fig. 9.15. The dependence of the gain factor of the open loop K_p on the signal-to-noise ratio h: 1 – optimum gain for circuits of the 1st order; 2 – optimum gain for a circuit of the 2nd order. --- change of gain from the circuit of automatic gain control.

In Fig. 9.15 there are also given curves of the dependences on h of the gain factor of the open loop, obtained due to the normalizing action of the AGC system. These curves are constructed on the assumption that in the absence of noise ($h \to \infty$) the gain factor is equal to the optimum. From the figure it is clear that when $h > 1$ and we have small ratios of bandwidths of the amplifier covered by automatic gain control and of the signal (y = 2 to 3) necessary change of the gain factor with sufficient accuracy is provided by the AGC system. For small h the system of automatic gain control leads to unnecessary lowering of the gain factor. Thus, in spite of the fact that with decrease of y fluctuation error grows (see Fig. 9.14), from the point of view of providing minimum total error it is necessary to select sufficiently small values of y.

The obtained relationships permit us to find error of speed meters under

conditions when it is possible to disregard parametric fluctuations. However in a number of cases such disregard is impermissible, since the spectral density of parametric fluctuations, as we have seen, can attain for the considered forms of frequency discriminators considerable magnitudes. Therefore, calculation of parametric fluctuations is of interest.

In accordance with formula (6.2.44) variance of total error during calculation of parametric fluctuations is defined as

$$\sigma^2=\sigma_0^2(1+2S_{\text{нар}}\Delta f_{\text{эф}}), \tag{9.5.25}$$

where σ_0^2 — the variance of error in the absence of parametric fluctuations, found above.

Considering that for frequency discriminators $S_{\text{нар}} = \varkappa/\Delta f_{\text{c}}$, where $\varkappa$ — a coefficient, varying in a wide range depending on the form and parameters of the frequency discriminator, and also on the magnitude of the signal-to-noise ratio h, we obtain

$$\sigma^2=\sigma_0^2\left(1+2\varkappa\frac{\Delta f_{\text{эф}}}{\Delta f_{\text{c}}}\right). \tag{9.5.26}$$

For large signal-to-noise ratios ($h \to \infty$) coefficient $\varkappa$ takes the following value:

— for an optimum discriminator $\varkappa = 0.5$;

— for a discriminators with a tuned loop and a phase shifter, and also with mixers and differentiation $\varkappa$ varies from 0.5 to 1 depending up $\alpha = \Delta f_{\text{к}}/\Delta f_{\text{c}}$:

— for a discriminator with detuned loops $\varkappa$ varies in the same range, depending upon the magnitude of detuning.

Therefore, with rapid fluctuations, when ratio $\Delta f_{\text{эф}}/\Delta f_{\text{c}}$ is small (it may attain magnitudes of the order of 10^{-3}), there are conditions in which parametric fluctuations can be disregarded. With slower fluctuations and for smaller h, at which $\varkappa$ grows; these conditions may not be observed. It is necessary, however, to recall that with $2\varkappa\frac{\Delta f_{\text{эф}}}{\Delta f_{\text{c}}}$ comparable with unity formula (9.5.26) ceases to be valid. Therefore, it gives the possibility of obtaining corrections due to parametric fluctuations only under the condition of smallness of these corrections, and the basic meaning of its application consists in determining conditions in which linear approximations are still valid.

Certain lowering of the influence of parametric fluctuations occurs from the AGC system. However, for Doppler speed meters this lowering is no longer found by formulas of Chapter VII inasmuch as parametric fluctuations (during rapid

fluctuations of the signal) are determined here not only by the random nature of changes of amplitude, but also by the random nature of changes of phase, not compensated by the AGC system.

9.5.2. Optimum Smoothing Circuits for Randomly Varying Speed

Linear smoothing circuits with constant parameters in certain cases analyzed in Chapter VI are optimum, i.e., ensure with application of the proper discriminater minimum error of measurement. This occurs when speed is either a stationary random process, or a random process with stationary increments.

Speed is a stationary random process, for instance, in the case of measurement of groundspeed by a Doppler dead-reckoner. With stationary random oscillations of the flying object relative to the horizontal line of flight, small stationary changes of its speed and sufficiently uniform relief of the locale, the measured components of groundspeed, just as the reflected signal, can be considered stationary processes.

The method of finding both the optimum filter and also errors of measurement obtained during its application is presented in § 6.8 for an arbitrary form of spectral density of the measured parameter. Let us consider the particular case of spectral density of speed of the form of (9.5.11), for which the frequency response of the optimum smoothing filter for a large time of observation, according to (6.8.11), has the form

$$G(i\omega)=\frac{2\sigma_0^2T}{1+\sqrt{1+2\sigma_0^2TK_{\text{опт}}}}\,\frac{1}{1+i\omega T}, \tag{9.5.27}$$

where σ_0^2 -- variance;

T — time of correlation of the process of change of speed;

$K_{\text{опт}}$ — gain factor of the optimum discriminator.

Variance of the total error of measurement of speed according to (6.8.13) is equal to

$$\sigma^2_{\text{смх}}=\frac{2\sigma_0^2}{1+\sqrt{1+2\sigma_0^2TK_{\text{опт}}}}. \tag{9.5.28}$$

Considering that in (9.5.28) there is implied application of an optimum frequency discriminator, and considering that its equivalent spectral density is determined by expression (9.3.11), we have

$$K_{\text{опт}} = \frac{1}{S_{\text{опт}}}\left(\frac{4\pi}{\lambda}\right)^2 = \left(\frac{4\pi}{\lambda}\right)^2 \frac{1}{2\Delta f_c} \times$$
$$\times \frac{h^2}{\sqrt{1+h}\,(1+\sqrt{1+h})^2}, \qquad (9.5.29)$$

whence

$$\sigma^2_{\text{вых}} = \frac{2\sigma_0^2}{1+\sqrt{1+\frac{T}{\Delta f_c}\left(\frac{4\pi\sigma_0}{\lambda}\right)^2 \frac{h^2}{\sqrt{1+h}\,(1+\sqrt{1+h})^2}}} \qquad (9.5.30)$$

For very small signal-to-noise ratios ($h \to 0$) quantity σ^2_{BHX} is equal to the a priori variance of change of speed; for large signal-to-noise ratios ($h \to \infty$)

$$\sigma^2_{\text{вых}} = \frac{2\sigma_0^2}{1+\sqrt{1+\frac{T}{\Delta f_c}\left(\frac{4\pi\sigma_0}{\lambda}\right)^2}}. \qquad (9.5.31)$$

The dependence of the mean square error of measurement of speed on the signal-to-noise ratio we shall show with an example. Considering $\sigma_0 = 10\ \frac{m}{sec}$, $\Delta f_c = 1000$ cps, $T = 10$ sec, and $\lambda = 4$ cm, for high-level noises ($h \to 0$) we obtain $\sigma_{BHX} = 10\ \frac{m}{sec}$; for low-level noises ($h \to \infty$) we obtain $\sigma_{BHX} \approx 0.8$ m/sec; the dependence of σ_{BHX} on h is shown in Fig. 9.16.

It is necessary to note that the obtained values of error of measurement of speed correspond to an optimum system, the gain factor of whose open loop should be equal to

$$K_p = \frac{2\sigma_0^2 T K_{\text{опт}}}{1+\sqrt{1+2\sigma_0^2 T K_{\text{опт}}}} \qquad (9.5.32)$$

and takes different values with change of the signal-to-noise ratio h. The graph of the dependence of K_p on h for the given example is shown in Fig. 9.17.

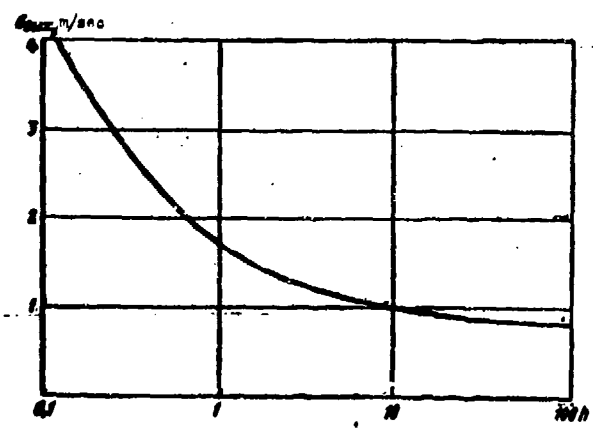

Fig. 9.16. Dependence of the mean square error of measurement of speed σ_{BHX} on the signal-to-noise ratio h for the example of an optimum meter when speed is a stationary process.

In the same figure there are shown curves of the dependence of the gain factor on h, determined by action of the automatic gain control system for three values of $y = \Delta f_y / \Delta f_c$. From

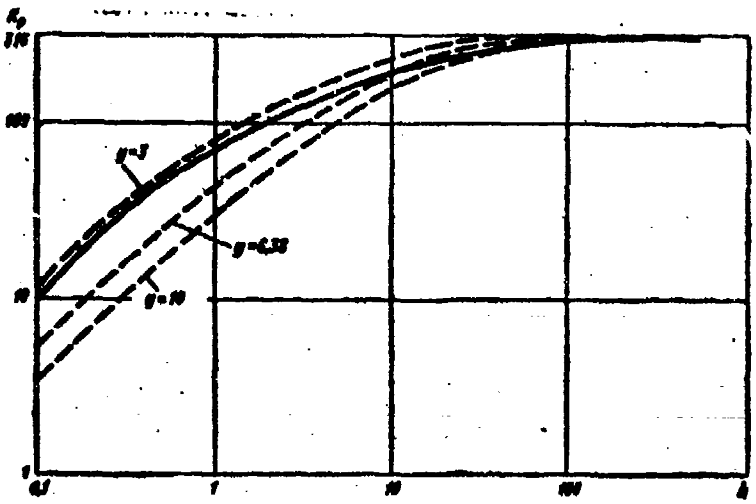

Fig. 9.17. Dependence of the gain factor of the open loop K_p on the signal-to-noise ratio h for the example of an optimum meter when speed is a stationary process: —— optimum gain; --- gain with application of automatic gain control.

consideration of these curves it may be concluded that satisfactory reproduction of the required law of change of amplification by automatic gain control is attained for values of h = 1 to 100 for y = 4 to 6.

In most cases the assumption of stationary random change of speed is not realized. Thus, in many cases the radar target can at any moment start to maneuver and never return to the past direction of motion, due to which the radial component of speed will change. However, sufficiently often change of speed can be considered a process with stationary increments. In particular, if changes of engine thrust, and consequently also of acceleration of the target are stationary, speed is nonstationary, but has a stationary first derivative, i.e., can be included in processes with stationary increments.

If we idealize the process of change of acceleration, considering it white noise with spectral density B_1 [m^2/sec^3], the speed is a Wiener process. For large time of observation the optimum smoothing filter, according to (6.8.28), is an integrator with a gain factor equal to $\sqrt{B_1/K_{\text{опт}}}$, where $K_{\text{опт}}$ – gain factor of the discriminator. The gain factor of the open loop here is defined as $\sqrt{B_1 K_{\text{опт}}}$ and should be different for different signal-to-noise ratios h. Thus, for an optimum discriminator in the same conditions as in the preceding case

$$K_p = \frac{4\pi}{\lambda}\sqrt{\frac{B_1}{2\Delta f_0}}\,x = K_0 x. \qquad (9.5.33)$$

The magnitude of variance of error of the meter turns out to be equal to

$$\sigma^2_{вых} = \sqrt{\frac{B_1}{K_{опт}}} = \frac{\lambda}{4\pi}\sqrt{2\Delta f_0 B_1}\,\frac{1}{x}, \qquad (9.5.34)$$

where

$$x = \frac{h}{(1+h)^{1/4}\left(1+\sqrt{1+h}\right)^{3/2}}. \qquad (9.5.35)$$

Relationship (9.5.35) is shown in Fig. 9.18. The dotted c ve in this figure shows that the required dependence is sufficiently well reproduced by the system of automatic gain control.

In order to illustrate the magnitude of errors of measurement of speed in the case when acceleration can be considered white noise we shall give the following example. Let us assume that wavelength of the radar $\lambda = 3.4$ cm, and the bandwidth

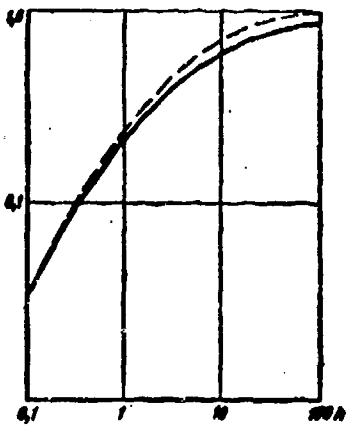

Fig. 9.18. Dependence of $x = K_p/K_0$ on the signal-to-noise ratio h when speed is a process with a stationary 1st derivative: —— optimum change of gain; --- change of gain from automatic gain control.

of fluctuations of the signal $\Delta f_c = 30$ cps. Errors of measurement of speed for a large signal-to-noise ratio ($h \to \infty$) and for different mean square values of speed developed in 1 sec are given in Table 9.2. In it there are also given corresponding errors of measurement of frequency and values of the effective bandwidth of the tracking meter, calculated from (9.5.2).

Table 9.2.

Mean square value of speed developed by the target in 1 sec, m/sec	0.5	5	50
B_1, m^2/sec^3	0.25	25	2500
Mean square error of measurement of frequency σ_f, cps	6	19	60
Mean square error of measurement of speed $\sigma_{вых}$, m/sec	0.1	0.32	1
Effective bandwidth of the system $\Delta f_{эф}$, cps	6	60	600

Acceleration of the target cannot always be assumed uncorrelated. However, if acceleration is finite, and its derivative is approximately stationary and changes rapidly, speed can be considered the double integral of white noise with a certain spectral density $B_2[m^2/sec^5]$. For a large time of observation according to (6.8.37) the optimum smoothing filter is a double integrator with correction. The gain factor of the open loop turns out to be equal to

$$K_{\text{р}}=\sqrt{B_2 K_{\text{опт}}} \tag{9.5.36}$$

and with the same the optimum frequency discriminator depends in the same way on the signal-to-noise ratio h as in the case of uncorrelated accelerations. The time constant of the correcting circuit is defined as

$$T_{\text{к}}=\sqrt[4]{\frac{4}{B_2 K_{\text{опт}}}} \tag{9.5.37}$$

and, finally, variance of error of measurement of speed according to (6.8.33) is equal to

$$\sigma^2_{\text{вых}}=\sqrt{2}\left(\frac{B_2}{K^3_{\text{опт}}}\right)^{1/4}. \tag{9.5.38}$$

The dependence of ratio $\sigma^2_{BHX}/\sigma^2_{BHX\ 0}$ on h (where $\sigma^2_{BHX\ 0}$ – variance as $h \to \infty$) obtained from (9.5.38) is shown in Fig. 9.19. It is valid for change of K_p and T_K with change of h in accordance with the above-mentioned formulas. In the case of fixed selection of K_p and T_K the system ceases to be optimum.

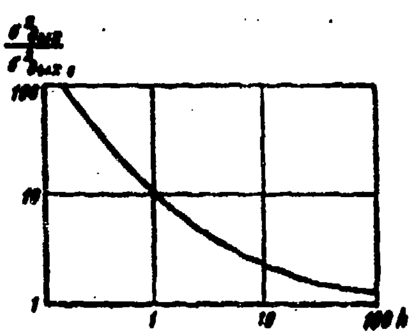

Fig. 9.19. Dependence of variance of error of measurement of speed σ^2_{BHX} on the signal-to-noise ratio h when speed is a process with a stationary 2nd derivative.

For illustration of the possibilities of an optimum system of measurement of speed when the derivative of acceleration is white noise, we turn again to the example of a radar for which $\lambda = 3.4$ cm, and the bandwidth of the received signal is $\Delta f_C = 30$ cps. Assigning the magnitudes of the mean square value of acceleration developed in 1 sec and calculating mean square error of measurement of speed as $h \to \infty$, we obtain the results presented in Table 9.3. In this table, as also in the preceding example, there are given values of error of measurement of frequency and the effective bandwidth of a tracking meter, calculated by formula (9.5.4).

Table 9.3.

Mean square value of acceleration developed by the target in 1 sec, m/sec	0.5	5	50
B_2, m^2/sec^3	0.25	25	2500
Mean square error of measurement of frequency σ_f, cps	3.3	5.8	10.5
Mean square error of measurement of speed σ_{BHX}, m/sec	0.056	0.1	0.18
Effective bandwidth of system $\Delta f_{эф}$, cps	2.6	8.2	26

From the table it is clear that the dependence of error of measurement on dynamic properties of the target, characterized by coefficient B_2, in this case is very weak.

9.5.3. Smoothing Circuits with Variable Parameters

In many cases the law of change of speed can be sufficiently approximated by a function of assigned form, but depending on a certain number of unknown parameters. Here, fairly frequently there are satisfied conditions in which approximation (6.3.42) is valid, and the measured speed is presented in the form

$$V(t)=\overline{V(t)}+\sum_{k=1}^{m}\mu_k f_k(t), \qquad (9.5.39)$$

where $\overline{V(t)}$ – mathematical expectation of the law of change of speed;

$f_k(t)$ – assigned functions;

μ_k – normally distributed random variables

$$(\overline{\mu_k}=0,\ \overline{\mu_i\mu_k}=M_{ik}).$$

As it was shown in Chapter VI, optimum smoothing circuits in such cases possess variable parameters, and their pulse responses in the [illegible] variant and error of measurement are determined by expressions [illegible] and [illegible], respectively. By virtue of the fact that these expressions include the gain factor of the discriminator $K_{д}(h)$ parameters of the smoothing circuits depend on the signal-to-noise ratio.

When $m = 1$

$$V(t)=\overline{V(t)}+\mu f(t). \qquad [illegible]$$

Considering $\mu^2 = \sigma_0^2$ we have for the pulse response of the smoothing circuit according to (6.8.49)

$$g(t, \tau) = \frac{\sigma_0^2 f(t) f(\tau)}{1 + K_{\text{опт}} \sigma_0^2 \int\limits_{t_0}^{\tau} f^2(s)\, ds} \tag{9.5.41}$$

and for error of measurement

$$\sigma^2_{\text{вых}}(t) = \frac{\sigma_0^2 f^2(t)}{1 + K_{\text{опт}} \sigma_0^2 \int\limits_{t_0}^{t} f^2(s)\, ds}. \tag{9.5.42}$$

Let us consider the following examples.

1. After lock-on the radar should pass to automatic target tracking. Let us assume that after lock-on there exists a certain initial error of tuning in frequency, and consequently also of measurement of speed. Let us assume also that speed varies according to a law, known with the error of this initial error (for instance, speed is constant during the time of measurement). Then $f(t) = 1$, and σ_0^2 – variance of the initial error of measurement of speed. Considering $t_0 = 0$, we obtain

$$g(t, \tau) = \frac{\sigma_0^2}{1 + K_{\text{опт}} \sigma_0^2 \tau}, \quad \sigma^2_{\text{вых}}(t) = \frac{\sigma_0^2}{1 + K_{\text{опт}} \sigma_0^2 t}. \tag{9.5.43}$$

The smoothing circuit consists of an amplifier with a variable gain factor $\sigma_0^2/1 + K_{\text{ОПТ}}\sigma_0^2 t$ and an integrator. Variance of error decreases in time for large times of observation as $1/K_{\text{ОПТ}}t$. In the given and subsequent examples we shall not give physical explanations of the meaning of optimality of smoothing circuits, since this question was generally considered in Chapter VI.

With application of an optimum frequency discriminator with a gain factor determined by (9.5.29) error of measurement of speed is defined as

$$\sigma_{\text{вых}}(t) = \frac{\sigma_0}{\sqrt{1 + \left(\frac{4\pi}{\lambda}\right)^2 \frac{\sigma_0^2 t}{2\Delta f_c} \varphi(h)}}, \tag{9.5.44}$$

where

$$\varphi(h) = \frac{h^2}{\sqrt{1+h}\,(1+\sqrt{1+h})^2}.$$

For the example already repeatedly considered $\lambda = 3.4$ cm, $\Delta f_c = 3$ cps, in Fig. 9.20 are curves $\sigma_{\text{вых}}(t)$ for three values of h and for values of initial error of measurement of speed corresponding to detuning of Δf by 40, 20 and 5 cps. In the given example error decreases rather rapidly, [illegible] several times [illegible] hundredths of a second.

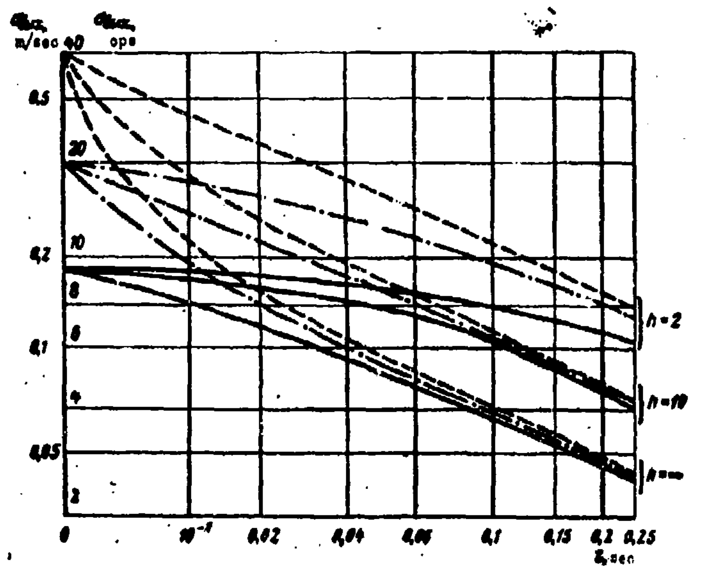

Fig. 9.20. Dependence of error of measurement of speed on time t for the example of an ptimum meter with f(t) = 1: —— Δf = 10 cps; -.- Δf = 20 cps; --- Δf = 40 cps.

.2. Let us assume that acceleration of an object whose speed is measured by a radar is constant in a certain stage of observation. Due to scattering of thrust of engines we can only know the mean value of acceleration and the magnitude of scattering of acceleration near this average. Here f(t) = t and from (9.5.41), (9.5.42) for $t_0 = 0$ we have

$$\left.\begin{aligned} g(t, \tau) &= \frac{\sigma_0^2 t\tau}{1 + K_{\text{опт}}\sigma_0^2 \frac{\tau^3}{3}}, \\ \sigma^2_{\text{вых}}(t) &= \frac{\sigma_0^2 t^2}{1 + K_{\text{опт}}\sigma_0^2 \frac{t^3}{3}}, \end{aligned}\right\} \qquad (9.5.45)$$

where σ_0^2 – variance of acceleration.

The optimum smoothing circuit here consists of a series-coupled amplifier with a gain factor varying in time according to the law $\dfrac{\sigma_0^2 t}{1 + K_{\text{опт}}\sigma_0^2 \frac{t^3}{3}}$, an integrator and an amplifier with a gain factor varying as t. Error of measurement in the beginning increases as $\sigma_0 t$, and for a large time of observation it decreases as $3/K_{\text{опт}} t$. At time

$$t^* = \sqrt[3]{\frac{6}{K_{\text{опт}}\sigma_0^2}} \qquad (9.5.46)$$

error takes its maximum value, equal to

$$\sigma_{\text{вых}\,m} = \frac{1}{\sqrt{3}}\left(\frac{6\sigma_a}{K_{\text{опт}}}\right)^{1/3}. \qquad (9.5.47)$$

and with the same assumptions about the frequency discriminator it is determined by the formula

$$\sigma_{\text{вых}\,m} = 1{,}03\left(\frac{\lambda}{4\pi}\right)^{2/3}\left[\frac{2\Delta f_c \sigma_a}{\varphi(h)}\right]^{1/3}. \qquad (9.5.48)$$

For case λ = 3.4 cm, Δf_c = 30 cps in Fig. 9.21 are curves of the dependence of $\sigma_{\text{BHX}\,m}$ on h for three mean square values of acceleration. It turns out that mean square error of measurement of speed in the given optimum system in a wide range of accelerations and signal-to-noise ratios (h = 1 to 100) is a quantity in its frequency expression corresponding to only part of the width of the signal spectrum.

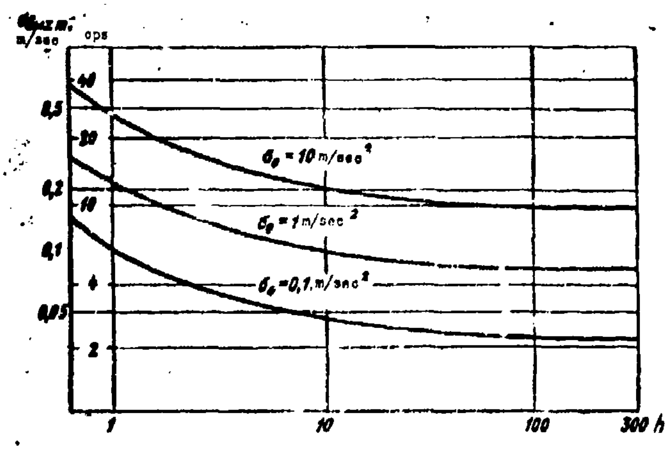

Fig. 9.21. Dependence of maximum error of measurement of speed $\sigma_{\text{BHX}\,m}$ on the signal-to-noise ratio h for the example of an optimum meter when f(t) = t.

3. If the target is a body moving rapidly and experiencing some drag in the atmosphere, then with observance of certain hardly limiting conditions one may assume that

$$V(t) = \overline{V(t)} - \mu \cdot e^{\alpha t}. \qquad (9.5.49)$$

Considering as before $\overline{\mu} = 0$ and $\overline{\mu^2} = \sigma_\mu^2$, according to (9.5.41) and (9.5.42) we obtain

and

$$\left.\begin{aligned} g(t,\ \tau) &= e^{\alpha t}\frac{\sigma_0^2 e^{\alpha\tau}}{1+\dfrac{K_{\text{опт}}\sigma_0^2}{2\alpha}(e^{2\alpha\tau}-1)} \\ \sigma^2_{\text{вых}}(t) &= \frac{\sigma_0^2 e^{2\alpha t}}{1+\dfrac{K_{\text{опт}}\sigma_0^2}{2\alpha}(e^{2\alpha t}-1)}. \end{aligned}\right\} \qquad (9.5.50)$$

The smoothing circuit consists of a series-coupled amplifier with variable gain $\dfrac{\sigma_0^2 e^{\alpha t}}{1+\dfrac{K_{\text{опт}}\sigma_0^2}{2\alpha}(e^{2\alpha t}-1)}$, an integrator and an amplifier with gain $e^{\alpha t}$.

For large times of observation,

$$g(t,\ \tau)\to\frac{2\alpha}{K_{\text{опт}}}e^{\alpha(t-\tau)}, \qquad (9.5.51)$$

and pulse response of the closed tracking meter

$$K_{\text{опт}}c(t,\ \tau)\to 2\alpha e^{-\alpha(t-\tau)}. \qquad (9.5.52)$$

Here variance of error of measurement of speed is

$$\sigma^2_{\text{вых}}(t)\to\frac{2\alpha}{K_{\text{опт}}}. \qquad (9.5.53)$$

Thus, if we are interes in error for large times of observation, the optimum tracking meter possesses cons parameters and is equivalent to an inertial link (RC-circuit) with time constant $1/\alpha$. Variance of error of the meter is equal here to $2\alpha S_{\text{опт}}(h)$, which determines the form of its dependence on the signal-to-noise ratio h. It is necessary to note that for realization of an optimum meter possessing the shown properties it is necessary in the smoothing circuit to have an amplifier whose gain factor is proportional to $1/K_{\text{опт}} = S_{\text{опт}}(h)$. If in the preceding examples the optimum gain factor of the open circuit decreased with decrease of h, and with some degree of accuracy this change was reproduced by the system of automatic gain control, in this case with a system of automatic gain control it is necessary to compensate its normalizing action so that the resultant gain factor remains constant.

The radial component of the speed of a target, measured by a Doppler speed meter, in many cases contains both components of assigned form, but depending on random parameters, and also purely random components. The form of smoothing circuits optimum for such cases already was discussed in Chapter VI. In the two-loop variant they contain two channels. One of these channels is intended basically for production of the component of the law of change of the measured variable which has the assigned form for finding true values of its parameters. The other channel

is intended basically for tracking the purely random part of the measured variable. However, other method of realizing smoothing circuits are possible. We meet one of them in the following example.

Let us find optimum smoothing circuits and corresponding potential accuracy of a speed meter when speed is

$$V(t)=\overline{V(t)}+\mu t+\xi(t), \tag{9.5.54}$$

where $\overline{V(t)}$ — known mathematical expectation of speed;

μ — random, normally distributed variable ($\overline{\mu} = 0$, $\overline{\mu^2} = \sigma_0^2$);

$\xi(t)$ — Wiener process with variance Bt.

Then, composing the correlation function of process $V(t)$, substituting it in equations (6.6.53) and (6.6.54) and solving them, for the pulse response of smoothing circuits we obtain

$$g(t, \tau)=\sqrt{\frac{B}{K_{\text{опт}}}}\times$$
$$\times\frac{\operatorname{sh}\sqrt{BK_{\text{опт}}}\tau\cdot\exp\left\{\int\limits_{\sqrt{BK_{\text{опт}}}\tau}^{\sqrt{BK_{\text{опт}}}t}\frac{\operatorname{sh}s\,ds}{\operatorname{ch}s-\operatorname{sh}s\,(s+B\sqrt{BK_{\text{опт}}}/\sigma_0^2)^{-1}}\right\}}{\operatorname{ch}\sqrt{BK_{\text{опт}}}t-\operatorname{sh}\sqrt{BK_{\text{опт}}}t\,(\sqrt{BK_{\text{опт}}}t+B\sqrt{BK_{\text{опт}}}/\sigma_0^2)^{-1}}. \tag{9.5.55}$$

In connection with the fact that $g(t, \tau)$ turns out to be equal to the product of a certain function of variable t and the function of variable τ, the smoothing circuits in the single-loop variant of tracking meter (Fig. 9.22) contain coupled in series:

an amplifier with variable gain

$$f_1(t)=\sqrt{\frac{B}{K_{\text{опт}}}}\operatorname{sh}\sqrt{BK_{\text{опт}}}t\cdot e^{-F(\sqrt{BK_{\text{опт}}}t)}, \tag{9.5.56}$$

an integrator and amplifier with variable gain

$$f_2(t)=\frac{e^{F(\sqrt{BK_{\text{опт}}}t)}}{\operatorname{ch}\sqrt{BK_{\text{опт}}}t-\operatorname{sh}\sqrt{BK_{\text{опт}}}t\,(\sqrt{BK_{\text{опт}}}t+B\sqrt{BK_{\text{опт}}}/\sigma_0^2)^{-1}}, \tag{9.5.57}$$

where

$$F(s)=\int\frac{\operatorname{sh}s\,ds}{\operatorname{ch}s-\operatorname{sh}s\,(s+B\sqrt{BK_{\text{опт}}}/\sigma_0^2)^{-1}}.$$

For small times of observation ($t \to 0$ and $\tau \to 0$) gain factors of amplifiers, depicted in Fig. 9.22, vary as

$$f_1(t)\approx\sqrt{\frac{B}{K_{\text{опт}}}}t \text{ и } f_2(t)\approx\frac{1}{t}.$$

For large times of observation ($t \to \infty$ and $\tau \to \infty$)

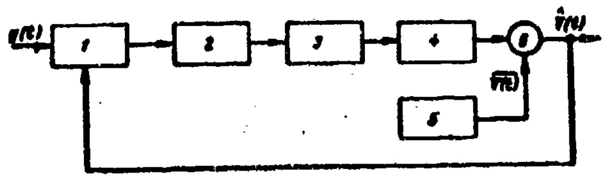

Fig. 9.22. Diagram of an optimum speed meter when

$$V(t)=\overline{V(t)}+\mu t+\xi(t):$$

1 – discriminator; 2 – amplifier with gain factor $f_1(t)$; 3 – integrator; 4 – amplifier with gain factor $f_2(t)$; 5 – generator of law of change of $\overline{V(t)}$; 6 – adder.

$$f_1(t)=\sqrt{\frac{B}{K_{\text{опт}}}} \text{ and } f_2(t)=1;$$

so that the smoothing circuit contains only an integrator with gain factor $\sqrt{B/K_{\text{ОПТ}}}$ It is easy to find that for intermediate values of t the gain factor $f_1(t)$ has a maximum at value $t = t_1$, determined by equation

$$\frac{1}{2}\operatorname{sh}2\sqrt{BK_{\text{опт}}}\,t=\sqrt{BK_{\text{опт}}}\left(t+\frac{B}{\sigma_0^2}\right),$$

and $f_2(t)$ has a minimum at $t = t_2$, determined from equation

$$\operatorname{th}\sqrt{BK_{\text{опт}}}\,t=\sqrt{BK_{\text{опт}}}\left(t+\frac{B}{\sigma_0^2}\right).$$

Laws of change of gain factors of the amplifiers are shown in Fig. 9.23.

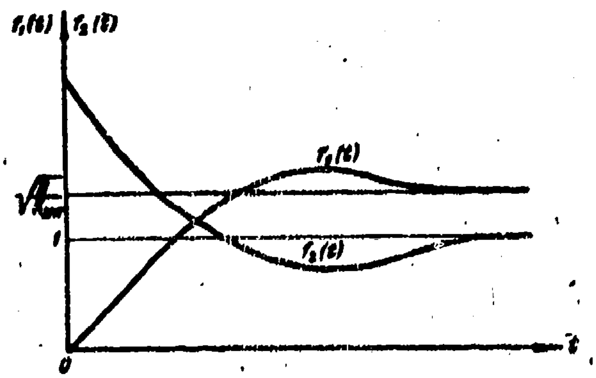

Fig. 9.23. Form of laws of change of gain factors $f_1(t)$ and $f_2(t)$.

Calculating by (6.6.48) variance of error of the considered optimum meter, we obtain

$$\sigma^2_{\text{вых}}(t)=\sqrt{\frac{B}{K_{\text{опт}}}}\times \qquad (9.5.58)$$

$$\times\frac{1}{\operatorname{cth}\sqrt{BK_{\text{опт}}}\,t-\left(\sqrt{BK_{\text{опт}}}\,t+B\sqrt{BK_{\text{опт}}}/\sigma_0^2\right)^{-1}}.$$

As $t \to \infty$ this variance takes value

$$\sigma^2_{\text{вых}}(\infty)=\sqrt{\frac{B}{K_{\text{опт}}}}. \qquad (9.5.59)$$

Quantity $\nu = \frac{B}{\sigma_0^2}\sqrt{BK_{\text{ОПТ}}}$ in the formulas is the ratio of $\sigma^2_{\text{ВЫХ}}(\infty)$ to variance of the linear component μt in (9.5.54) over interval of time (from moment of beginning of observation) $5 = (BK_{\text{ОПТ}})^{1/2}$, equal to the set-up time in the circuit, calculated for a Wiener process.

In Fig. 9.24 according to (9.5.58) there are shown curves of the dependence of $\sigma^2_{\text{ВЫХ}}(t)/\sigma^2_{\text{ВЫХ}}(\infty)$ on time for different values of parameter ν. The dotted line shows the asymptotic value and the locus of maximum values.

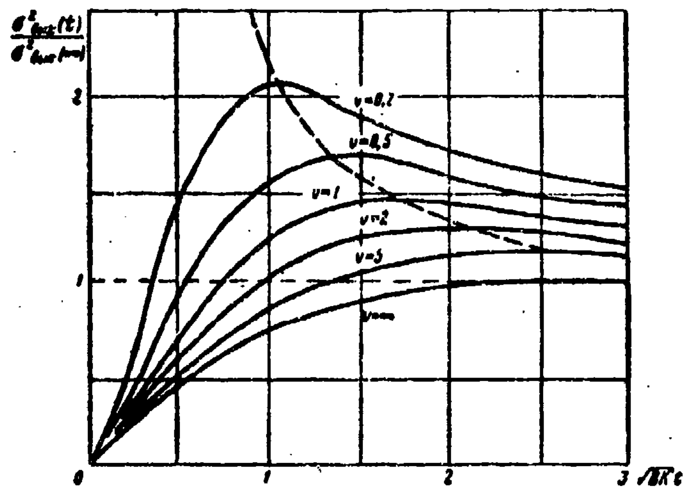

Fig. 9.24. Dependence of variance of error on time t for the example of a meter of speed, determined as $V(t) = \overline{V(t)} + \mu t + \xi(t)$.

The dependence of $\sigma_{\text{ВЫХ}}(t)$ on the signal-to-noise ratio h is easy to find, assigning $K_{\text{ОПТ}}$, for instance, in the form of (9.5.29). For $\sigma_{\text{ВЫХ}}(\infty)$ here results obtained above for an optimum meter in case of speed which is a Wiener process are valid.

9.5.4. The Influence of Errors in a Priori Data on Accuracy of Measurement

Optimum speed meters, errors of which are analyzed above, can be realized only for known a priori statistics both of signals and also of their parameters. Change of a priori statistics of the signal leads to variation of parameters of the optimum discriminator, whereas change of a priori statistics of the measured variable (speed) leads to change of smoothing circuits. Usually "a priori difficulty" with respect to statistics of the measured parameter are stronger than with respect to the received signal. Therefore, consideration of the case of a discriminator which does not differ in its properties from the optimum and of nonoptimal smoothing circuits is of interest. Here we assume that smoothing circuits are synthesized preceeding from certain a priori statistics of the law of change of speed, but in fact these statistics are different.

We shall consider this question with examples.

Let us assume that accelerations of the target are random, stationary and uncorrelated. Then speed is a Wiener process. However, variance of this process $B_{10}t$ is difficult to know exactly. Therefore, we consider that smoothing circuits

are synthesized counting on some other variance $B_1 t$. As it was shown above, as the smoothing filter in this case one should apply an integrator with gain factor $\sqrt{B_1/K_{опт}}$. Error of tracking in steady-state operating conditions will be composed of fluctuation and dynamic errors. Variance of the first of them is defined as

$$\sigma^2_{фл} = 2S_{вкв}\Delta f_{эф} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{K_{опт}}\left|\frac{K_{опт}G(i\omega)}{1+K_{опт}G(i\omega)}\right|^2 d\omega = \frac{1}{2}\sqrt{\frac{B_{10}}{K_{опт}}}\sqrt{\frac{B_1}{B_{10}}}, \qquad (9.5.60)$$

where $G(i\omega)$ – frequency response of the integrator.

Variance of dynamic error* can be found, proceeding from the following reasonings. Dynamic error is defined as

$$\varepsilon_{дин}(t) = \int_{t_0}^{t}\vartheta(t-\tau)V(\tau)\,d\tau = \int_{t_0}^{t}\vartheta(t-\tau)\,d\tau\int_{t_0}^{\tau}\xi(s)\,ds = \int_{t_0}^{t}\varkappa(t-\tau)\xi(\tau)\,d\tau, \qquad (9.5.61)$$

where $V(t)$ – law of change of the random component of speed, consisting of the integral of white noise $\xi(t)$, having spectral density B_{10}: $\vartheta(t)$ – pulse response of the closed system, if we consider mismatch the output variable

$$\varkappa(t) = \int_0^t \vartheta(\tau)\,d\tau.$$

Variance of dynamic error

$$\sigma^2_{дин} = \overline{\varepsilon^2_{дин}(t)} = B_{10}\int_{t_0}^{t}\varkappa^2(t-\tau)\,d\tau \approx \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{B_{10}}{\omega^2}\left|\frac{1}{1+K_{опт}G(i\omega)}\right|^2 d\omega = \frac{1}{2}\sqrt{\frac{B_{10}}{K_{опт}}}\sqrt{\frac{B_{10}}{B_1}}, \qquad (9.5.62)$$

where we use the circumstance that the Fourier transform of $\vartheta(t)$ is $1/1 + K_{опт}G(i\omega)$.

Total error has variance $\sigma^2_{вых} = \sigma^2_{фл} + \sigma^2_{дин}$. Its ratio to minimum possible $\sigma^2_{вых\ 0} = \sqrt{B_{10}/K_{опт}}$, obtained for $B_1 = B_{10}$, is equal to

$$\frac{\sigma^2_{вых}}{\sigma^2_{вых\ 0}} = \frac{1}{2}\sqrt{\frac{B_1}{B_{10}}} + \frac{1}{2}\sqrt{\frac{B_{10}}{B_1}}. \qquad (9.5.63)$$

This formula also determines increase of error due to a priori data incorrectly presented during synthesis of the system.

*Here, in accordance with Chapter VI, we applied the statistical approach to dynamic error.

If the measured speed is the double integral of white noise, absolutely analogously it is possible to derive formula

$$\frac{\sigma^2_{\text{вых}}}{\sigma^2_{\text{вых}\,0}}=\frac{3}{4}\left(\frac{B_2}{B_{20}}\right)^{1/4}+\frac{1}{4}\left(\frac{B_{20}}{B_2}\right)^{3/4}, \tag{9.5.64}$$

where B_{20} — true spectral density of white noise, the double integral of which is speed, and B_2 — its value assumed during synthesis of the filter.

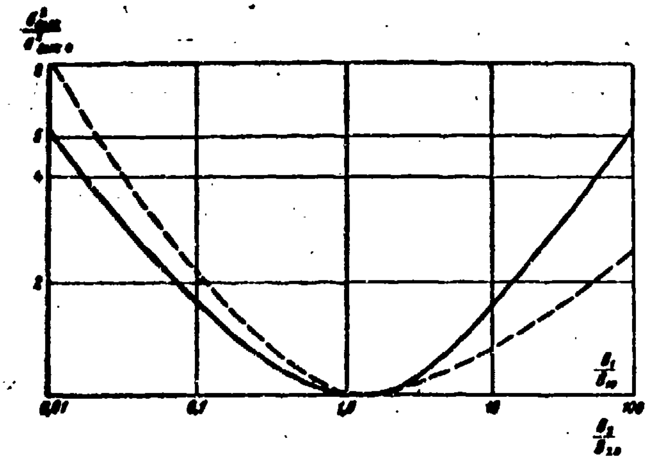

Fig. 9.25. Curves of the dependence of increase of error of measurement of speed on error in a priori data presented during synthesis: —— B_1/B_{10}; --- B_2/B_{20}.

In Fig. 9.25 formulas (9.5.63) and (9.5.64) are presented graphically. From the graphs we see the low criticality of the system of measurement of speed to selection of magnitudes of B_1 and B_2, respectively. Thus, for $B_1 = 10B_{10}$ or $B_1 = 0.1B_{10}$ (which are equivalent to expansion or narrowing of bandwidth of the system by a factor of 3.2 as compared to the optimum) error increases by 32%, and for $B_2 = 16B_{20}$ (i.e., with expansion of the band by a factor of 2) this growth comprises 24% in all. With decrease of B_2/B_{20} quantity $\sigma^2_{\text{вых}}$ grows somewhat more strongly, so that in systems of the second order certain expansion of bandwidth is preferable if dynamic properties of the target are not known exactly.

Let us consider now the case when speed varies as

$$V(t)=\overline{V(t)}+\mu t+\xi(t),$$

where $\xi(t)$ — Wiener process with variance Bt;

μ — normally distributed random variable ($\overline{\mu} = 0$, $\overline{\mu^2} = \sigma_0^2$).

Let us assume that this process does not influence the optimum system of Fig. 9.22, but a meter designed for the fact that speed is a Wiener process. Then the smoothing filter is an integrator with gain $\sqrt{B/K_{\text{опт}}}$. Calculation of variance of total error for large times of observation ($t \to \infty$) gives

$$\sigma^2_{\text{вых}}=\sigma^2_{\text{фл}}+\sigma^2_{\text{дин}}=\sqrt{\frac{B}{K_{\text{опт}}}}\left(1+\frac{1}{\nu}\right),$$

from which the ratio of variance of error to its minimum possible value (9.5.59), peculiar to an optimum system, is defined as

$$\frac{\sigma^2_{\text{вых}}}{\sigma^2_{\text{вых}\,0}} = 1 + \frac{1}{\nu}, \tag{9.5.65}$$

where

$$\nu = \frac{B}{\sigma_0^2}\sqrt{BK_{\text{опт}}}.$$

When $\nu < 1$ growth of error is substantial. If we assume that in the considered speed meter there is not introduced the mean value of $\overline{V(t)}$, for $\overline{V(t)} = at$ error will increase additionally magnitude $a/\sqrt{BK_{\text{опт}}}$.

For the above-considered example of a radar with wavelength $\lambda = 3.4$ cm with signal bandwidth $\Delta f_c = 30$ cps and a large signal-to-noise ratio $(h \to \infty)$ increase of errors due to disregarding the quasi-regular component of speed and failure to insert the mean value is illustrated by Table 9.4.

Table 9.4

B	σ_0	Minimum error		Error with application of an integrator for smoothing		Error with application of an integrator and without insertion of the mean value, $a = 1$ m/sec^2		$a = 10$ m/sec^2	
m^2/sec^3	m/sec^2	m/sec	cps	m/sec	cps	m/sec	cps	m/sec	cps
$0{,}25\cdot10^{-2}$	0,0324	0,0324	1,9	0,035	2,06	0,455	26,8	4,235	250
	0,1			0,053	3,1	0,473	27,8	4,253	250
	0,324			0,14	8,2	0,56	33	4,34	255
$0{,}25\cdot10^{-1}$	0,1	0,0576	3,4	0,059	3,5	0,192	11,3	1,39	81,8
	0,324			0,072	4,25	0,205	12,1	1,4	82,2
	1			0,014	8,55	0,278	16,3	1,47	86,5
0,25	0,324	0,105	6	0,106	6,04	0,148	8,7	0,526	31
	1			0,113	6,48	0,155	9,1	0,533	31,4
	3,24			0,175	10	0,217	12,8	0,595	35

As can be seen from the table, for comparatively low rates of change of the mean value of measured speed (acceleration of 1 and 10 m/sec^2) errors of measurement, expressed in units of Doppler frequency shift, attain substantial magnitudes, exceeding the width of the spectrum of fluctuations of the signal. Due to the narrow bands of filters of the discriminator breakoff of tracking is possible here. To avoid it, it is necessary to widen the effective band of the system, increasing fluctuation error of measurement, or to switch to smoothing circuits containing two integrators. Introduction of the mean value and use of a smoothing filter with variable parameters permits us to narrow the effective bandwidth and to obtain minimum errors given in the table.

§ 9.6. Breakoff of Tracking in Doppler Speed Meters

With intense noises and interferences in speed meters there occur nonlinear phenomena, leading ultimately to breakoff of tracking. Such phenomena were studied in Chapter VI for tracking meters of any parameters of motion of a target. There we discussed two criteria of breakoff of tracking: the criterion of sharp decrease of average time to the first breakoff and the criterion of sharp increase of fluctuation error of measurement in steady-state operating conditions. With sinusoidal approximation of the discrimination characteristic we found the critical value of the ratio of the half-width of the selected domain Δ to the mean square value of fluctuating error of a linearized system $\sigma_{Л}$ at which breakoff of tracking occurs.

For the criterion of the average time to breakoff this ratio is $(\Delta/\sigma_{Л})_{кр} \approx 5$, and for the criterion of steady-state error it is $(\Delta/\sigma_{Л})_{кр} \approx 10$.

Let us consider the phenomenon of breakoff of tracking for Doppler speed meters, considering application of the forms of frequency discriminators investigated above. Here, besides using results of Chapter VI just now mentioned we analyze nonlinear phenomena in certain speed meters without using a sinusoidal approximation of the discrimination curve, which gives us the possibility of estimating accuracy of finding critical intensities of interferences.

We turn, first of all, to a speed meter with an optimum frequency discriminator. Its discrimination characteristic is determined by expression (9.3.6).

Considering that the spectral density of fluctuations of the signal is described by formula (9.3.10), after calculations we obtain

$$a(\delta)=\frac{1}{2\Delta f_c}\,h^2\,\frac{1+\sqrt{1+h}}{\sqrt{1+h}}\,\frac{\delta}{\left[\left(\frac{\delta}{2\Delta f_c}\right)^2+\left(1+\sqrt{1+h}\right)^2\right]^2}. \qquad (9.6.1)$$

We assume that there is realized a discriminator completely optimum only for a signal-to-noise ratio $h = h_1$ and untunable with change of h.

Then from (9.6.1) and (9.3.11) the equivalent discrimination characteristic is

$$a_{экв}(\delta)=\frac{a(\delta)}{K_Д}=a_1^4\,\frac{\delta}{\left[\left(\frac{\delta}{2\Delta f_c}\right)^2+a_1^2\right]^2}, \qquad (9.6.2)$$

where

$$a_1=1+\sqrt{1+h_1}. \qquad (9.6.3)$$

In the case of application of a sinusoidal approximation the equivalent discrimination characteristic is

$$a_{\text{эквs}}(\delta) = \frac{\Delta}{\pi}\sin\frac{\pi\delta}{\Delta}. \tag{9.6.4}$$

Quantity Δ can be found by equating areas of curves (9.6.2) and (9.6.4) in the domain $\delta > 0$:

$$\int_0^\infty a_{\text{экв}}(\delta)\,d\delta = \int_0^\infty a_{\text{эквs}}(\delta)\,d\delta,$$

from which we obtain

$$\Delta = \pi\Delta f_c a_1. \tag{9.6.5}$$

There can be applied other methods of approximation of the discrimination characteristic by a sinusoid. In particular, it is possible to equate abscissas of maxima.

Then

$$\Delta = \frac{4}{\sqrt{3}}a_1\Delta f_c = 2{,}32\Delta f_c a_1. \tag{9.6.5'}$$

Applying the criterion of average time to breakoff, we have

$$\Delta^2 = 25\sigma_д^2.$$

Substituting in this formula expression (9.7.5) for variance of fluctuation error of a linearized speed meter with application of an integrator as the smoothing circuit, it is easy to find the relationship for the critical signal-to-noise ratio. For the first method of approximation it has the form

$$\varphi(h_{\text{кр}}) = \frac{a_1^2}{10}\frac{\Delta f_c}{\Delta f_{\text{эф}0}}, \tag{9.6.6}$$

for the second method of approximation it has the form

$$\varphi(h_{\text{кр}}) = \frac{a_1^2}{18{,}7}\frac{\Delta f_c}{\Delta f_{\text{эф}0}}. \tag{9.6.6'}$$

Here $\Delta f_{\text{эф}\ 0}$ — effective bandwidth of the tracking meter in the absence of noises, and dependence

$$\varphi(h) = \frac{\sqrt{1+h}(1+\sqrt{1+h})^2}{h}\cdot\frac{1}{h+y} \tag{9.6.7}$$

is shown in Fig. 9.14.

Let us assume for instance, that $\Delta f_c = 30$ cps, $\Delta f_{\text{эф}\ 0} = 1$ cps, $y = 3$ and $h_I = 1$; then from the curve for $y = 3$ in Fig. 9.14 we obtain $h_{\text{кр}\ I} = 0.17$ with the first method of approximation and $h_{\text{кр}\ II} = 0.35$ for the second method.

When using the criterion of steady-state error

$$\Delta^2 = 100\sigma_д^2$$

the equation for $h_{\text{кр}}$ takes the form

$$\varphi(h_{\text{кр}}) = \frac{a_1^2}{40} \frac{\Delta f_c}{\Delta f_{\text{эфф}}} \qquad (9.6.8)$$

for the first method of approximation, and

$$\varphi(h_{\text{кр}}) = \frac{a_1^2}{75} \frac{\Delta f_c}{\Delta f_{\text{эфф}}} \qquad (9.6.8')$$

for the second method of approximation.

For the preceding example we obtain $h_{\text{кр I}} = 1$ and $h_{\text{кр II}} = 6$. In connection with the fact that between the values of $h_{\text{кр}}$ obtained using different criteria there exists a marked difference in cases when from conditions of work of system it is impossible to give obvious preference to the criterion of average time to breakoff, one should use equations (9.6.8).

Let us find the equation for $h_{\text{кр}}$ without resorting to sinusoidal approximation of the discrimination characteristic. Using the criterion of steady-state error on the basis of the general formulas (6.3.22) assuming the absence of dynamic erro , for variance of fluctuating error we have

$$\sigma^2 = \frac{\int\limits_{-l}^{l} \delta^2 \exp\left\{-\frac{1}{\sigma_{\text{л}}^2}\int a_{\text{экв}}(\delta')\, d\delta'\right\} d\delta}{\int\limits_{-l}^{l} \exp\left\{-\frac{1}{\sigma_{\text{л}}^2}\int a_{\text{экв}}(\delta')\, d\delta'\right\} d\delta}, \qquad (9.6.9)$$

where l — level of limitation.

Substituting (9.6.2) in (9.6.9), we obtain

$$\sigma^2 = \sigma_{\text{л}}^2 \frac{\int\limits_{-l_1}^{l_1} x^2 e^{\frac{\mu}{\sqrt{1+x^2/2\mu}}}\, dx}{\int\limits_{-l_1}^{l_1} e^{\frac{\mu}{\sqrt{1+x^2/2\mu}}}\, dx}, \qquad (9.6.10)$$

where

$$\left.\begin{aligned} \mu &= \frac{a_1^2 (2\Delta f_c)^2}{2\sigma_{\text{л}}^2}, \\ l_1 &= \frac{l\sqrt{2\mu}}{a_1 2\Delta f_c}. \end{aligned}\right\} \qquad (9.6.11)$$

Calculating (9.6.10) approximately for large μ (low level of noises), we have

$$\frac{\sigma^2}{\sigma_{\text{л}}^2} \simeq \frac{1 + \frac{15}{4\mu} + \frac{525}{32\mu^2} - \frac{8505}{32\mu^3} + \frac{363825}{128\mu^4}}{1 + \frac{3}{4\mu} + \frac{45}{32\mu^2} - \frac{735}{32\mu^3} + \frac{27405}{128\mu^4}}. \qquad (9.6.12)$$

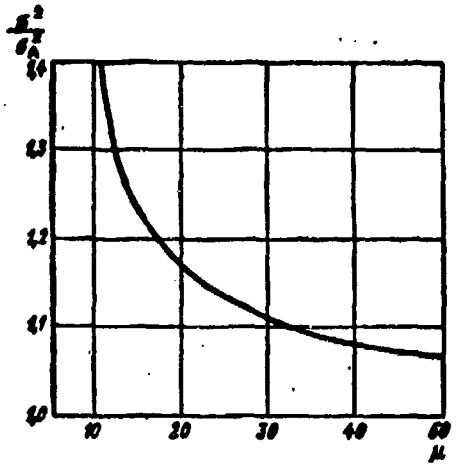

Fig. 9.26. The dependence of increase of variance of error of measurement of speed on parameter μ in the case of application of an optimum frequency discriminator.

Dependence (9.6.12) is shown in Fig. 9.26. Considering that for $\mu = 10$ this dependence already gives clearly minimized values, it is necessary to consider that $\mu_{кр} = 15$ to 20. If we consider $\mu_{кр} = 20$, for $h_{кр}$ we again obtain equation (9.6.8).

Let us analyze breakoff of tracking during application of discriminators with a tuned circuit and a phase shifter and also with mixers and differentiation. According to (9.4.13) the equivalent discrimination characteristic of these discriminators has the form

$$a_{экв}(\delta) = \frac{\delta}{1+\left[\frac{\delta}{2(\Delta f_c+\Delta f_к)}\right]^2}. \qquad (9.6.13)$$

Approximating (9.6.13) by a sinusoid, with the help of equating of abscissas of maxima* we have

$$\Delta = 4(\Delta f_c + \Delta f_к), \qquad (9.6.14)$$

from which, during application of the criterion of steady-state error ($\Delta = 10\sigma_л$) and using expression (9.4.9) for equivalent spectral density, we obtain

$$F(h_{кр}) = 0{,}04\alpha(1+\alpha)\frac{\Delta f_c}{\Delta f_{эфф}}, \qquad (9.6.15)$$

where

$$F(h) = \frac{h}{h+\gamma}\left[1+\frac{2}{h}(1+\alpha)+\frac{1}{h^2}(1+\alpha)^2\right], \qquad (9.6.16)$$

and $\alpha = \Delta f_к/\Delta f_c$.

Relationship F(h) for $\gamma = 3$ and $\alpha = 1$ is shown in Fig. 9.27. For the above-mentioned example from the curve of Fig. 9.27 we find $h_{кр} = 1.5$.

If we do not use sinusoidal approximation of the discrimination characteristic, the law of distribution of probabilities for mismatch is defined as

$$W(\delta) = Ce^{-\frac{1}{\sigma_л^2}\int a_{экв}(\delta)d\delta} = \frac{C}{\left[1+\left(\frac{\delta}{2(\Delta f_c+\Delta f_к)}\right)^2\right]^{\frac{4(\Delta f_c+\Delta f_к)^2}{\sigma_л^2}}}. \qquad (9.6.17)$$

*The method of approximation based on equating areas of curves is inapplicable in this case, since the integral of (9.6.13) goes to infinity.

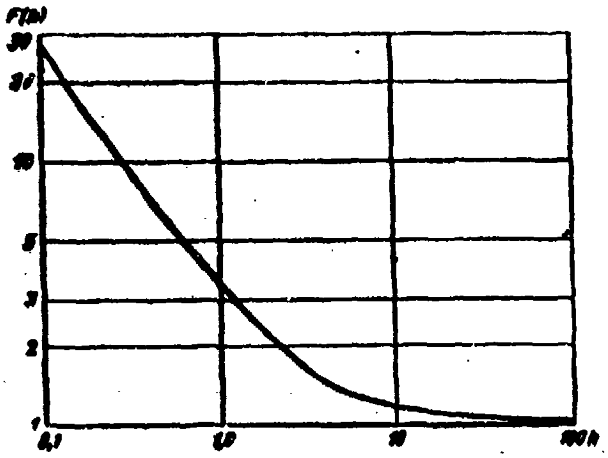

Fig. 9.27. Relationship F(h), illustrating formula (9.6.16).

Designating

$$\left.\begin{aligned} \frac{r+1}{2} &= \frac{4(\Delta f_c + \Delta f_к)^2}{2\sigma_л^2} \\ \text{and} \quad \frac{x^2}{n} &= \frac{\delta^2}{4(\Delta f_c + \Delta f_к)^2}, \end{aligned}\right\} \qquad (9.6.18)$$

we reduce (9.6.17) to a Student distribution [8].

Then, with no limitation in the smoothing circuits, we obtain

$$\sigma^2 = \overline{\delta^2} = \frac{4(\Delta f_c + \Delta f_к)^2}{n-2} = \frac{\sigma_л^2}{1 - \dfrac{3\sigma_л^2}{4(\Delta f_c + \Delta f_к)^2}}. \qquad (9.6.19)$$

Dependence

$$\frac{\sigma^2}{\sigma_л^2} = \frac{1}{1 - \frac{3}{4}\nu^2}, \qquad (9.6.20)$$

where

$$\nu = \frac{\sigma_л}{\Delta f_c + \Delta f_к},$$

is shown in Fig. 9.28.

Proceeding from this dependence, critical value of $\nu_{кр}$ = 0.8 to 0.9. Considering $\nu_{кр}$ = 0.8, we find

$$F(h_{кр}) = 0{,}16a(1+a)\frac{\Delta f_c}{\Delta f_{эфф}}. \qquad (9.6.21)$$

For the example considered in this paragraph from the curve of Fig. 9.28 we obtain $h_{кр} \approx 0.3$.

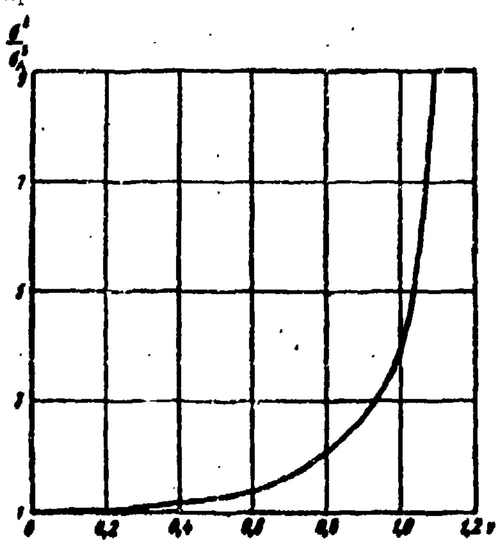

Fig. 9.28. Dependence of increase of variance of error of measurement of speed on parameter ν when using a discriminator with a tuned circuit and phase shifter or a discriminator with mixers and differentiation.

From the given examples it is clear that even in the framework of one criterion of breakoff of tracking, depending upon the form and method of approximation of the discrimination characteristic we can obtain

rather substantial scattering in determination of the critical value of the signal-to-noise ratio $h_{\text{кр}}$. Therefore, when we have formulas for $h_{\text{кр}}$ obtained without any approximations, one should use them. Results obtained by sinusoidal approximation of the discrimination characteristic can serve only for estimating the order of quantity $h_{\text{кр}}$.

§ 9.7. Nontracking Speed Meters

In multipurpose radars, to which class most detection and acquisition radars belong, they often use nontracking meters. Therefore, consideration of the question of accuracy attainable when using nontracking Doppler speed meters, and of methods of constructing such meters merits our attention.

In accordance with Chapter VI (Paragraph 6.6.5) optimum nontracking meters should be constructed by the scheme of Fig. 6.17 and should consist of an estimator unit, issuing the maximum likelihood estimate of measured speed at every given moment of time, an accuracy unit which with normally satisfactory limiting assumptions can be absent, and a certain linear filter. The estimator unit can be approximately presented in the form of a multichannel system for processing the received radio signal, channels of which are mutually detuned in frequency (theoretically there should be infinitely many of them). Every channel should be constructed in such a way that the voltage at its output, formed during time T, for which the measured speed (Doppler shift of frequency) does not change, is proportional to the value of the likelihood function for some value of $\omega_{\text{Д}}$. Selection of the maximum of the obtained voltages gives the maximum likelihood estimate of speed.

The form of each channel here is determined in Chapter IV and is illustrated by Fig. 9.29. As also in discriminators of tracking meters, in the channel there occurs shifting of phase modulation by heterodyning and multiplication by a function expressing the law of amplitude modulation (gating — for pulse modulation). Thus processed, the signal is subjected to narrow-band filtration and then square-law detection. The form of the filter is different for fast and slow fluctuations the low-frequency equivalent of its frequency response is determined by (9.2.3); for slow fluctuations this low-frequency equivalent corresponds to an integrator. In

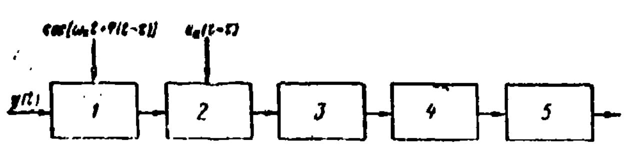

Fig. 9.29. Scheme of a channel of an optimum nontracking speed meter: 1 — mixer; 2 — amplifier; 3 — filter; 4 — square-law detector; 5 — integrator (absent with slow fluctuations).

practice, parameters of such systems of course differ from the optimum, which leads to lowering of accuracy. We, however, shall limit ourselves to analysis of the potential accuracy peculiar to the described optimum circuit of signal processing.

In accordance with the analysis in Chapter VI (Paragraph 6.6.9), with measurement by the method of maximum likelihood of speed, constant in a certain interval T, variance of error of measurement with broad assumptions is close to variance of the efficient estimate and is defined as

$$\sigma^2_{\text{эф}} = \frac{S_{\text{опт}}}{T}, \qquad (9.7.1)$$

where $S_{\text{опт}}$ – equivalent spectral density of the optimum discriminator of the tracking speed meter.

According to (9.3.1)

$$\sigma^{-2}_{\text{эф}} = -\frac{1}{2}\int_0^T\int_0^T \frac{\partial W(t_1, t_2, \omega_{\text{Д}})}{\partial \omega_{\text{Д}}} \frac{\partial R(t_1, t_2, \omega_{\text{Д}})}{\partial \omega_{\text{Д}}} dt_1 dt_2, \qquad (9.7.2)$$

where $R(t_1, t_2, \omega_{\text{Д}})$ – correlation function of the signal, determined by (9.2.1), and $W(t_1, t_2, \omega_{\text{Д}})$ in accordance with (6.7.20) is found from solution of equation

$$\int_0^T R(t_1, s) W(s, t_2)\, ds = \delta(t_1 - t_2). \qquad (9.7.3)$$

Presenting $W(t_1, t_2)$ in the form

$$W(t_1, t_2) = -w(t_1, t_2)\,\mathrm{Re}\{u(t-t_1)\,u^*(t-t_2)\,e^{i(\omega_0+\omega_{\text{Д}})(t_1-t_2)}\} + \frac{1}{N_0}\delta(t_1 - t_2), \qquad (9.7.4)$$

where $u(t) = u_a(t)e^{i\psi(t)}$, we transform (9.7.3) and (9.7.2) to

$$\frac{P_c}{2}\int_0^T \rho(t_1 - s)\, w(s, t_2)\, ds + N_0 w(t_1, t_2) = \frac{P_c}{N_0}\rho(t_1 - t_2) \qquad (9.7.5)$$

and

$$\sigma^{-2}_{\text{эф}} = \frac{P_c}{4}\int_0^T\int_0^T (t_1 - t_2)^2 \rho(t_1 - t_2)\, w(t_1, t_2)\, dt_1 dt_2, \qquad (9.7.6)$$

where P_c – mean signal power;

N_0 – spectral density of noise;

$\rho(t_1 - t_2)$ – correlation function of fluctuations.

Approximate solution of (9.7.5) and calculation of (9.7.6) can be performed immediately in asymptotic cases of fast and slow fluctuations. For fast fluctuations ($\Delta f_c T \gg 1$) this solution is given by formula (9.7.1) with substitution in it of $S_{\text{опт}}$, found in § 9.3. As a result

$$\sigma^{-2}_{\text{эф}} = \frac{T}{2\pi}\int_{-\infty}^{\infty}\left[\frac{hS'_0(\omega)}{1 + hS_0(\omega)}\right]^2 d\omega \qquad (9.7.7)$$

and with spectral density of fluctuations $S_0(\omega)$, determined by (9.3.10),

$$\sigma^2_{\text{эф}} = \frac{2\Delta f_c}{T} \frac{\sqrt{1+h}\,(1+\sqrt{1+h})^2}{h^2}, \tag{9.7.8}$$

where Δf_c – effective bandwidth of signal fluctuations.

For slow fluctuations ($\Delta f_c T \ll 1$), assuming that during time T the coefficient of correlation $\rho(t_1, t_2) = 1$, we obtain

$$\sigma^2_{\text{эф}} = \frac{6}{T^2} \frac{1+\mu}{\mu^2}, \tag{9.7.9}$$

where

$$\mu = \frac{P_c T}{2N_0} = \Delta f_c T h. \tag{9.7.10}$$

However, asymptotic results are insufficient, first, because we are interested in error of measurement in intermediate cases and, secondly, formula (9.7.9) needs more precise definition of conditions of its applicability, since as $\mu \to \infty$ according to this formula $\sigma^2_{\text{эф}} \to 0$, while for fast fluctuations $\sigma^2_{\text{эф}}$ differs from zero. To find $\sigma^2_{\text{эф}}$ for arbitrary time of correlation of the signal it is necessary to assign the concrete form of the correlation function. Producing calculations for exponential correlation

$$\rho(t) = e^{-2\Delta f_c |t|}, \tag{9.7.11}$$

we obtain

$$\begin{aligned}
\sigma^{-2}_{\text{эф}} = \frac{h^2}{4\Delta f_c^2} \Bigg\{ & \frac{1}{\lambda(\lambda+1)^2 \left[1 - \left(\frac{\lambda-1}{\lambda+1}\right)^2 e^{-2\lambda a}\right]} \times \\
& \times \left[a - \frac{5\lambda^3 - 9\lambda^2 + 3\lambda + 1}{2\lambda(\lambda-1)^2(\lambda+1)} + 2\lambda e^{-(\lambda+1)a} \frac{\lambda - 3 + (\lambda^2 - 1)a}{(\lambda-1)^2(\lambda+1)} + \right. \\
& \left. + \frac{e^{-2\lambda a}}{2\lambda} \left(\frac{\lambda+1}{\lambda-1}\right)^2\right] - \frac{1}{\lambda(\lambda-1)^2 \left[\left(\frac{\lambda+1}{\lambda-1}\right)^2 e^{2\lambda a} - 1\right]} \times \\
& \times \left[a + \frac{5\lambda^3 + 9\lambda^2 + 3\lambda - 1}{2\lambda(\lambda+1)^2(\lambda-1)} - 2\lambda e^{(\lambda-1)a} \frac{\lambda + 3 - (\lambda^2 - 1)a}{(\lambda+1)^2(\lambda-1)} - \right. \\
& \left. - \frac{e^{2\lambda a}}{2\lambda} \left(\frac{\lambda-1}{\lambda+1}\right)^2\right] \Bigg\},
\end{aligned} \tag{9.7.12}$$

where $\lambda = \sqrt{1+h}$,

$$a = 2\Delta f_c T. \tag{9.7.13}$$

As $h \to \infty$ this expression leads to $\sigma^2_{\text{эф}} = 2\Delta f_c / T$ for any Δf_c, i.e., variance of the efficient estimate of frequency (speed) in the case of absence of noises both for fast and for slow fluctuations of the signal is identical. In the case of slow fluctuations $\lambda a = 2\Delta f_c T\sqrt{1+h}$ is a small quantity. From (9.7.12) approximately for $h \gg 1$ we obtain

$$\sigma_{э\phi}^{-2} = \frac{T^2\mu^2}{6(1+\mu)}\left(1 - \mu\frac{16\Delta f_c T}{5}\right), \tag{9.7.14}$$

from which it follows that asymptotic formula (9.7.9) is valid only when $\mu \ll 5/16\Delta f_c T$, i.e., for not too large a signal-to-noise ratio h.

For large λa, occurring either in the case of fast fluctuations or during sufficiently large signal-to-noise ratios h, formula (9.7.12) is simplified, taking form

$$\sigma_{э\phi}^{-2} = \frac{h^2 T}{2\Delta f_c}\frac{1}{\lambda(\lambda+1)^2}\left[1 - \frac{2\lambda^3-3\lambda^2+1}{2\Delta f_c T\lambda(\lambda+1)(\lambda-1)^2} + \frac{2\lambda^2 e^{-(\lambda+1)2\Delta f_c T}\left[\lambda-3+2\Delta f_c T(\lambda^2-1)\right]}{2\lambda\Delta f_c T(\lambda-1)^2(\lambda+1)}\right]. \tag{9.7.15}$$

The obtained expressions determine potential accuracy of nontracking Doppler speed meters in case of absence of averaging of data from the output of the estimator unit. For illustration of errors of measurement of speed, obtained for different values of $\Delta f_c T$ and h, in Fig. 9.30 we give a graph of

$$\frac{\sigma_{э\phi}^2(\Delta f_c T,\ h)}{\frac{2\Delta f_c}{T}} = \frac{\sigma_{э\phi}^2(\Delta f_c T,\ h)}{\sigma_{э\phi}^2(\infty,\ \infty)}$$

depending upon the signal-to-noise ratio h for various $\Delta f_c T$. As already noted, with increase of h quantity $\sigma_{э\phi}^2$ seeks $2\Delta f_c/T$. However, for finite h the obtained

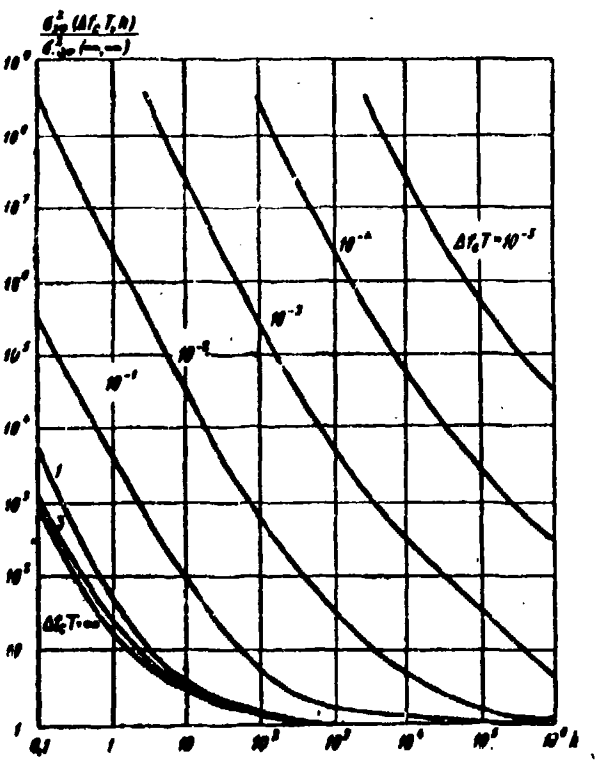

Fig. 9.30. Dependence of variance of error of an optimum nontracking speed meter on the signal-to-noise ratio for different products of bandwidth of fluctuations of signal by the time of observation $\Delta f_c T$.

variance of error of measurement strongly depends on $\Delta f_c T$ if $\Delta f_c T \ll 1$. Therefore, one should select time of estimate T in such a manner that there is observed the condition $\Delta f_c T \gg 1$. As can be seen from the graph, considerable increase of $\Delta f_c T$ as compared to unity is not required.

When for measurement of speed

there are used data from the output of the estimator unit, experiencing additional smoothing by some filter, the obtained variances of efficient estimates characterize the intensity of the random components at the input of this filter. Here if the time of formation of the estimate T and the time of correlation of signal fluctuations are small as compared to the inertia of the filter, then, as also in the case of measurement of distance, it is possible to approximately consider that at the input of the filter there acts white noise with spectral density

$$S_{\text{вх}} = \sigma^2_{\text{эф}} T. \tag{9.7.16}$$

For fast fluctuations $S_{\text{ВХ}} = S_{\text{опт}}$, and, consequently, with proper selection of smoothing circuits tracking and in nontracking optimum meters ensure identical accuracy, which was noted in Chapter VI.

The obtained results permit us to estimate the accuracy with which it is possible to measure speed on the basis of the Doppler effect with incoherent pulse radiation. If for pulse duration $\tau_{\text{И}}$, as normally occurs, fluctuations of signal are considered slow, then variance of the efficient estimate of Doppler frequency shift, obtained during the time of the pulse, is, according to (9.7.9), found as

$$\sigma^2_{\text{эф и}} = \frac{6}{\tau^2_{\text{и}}} \frac{1+q}{q^2}, \tag{9.7.17}$$

where

$$q = \frac{P_{\text{и}} \tau_{\text{и}}}{2N_0} = \frac{P_c T_r}{2N_0},$$

$P_{\text{И}}$ – power in pulse;

T_r – period of repetition of pulses.

Expression (9.7.17) determines accuracy of a unit measurement of Doppler frequency. If these measurements are cross-correlated and we average them for time T,

$$\sigma^2_{\text{эф}} = \sigma^2_{\text{эф и}} \frac{T_r}{T}. \tag{9.7.18}$$

For large signal-to-noise ratios ($q \gg 1$)

$$\sigma^2_{\text{эф}} = \frac{6}{\tau^2_{\text{и}}} \frac{1}{q} \frac{T_r}{T} = \frac{6}{\tau^2_{\text{и}}} \frac{1}{\mu}. \tag{9.7.19}$$

The obtained accuracy of measurements for the usual relationships of parameters is very low. Thus, for instance, when $\tau_{\text{И}} = 1$ μsec, $q = 10^3$ and $T/T_r = 10^3$ we obtain $\sigma_{\text{эф}} = 390$ cps, i.e., error is great even for a very large signal-to-noise ratio. Therefore, for sufficiently accurate measurement of radial speed by the Doppler effect it is necessary to have a coherent signal.

§ 9.8. Measurement of Radial and Tangential Components of Speed as Derivatives of Distance and Angular Coordinates

In the preceding paragraphs we considered in detail questions of measurement of speed on the basis of the Doppler effect. Not always, however, is this method of measurement of speed applicable. As we have seen, in the case of widely-used incoherent pulse radiation with the usual combinations of parameters there is unsatisfactory accuracy of measurement. Furthermore, we are not alway interested in the radial component of speed alone; in certain cases it is also necessary to measure its tangential component. Therefore, we shall investigate methods of measurement of components of speed of target, based on the fact that these components are derivatives of distance and angular coordinates.

We shall consider that there are applied tracking range meters and goniometers (Fig. 9.31), general analysis of which was conducted in Chapter VI. We designate $\lambda(t)$ — a parameter varying in time (distance, angle), to be measured. The component

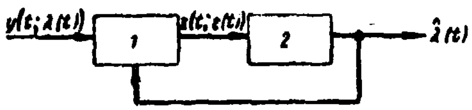

Fig. 9.31. Tracking range meter and goniometer: 1 — discriminator; 2 — smoothing circuits.

of speed interesting us is the derivative of this parameter $\lambda'(t)$. Parameter $\lambda(t)$ is coded in the received radar signal, mixed with noise $y[t, \lambda(t)]$. As a result of the influence of this signal on the discriminator there will be formed mismatch signal $z[t, \varepsilon(t)]$, where $\varepsilon(t) = \lambda(t) - \hat{\lambda}(t)$, and $\hat{\lambda}(t)$ is the output of the range meter or goniometer. The mismatch signal influences smoothing circuits of some form, forming $\hat{\lambda}(t)$.

Such a meter can be both optimum, or nonoptimal depending upon selection of the discriminator and smoothing circuits. Investigation of different range discriminators is conducted in Chapters VII and VIII, and angular discriminators are investiaged in Chapters X and XI. In all cases for low levels of noises or interferences there are small mismatches, for which the discriminator can be linearized, i.e., $z(t, \varepsilon(t))$ can be presented in the form

$$z(t, \varepsilon(t)) \approx K_{\text{Д}}[\varepsilon(t) + \xi(t)], \tag{9.8.1}$$

where $K_{\text{Д}}$ — gain factor of the discriminator;

$\xi(t)$ — equivalent noise, which, thanks to the inertia of the smoothing circuits, can be considered "white" and is characterized by its spectral density $S_{\text{экв}}$.

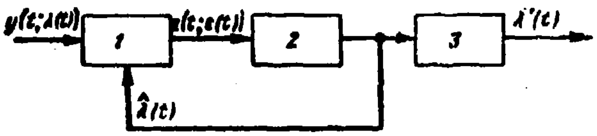

Fig. 9.32. A speed meter based on differentiation of a coordinate: 1 — discriminator; 2 — smoothing circuits; 3 — differentiator.

It is required to measure not only coordinate $\lambda(t)$, but also speed $\lambda'(t)$, this can be done by passing the output quantity of the tracking meter $\hat{\lambda}(t)$ through a filter which to some degree of accuracy carries out differentiation (Fig. 9.32). Often the voltages at certain points of filters of range finders and goniometers are proportional to the derivative of the measured coordinate; in these cases there is no need for additional differentiating filters. Of interest is investigation of accuracy of measurement of speed by the shown method for various forms of smoothing circuits; also of interest is synthesis of an optimum meter of speed as the derivative of a coordinate and finding the potential accuracy of such measurements. These problems are the content of the present paragraph.

9.8.1. Accuracy of Measurement of Speed by a System with Constant Parameters

As follows from the preceding chapters, in many cases both the ones most widespread in practice, and also optimum smoothing filters are linear and possess constant parameters. We shall investigate accuracy of measurement of speed by such systems.

If we designate by G(p) the transfer function of a smoothing filter of a range meter or goniometer, with observance of condition (9.8.1) the Laplace transform of the output variable of the meter will be recorded in the form

$$\hat{\Lambda}(p) = \frac{K_{\text{д}} G(p)}{1 + K_{\text{д}} G(p)} [\Lambda(p) + \Xi(p)], \tag{9.8.2}$$

where $\Lambda(p)$ and $\Xi(p)$ — Laplace transforms of $\lambda(t)$ and $\xi(t)$, respectively.

We consider that formation of the derivative is also carried out by a linear filter with constant parameters with transfer function H(p). The Laplace transform of the output variable of this filter takes the form

$$\hat{\dot{\Lambda}}(p) = \frac{K_{\text{д}} G(p) H(p)}{1 + K_{\text{д}} G(p)} [\Lambda(p) + \Xi(p)]. \tag{9.8.3}$$

Considering that $\dot{\Lambda}(p) = p\Lambda(p)$, for error of measurement of speed transformed according to Laplace we have

$$D(p) = \dot{\Lambda}(p) - \hat{\dot{\Lambda}}(p) = \frac{p + [p - H(p)] K_{\text{д}} G(p)}{1 + K_{\text{д}} G(p)} \Lambda(p) - \\ - \frac{K_{\text{д}} G(p) H(p)}{1 + K_{\text{д}} G(p)} \Xi(p). \tag{9.8.4}$$

The first term in the direct part of (9.8.4) depicts dynamic error, and the

second – fluctuation error. Then dynamic error of measurement of speed can be found by the formula

$$\Delta_{\text{дин}}(t)=\int_0^t v(t-\tau)\lambda(\tau)\,d\tau, \qquad (9.8.5)$$

where v(t) – reverse Laplace transform of

$$V(p)=\frac{p+[p-H(p)]K_{\text{д}}G(p)}{1+K_{\text{д}}G(p)}. \qquad (9.8.6)$$

Variance of fluctuation error of measurement of speed in steady-state operating conditions ($t\to\infty$), according to (9.8.4), is defined as

$$\overline{\Delta^2_{\phi x}}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{K_{\text{д}}G(i\omega)H(i\omega)}{1+K_{\text{д}}G(i\omega)}\right|^2 S_{\text{экв}}\,d\omega. \qquad (9.8.7)$$

As also earlier, dynamic error can be considered statistically, or for unknown statistics of $\lambda(t)$ we can determine it from model disturbances. In the statistical approach we consider assigned the mathematical expectation of coordinate $\overline{\lambda(t)}$ and its correlation function $R_\lambda(t, \tau)$. Then statistical characteristics of dynamic error of measurement of speed can be calculated by formulas known from the theory of random processes [9]. In particular, when $R_\lambda(t, \tau) = R_\lambda(t - \tau)$ (condition of stationariness of the random part of $\lambda(t)$) variance of dynamic error in steady-state operating conditions is

$$\sigma^2_{\text{дин}}=\frac{1}{2\pi}\int_{-\infty}^{\infty}|V(i\omega)|^2 S_\lambda(\omega)\,d\omega, \qquad (9.8.8)$$

where $S_\lambda(\omega)$ – Fourier transform of $R_\lambda(\tau)$.

The mathematical expectation of dynamic error is

$$\overline{\Delta_{\text{дин}}(t)}=\int_0^t v(t-\tau)\overline{\lambda(\tau)}\,d\tau. \qquad (9.8.9)$$

In the nonstatistical approach to dynamic error of measurement of speed we present coordinate $\lambda(t)$ in the form of a polynomial of a certain degree n. We shall limit our consideration to steady-state operating conditions. Then, using the formula known from theory of dynamic systems [40], we shall present dynamic error in the form

$$\Delta_{\text{дин}}(t)=\sum_{k=0}^{\infty} v_k\lambda^{(k)}(t), \qquad (9.8.10)$$

where $\lambda^{(k)}(t) = d^k\lambda(t)/dt^k$ – derivatives of some assigned model law of change of coordinate $\lambda(t)$; coefficients $v_k=\frac{1}{k!}\left.\frac{d^kV(p)}{dp^k}\right|_{p=0}$.

With presentation of $\lambda(t)$ by a polynomial of degree n series (9.8.10) includes only the first n + 1 terms, and the requirement of finiteness of $\Delta_{\text{дин}}(t)$ as $t \to \infty$ leads to condition $v_0 = v_1 = \ldots = v_{n-1} = 0$.

Usually to guarantee smallness of dynamic errors of measurement of coordinate $\lambda(t)$ there are applied smoothing circuits possessing astaticism of a certain order m, i.e.,

$$G(p)=\frac{G_1(p)}{p^m}, \tag{9.8.11}$$

where $G_1(p)$ does not contain factors $1/p$.

Substituting (9.8.11) in (9.8.6), we have

$$V(p)=\frac{p^{m+1}+[p-H(p)]\,K_\text{д}G_1(p)}{p^m+K_\text{д}G_1(p)}.$$

We require that

$$p-H(p)=p^{m+1}H_1(p), \tag{9.8.12}$$

where $H_1(0)$ — finite quantity; then

$$V(p)=p^{m+1}\frac{1+K_\text{д}H_1(p)G_1(p)}{p^m+K_\text{д}G_1(p)} \tag{9.8.13}$$

and the first m + 1 coefficients in (9.8.10) are equal to zero ($v_0 = v_1 = \ldots = v_m = 0$).

If the polynomial by which we present coordinate $\lambda(t)$ has a degree not higher than m, there is ensured zero dynamic error of measurement of speed. When n = m + 1 dynamic error is determined by coefficient

$$v_{m+1}=\frac{1+K_\text{д}H_1(0)G_1(0)}{K_\text{д}G_1(0)}. \tag{9.8.14}$$

Condition (9.8.12) is satisfied, in particular, with application of a purely differentiating filter, when $H_1(p) = 0$ and $H(p) = p$. Here, coefficient v_{m+1}, and consequently, dynamic error, too, reaches its minimum at n = m + 1. This, however, does not mean that it is necessary to apply ideal differentiation of distance and angles, since from (9.8.7) it follows that fluctuation error decreases if $|H(i\omega)/i\omega|$ has a steep drop at high frequencies. When $H(i\omega) = i\omega$ this dip, in general, is absent, and for certain forms of frequency response of the filter of the tracking meter of coordinate $G(i\omega)$ variance of fluctuating error of measurement of speed is formally obtained as infinite. Turning to filters, not carrying out ideal differentiation, one should note that for large m filters can be rather complicated. Then, proceeding to certain increase of dynamic errors, it is reasonable to impose on the filter forming the estimate of speed the condition

$$p - H(p) = p^{k+1} H_1(p), \qquad (9.8.15)$$

where $H_1(0)$ — finite quantity, and $k < m$. Here, $v_0 = v_1 = \ldots = v_k = 0$, and

$$v_{k+1} = H_1(0). \qquad (9.8.16)$$

The question of selection of the filter for formation of the speed estimate we shall consider later in the course of synthesis of an optimum meter; however, we first shall examine with examples errors of measurement of speed for certain normal forms of filters.

1. Let us assume that the meter of distance or angle as its filter contains an integrator, i.e., $K_Д G(p) = K/p$, where K — overall gain factor of the open tracking system, having dimensionality 1/sec. Let us assume that as the filter intended for measurement of speed there is used a differentiating RC-circuit, i.e., $H(p) = p/1 + pT$. Consideration of this case gives us the possibility of obtaining all the necessary results for pure differentiation, too, if we set $T = 0$.

According to (9.8.7) variance of fluctuating error

$$\overline{\Delta^2_{\phi л}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{K^2\omega^2 S_{экв}\, d\omega}{(K^2+\omega^2)(1+\omega^2T^2)} = \frac{K S_{экв}}{2}\,\frac{K}{T(1+KT)} = \sigma^2_{\phi л} K^2 \frac{1}{KT(1+KT)}, \qquad (9.8.17)$$

where $\sigma^2_{\phi л} = S_{экв}\frac{K}{2} = 2S_{экв}\Delta f_{эф}$ — variance of fluctuating error of measurement of coordinate $\lambda(t)$;

$\Delta f_{эф} = \frac{K}{4}$ — effective bandwidth of the closed tracking system.

From expression (9.8.17) it follows that fluctuating error of measurement of speed decreases with growth of the time constant T of the differentiating filter. With pure differentiation ($T = 0$) variance of fluctuating error turns into infintiy. This is obtained, of course, due to the accepted idealizations, since we considered $\xi(t)$ white noise. In reality, due to certain inertia of the discriminator this is not white noise. We can approximately assume that the spectral density of $\xi(t)$ is constant in the band of the discriminator $\Delta f_Д$, the magnitude of which is determined by the form of operations and parameters of every given discriminator, and is equal to $S_{экв}$, and beyond this band it is equal to zero. Then integration in (9.8.17) must be produced within limits $(-2\pi\Delta f_Д, +2\pi\Delta f_Д)$, and

$$\overline{\Delta^2_{\phi л}} = \sigma^2_{\phi л} K^2 \frac{1}{KT(1-K^2T^2)}\left(\frac{2}{\pi}\,\mathrm{arc\,tg}\, 2\pi\Delta f_Д T - KT\right). \qquad (9.8.18)$$

With ideal differentiation, i.e., when T = 0, we obtain

$$\overline{\Delta^2_{\phi\pi}} = \sigma^2_{\phi\pi} K(4\Delta f_{\text{Д}} - K). \tag{9.8.19}$$

Thus, variance of fluctuating error of measurement of speed in this case, too, is finite and is determined by the bandwidth of the discriminator $\Delta f_{\text{Д}}$. With a coherent signal $\Delta f_{\text{Д}}$ is a quantity of the order of the bandwidth of the narrow-band filters of the discriminator; for an incoherent signal $\Delta f_{\text{Д}} = F_{r}$, where F_{r} — frequency of repetition.

In Fig. 9.33 are curves of the dependence of the relative fluctuation error of measurement of speed $\sqrt{\overline{\Delta^2_{\text{фл}}}}/\sigma_{\text{фл}}K$ on KT for various ratios of bandwidths of the discriminator and the closed system of measurement of the coordinate $\Delta f_{\text{Д}}/\Delta f_{\text{эф}} = 4\Delta f_{\text{Д}}/K$. Curves are constructed for values $\Delta f_{\text{Д}}/\Delta f_{\text{эф}} \gg 1$. To estimate the order of magnitudes of fluctuating errors of measurement of speed we consider that very often $\Delta f_{\text{эф}}$ = (1 to 10) cps, i.e., (K = 4 to 40) 1/sec. Besides, if we express $\sigma_{\text{фл}}$ in meters, to obtain $\sqrt{\overline{\Delta^2_{\text{фл}}}}$ m/sec we must multiply $\sigma_{\text{фл}}$ by a coefficient, usually numerically equal to several units or even tens. Thus, for instance, when K = 10 1/sec ($\Delta f_{\text{эф}}$ = 2.5 cps) and with KT = 1 from the graph of Fig. 9.33 we determine that $\sqrt{\overline{\Delta^2_{\text{фл}}}}$ = 7σ [m/sec].

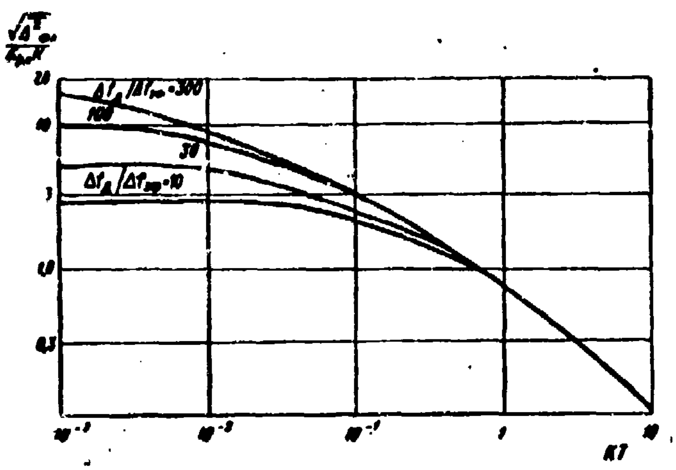

Fig. 9.33. Dependence of relative fluctuating error of measurement of speed on the product of the gain factor of an open system K for the time constant of the differentiator T with various $\Delta f_{\text{Д}}/\Delta f_{\text{эф}}$.

We now turn to consideration of dynamic errors. Substituting expressions of G(p) and H(p) in (9.8.6), we have

$$V(p) = p^2 \frac{1 + KT + pT}{(K + p)(1 + pT)}, \tag{9.8.20}$$

from which $v_0 = v_1 = 0$, and coefficient v_2 in formula (9.8.10) for dynamic error is equal to

$$v_2 = \frac{1 + KT}{K}. \tag{9.8.21}$$

Then, for instance, when $\lambda(t) = \lambda_0 + \lambda_1 t + \frac{1}{2}\lambda_2 t^2$

$$\Delta_{дин} = \frac{\lambda_2}{K}(1 + KT). \qquad (9.8.22)$$

From the obtained expressions it follows that $\Delta_{дин}$ decreases with decrease of T. However, this dependence is relatively weak, therefore one should not select magnitudes of KT which are too small, in order not to increase fluctuating error to a considerable degree. Selection of the magnitude of KT is best produced proceeding from the requirement of minimum total error of measurement of speed. For instance, if $\lambda(t)$ is quadratic, $\Delta_{дин}$ is determined by expression (9.8.22), and the optimum value of KT is found from condition $\Delta^2 = \Delta^2_{дин} + \overline{\Delta^2_{фл}} = \min$. Here we obtain the dependence of $(KT)_{опт}$ on $x = \sigma_{фл}K^2/\lambda_2$, shown in Fig. 9.34. For small x quantity $(KT)_{опт}$ is approximately proportional to x; for large x it is proportional to $\sqrt{x}$. In the same figure we show the dependence of the relative magnitude of total error $\sqrt{\Delta^2_{дин} + \overline{\Delta^2_{фл}}}/K\sigma_{фл}$ on x. If in this system gain of the open loop is selected optimally from the condition of minimum error of measurement of coordinate $\lambda(t)$, then, as it is easy to show,

$$K = \left(\frac{4\lambda_1^2}{S_{экв}}\right)^{1/3}$$

and

$$\sigma_{фл}K = \sqrt{2}\lambda_1.$$

Then from Fig. 9.34 it is clear that for small x, taking place with high accelerations of the target λ_2, error of measurement of speed can exceed λ_1, i.e., measurement of speed loses meaning. For small accelerations, i.e., large x and with optimum selection both of K, and also of KT, it is easy to find that error of measurement of speed is equal to

$$\Delta \approx 1{,}35\lambda_1^{1/6}\lambda_2^{1/2}S_{экв}^{1/6}. \qquad (9.8.23)$$

The magnitude of error has considerably stronger dependence on acceleration λ_2 than on λ_1 and $S_{экв}$. If, for instance, in a goniometer system $S_{экв} = 0.1$ deg^2/cps and $\lambda_1 = 2$ deg/sec, $\Delta = 0.3$ deg/sec with acceleration

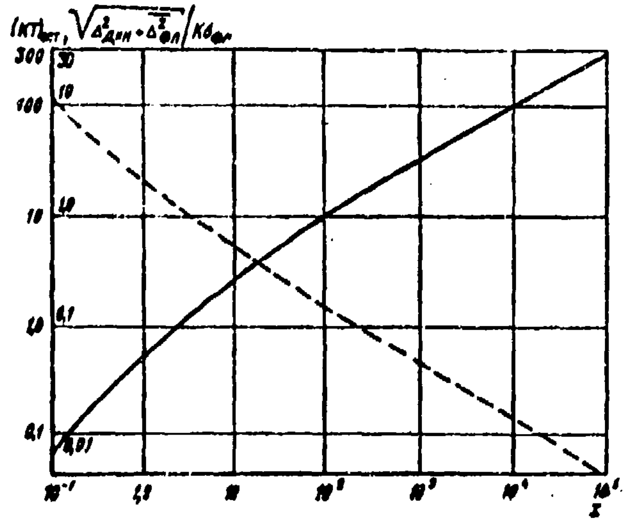

Fig. 9.34. Dependence of the optimum value of product $(KT)_{опт}$ (solid curve) and relative total error of measurement of speed (dotted curve) on parameter $x = \sigma_{фл}K^2/\lambda_2$.

$\lambda_2 = 0.1$ deg/sec^2; $\Delta \doteq 0.2$ deg/sec with $\lambda_2 = 0.04$ deg/sec^2 and $\Delta = 0.1$ deg/sec with $\lambda_2 = 10^{-2}$ deg/sec^2.

2. Let us assume that the smoothing filter of the tracking meter of distance or angle is a double integrator with correction. Then

$$K_Д G(p) = \frac{K(1 + pT_к)}{p^2}, \tag{9.8.24}$$

where K — gain factor of the open loop, having dimensionality 1/sec^2;

$T_к$ — time constant of the correcting circuit.

We shall consider that for formation of the estimate of speed, as also in the preceding example, there is used a differentiating RC-circuit, i.e., $H(p) = p/1 + pT$. Below we shall show that the greatest accuracy of measurement of speed is attained at $T = T_к$; therefore, we assume that this condition is realized. Considering equaltity $K_Д G(p)H(p) = K/p$, we note that with observance of the mentioned conditions there is no need to install a special differentiating filter. As the estimate of speed $\lambda'(t)$ it is possible to use the output of first integrator of the smoothing circuits of the tracking system (Fig. 9.35). Here, formulas errors of measurement,

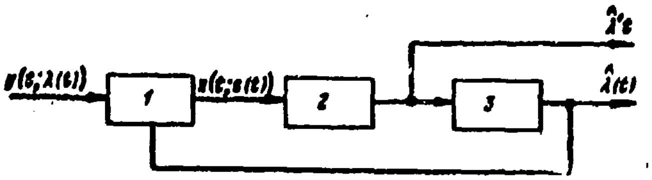

Fig. 9.35. Meter of coordinate $\lambda(t)$ and the rate of its change $\lambda'(t)$: 1 — discriminator; 2 — integrator; 3 — integrator with correction.

obtained above, remain valid. According to (9.8.7) variance of fluctuating error of measurement of speed

$$\overline{\Delta^2_{фл}} = S_{экв}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{K^2\omega^2 d\omega}{(\omega^2 - K)^2 + K^2T_к^2\omega^2} = \frac{KS_{экв}}{2T_к} =$$

$$= 16\sigma^2_{фл}\Delta f^2_{эф}\frac{KT_к^2}{(1 + KT_к^2)^2}, \tag{9.8.25}$$

where $\sigma^2_{фл} = 2S_{экв}\Delta f_{эф}$ — variance of fluctuating error of measurement of coordinate $\lambda(t)$;

$\Delta f_{эф} = (1 + KT^2_к)/4T_к$ — effective bandwidth of the closed tracking system.

The dependence of the relative magnitude of fluctuating error $\sqrt{\overline{\Delta^2_{фл}}}/\sigma_{фл}\Delta f_{эф}$ on the dimensionless parameter $KT^2_к$ is shown in Fig. 9.36. Error reaches a maximum at $KT^2_к = 0.5$. If we select $KT^2_к \approx 1$ to 6 and have an effective bandwidth of the system $\Delta f_{эф}$ = (1 to 10) cycles, fluctuating error of measurement of speed $\sqrt{\overline{\Delta^2_{фл}}}$ = = (0.5 to 15)$\sigma_{фл}$, i.e., is a smaller magnitude than in a first-order system. For

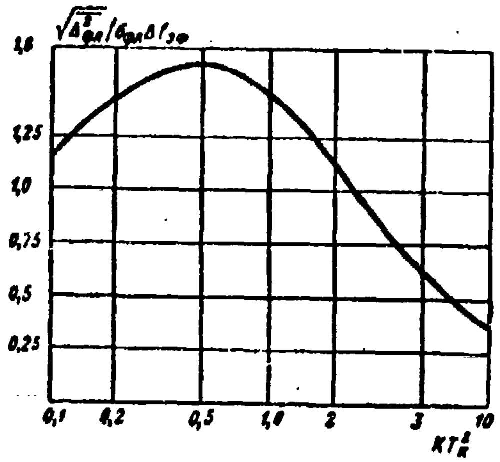

Fig. 9.36. Dependence of relative fluctuating error of measurement of the speed of a target by a system with a smoothing filter consisting of two integrators with correction on parameter $KT^2_{\text{к}}$.

instance, when $\Delta f_{\text{эф}} = 2.5$ cps and $KT^2_{\text{к}} = 4$, which corresponds to transition of the tracking system from oscillatory to aperiodic, $\sqrt{\overline{\Delta^2_{\text{фл}}}} = 1.8\sigma_{\text{фл}}$; then, as in the example given for the first-order system, this error composes $7\sigma_{\text{фл}}$.

Calculating dynamic error, we have

$$V(p) = p^2 \frac{p + KT_{\text{к}}}{p^2 + K(1 + pT_{\text{к}})}, \; v_0 = v_1 = 0, \; v_2 = T_{\text{к}}. \tag{9.8.26}$$

With a square law of change of $\lambda(t)$ steady-state error is equal to

$$\Delta_{\text{дин}} = T_{\text{к}}\lambda_2. \tag{9.8.27}$$

We will not study finding of optimum relationships between parameters of system here, leaving this question to synthesis of optimum speed meters.

Of interest is consideration of an ideal differentiating filter for the same system. In this case

$$V(p) = \frac{p^3}{p^2 + K(1 + pT_{\text{к}})}, \; v_0 = v_1 = v_2 = 0, \; v_3 = \frac{1}{K}. \tag{9.8.28}$$

Dynamic error decreases as compared to the preceding case; and for quadratic $\lambda(t)$ it is generally absent. However, fluctuating error increases and, if we do not consider finiteness of the width of spectrum $\xi(t)$, it turns into infinity. Allowing as before for the form of finiteness of the discriminator band, we find

$$\overline{\Delta^2_{\text{фл}}} = K^2T^2_{\text{к}} S_{\text{экв}}\Delta f_{\text{Д}} = \sigma^2_{\text{фл}}\Delta f^2_{\text{эф}} \frac{\Delta f_{\text{Д}}}{\Delta f_{\text{эф}}} \frac{8(KT^2_{\text{к}})^2}{(1 + KT^2_{\text{к}})^2}. \tag{9.8.29}$$

Fluctuating error is increased as compared to the case $T = T_{\text{к}}$ by a factor of $\sqrt{0.5KT^2_{\text{к}}(1 + KT^2_{\text{к}})\Delta f_{\text{Д}}/\Delta f_{\text{эф}}}$. With the above-mentioned selection of magnitudes of $KT^2_{\text{к}}$ this comprises (1 to 5) $\sqrt{\Delta f_{\text{Д}}/\Delta f_{\text{эф}}}$, which corresponds to large loss in the accuracy of measurement of speed.

3. A form of smoothing circuits very wide-spread in practice is a circuit with transfer function

$$K_{\text{к}}G(p) = \frac{K(1 + pT_{\text{к}})}{p(1 + pT_1)(1 + pT_2)}. \tag{9.8.30}$$

Time constants here usually satisfy condition $T_1 > T_{\text{к}} > T_2$ at which stability

of the system is ensured for any K. Frequently ratio $T_2/T_к$ is selected in the range 0.1 to 0.5 and $T_1/T_к$ in the range 2 to 50. The order of magnitudes of $KT_к$ and T_1/T_2 is usually identical. The effective bandwidth of the closed system turns out to be equal to

$$\Delta f_{эф} = \frac{K}{4} \frac{T_1 + T_2 + KT_к^2}{T_1 + T_2 + K(T_1 T_к + T_2 T_к - T_1 T_2)}. \tag{9.8.31}$$

Finding errors of measurement of speed for such a system on the assumption that there is executed ideal differentiation (H(p) = p), we have

$$\begin{aligned} \overline{\Delta^2_{фл}} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{K^2 \omega^2 (1 + \omega^2 T_к^2) S_{экв} d\omega}{|i\omega(1 + i\omega T_1)(1 + i\omega T_2) + K(1 + i\omega T_к)|^2} = \\ &= \frac{K^2 S_{экв} [T_1 T_2 + T_к^2 (1 + KT_к)]}{2 T_1 T_2 [(T_1 + T_2)(1 + KT_к) - KT_1 T_2]} = \\ &= \sigma^2_{фл} K^2 \frac{1 + KT_к + \frac{T_1}{T_к} \frac{T_2}{T_к}}{KT_к \left(KT_к + \frac{T_1}{T_к} + \frac{T_2}{T_к} \right)}. \end{aligned} \tag{9.8.32}$$

Calculating coefficients of dynamic error, we obtain

$$v_0 = v_1 = 0, \quad v_2 = \frac{1}{K}, \tag{9.8.33}$$

i.e., for quadratic $\lambda(t)$ dynamic error is

$$\Delta_{дин} = \frac{\lambda_2}{K}, \tag{9.8.34}$$

as also in a system with a single integrator. However, by selection of time constants T_1, T_2 and $T_к$ in the considered case with an assigned effective bandwidth we can obtain a large value of coefficient K and thereby decrease dynamic error. In practice it is possible to increase gain K by a factor of approximately $T_1/T_к$ as compared to a system with one integrator. The dependence of relative fluctuating error on parameter $KT_к$ for different $T_1/T_к$ and $T_2/T_к$ is shown in Fig. 9.37. It is easy to see that with change of $KT_к$ in a wide range $\sqrt{\overline{\Delta^2_{фл}}} =$ = (0.1 to 0.5)$\sigma_{фл}$K. The magnitude of error decreases with growth of $T_1/T_к$ and with decrease of $T_2/T_к$; however, these dependences are weak. Although the considered system with ideal differentiation for formation of the estimate of speed ensures higher accuracy than a system with a single integrator, this accuracy is nevertheless low. Thus, for instance, when K = (100 to 500)1/sec fluctuating error $\sqrt{\overline{\Delta^2_{фл}}}$ = (10 to 250)$\sigma_{фл}$, which is usually absolutely unacceptable.

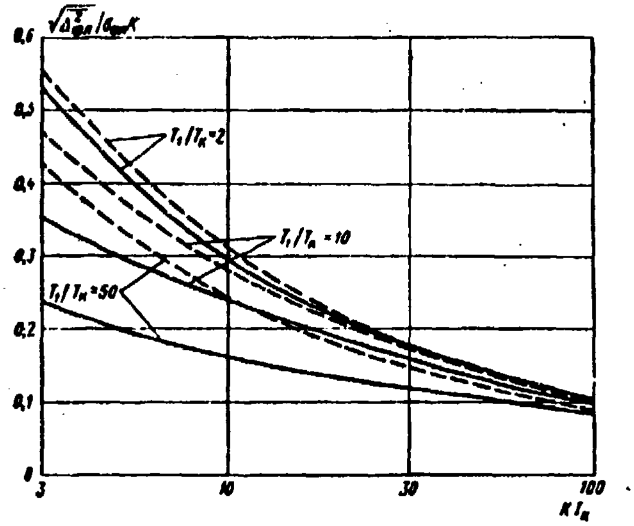

Fig. 9.37. Dependence of relative fluctuating error of measurement of speed by a system with a smoothing filter of the second order and ideal differentiation on parameter KT_K : —— $T_2/T_K = 0.1$; --- $T_2/T_K = 0.5$.

Fluctuating error can be substantially lowered by applying, e.g., a differentiating RC-circuit [$H(p) = p/1 + pT_K$].

Then

$$\overline{\Delta^2_{\text{фл}}} = \sigma^2_{\text{фл}} K^2 \frac{1}{KT_K\left(KT_K + \frac{T_1}{T_K} + \frac{T_2}{T_K}\right)}. \quad (9.8.35)$$

In Fig. 9.38 there are shown curves of the dependence of relative fluctuating error on KT_K for various T_1/T_K and for $T_2/T_K = 0.2$. With those same values of parameters as in the preceding case fluctuating error is much less $\sqrt{\overline{\Delta^2_{\text{фл}}}} = (0.01$ to $0.2)\sigma_{\text{фл}}K$; however, dynamic error increases. The first coefficient of error differing from zero $v_2 = 1 + KT_K/K$, i.e., with quadratic $\lambda(t)$, error $\Delta_{\text{дин}}$ increases by $1 + KT_K$ as compared to the preceding case.

Let us consider numerical examples:

a) Let us assume that we are measuring radial speed as the derivative of the range. Let us assume that the range finder has a smoothing filter of the last-considered type with parameters $K = 200$ 1/sec, $T_1 = 1$ sec, $T_2 = 0.01$ sec,

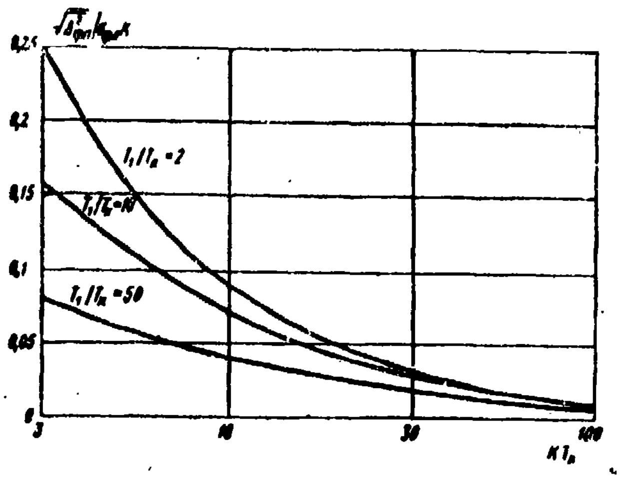

Fig. 9.38. Dependence of relative fluctuating error of measurement of speed by a system with a smoothing filter of the second order and a differentiating RC-circuit on parameter KT_K.

$T_K = 0.1$ sec, and the transfer function of the differentiating filter is $H(p) = p/1 + pT_K$. Let us assume that radial speed of the target can attain a value

$\lambda_1 = V_r$ = 1000 m/sec, and radial acceleration can reach λ_2 = 10 m/sec^2. For the selected values of the parameters $\Delta f_{\text{эф}}$ = 8 cps,

$$\sigma_{\text{фл}} = 4\sqrt{S_{\text{экв}}}\ [\text{m}],\quad \sqrt{\overline{\Delta^2_{\text{фл}}}} = 32{,}6\sqrt{S_{\text{экв}}}\left[\frac{\text{m}}{\text{sec}}\right],$$

where $S_{\text{экв}}$ is expressed in m^2/cps. Dynamic errors of measurement of distance and speeds are equal to

$$\varepsilon_{\text{дин}} = (5 + 0{,}05t)\ \text{m},\quad \Delta_{\text{дин}} = 1{,}05\,\frac{\text{m}}{\text{sec}},$$

and values of fluctuating errors are given in Table 9.5a.

Table 9.5a

$S_{\text{экв}}, \frac{\text{m}^2}{\text{cps}}$	1	3	10	30	100
$\sigma_{\text{фл}}$, m	4	6,9	12,6	21,8	40
$\sqrt{\overline{\Delta^2_{\text{фл}}}}, \frac{\text{m}}{\text{sec}}$	32, 6	56,5	103	178	326

b) Let us assume that the tangential component of speed is measured by differentiation of the angular coordinate, and filters have the same form and the same parameters as in the preceding example. Let us assume that angular velocity is characterized by quantity λ_1 = 2 deg/sec, and angular acceleration is characterized by λ_2 = 0.1 deg/sec^2. Here, dynamic errors of measurement of the angle and angular velocities are equal to

$$\varepsilon_{\text{дин}} = (1 + 0{,}05t)\,10^{-2}\ \text{deg},\quad \Delta_{\text{дин}} = 1{,}05\cdot 10^{-2}\ \frac{\text{deg}}{\text{sec}},$$

and fluctuating errors are found by the same formulas, in which $S_{\text{экв}}$ is expressed in deg^2/cps. Their values are given in Table 9.5b.

Table 9.5b

$S_{\text{экв}}, \frac{\text{deg}^2}{\text{cps}}$	10^{-3}	3.10^{-3}	10^{-2}	3.10^{-2}	10^{-1}
$\sigma_{\text{фл}}$, deg	0,126	0,218	0,4	0,69	1,26
$\sqrt{\overline{\Delta^2_{\text{фл}}}}, \frac{\text{deg}}{\text{sec}}$	1,03	1,78	3,26	5,65	10,3

From the given examples it is clear that in a number of cases accuracy of measurement of speed by the considered method is low. However, it essentially

depends on the method of construction of systems and selection of their parameters. For estimating potential possibilities of measurement of speed as the derivative of the coordinate it is necessary to turn to an optimum meter and its properties.

9.8.2. Optimum Speed Meter

Solution of problem of synthesis of an optimum system of measurement of speed as the derivative of a coordinate with Gaussian statistics of the coordinate $\lambda(t)$ ensues from results obtained in Chapter VI (Paragraphs 6.6.9 and 6.8.5) concerning optimum systems of measurement of functionals of parameters of a random process.

According to these results the optimum speed meter is the scheme of Fig. 9.39. It consists of an optimum meter of the actual coordinate $\lambda(t)$, the principle of construction and action of which already has been repeatedly discussed, and an additional filter, serving to form the estimate of speed $\hat{\lambda}'(t)$, to whose input there is fed voltage from the output of the discriminator $z(t)$. Generally speaking, there should also take place formation of the mathematical expectation of speed to which the output quantity of the filter is added.

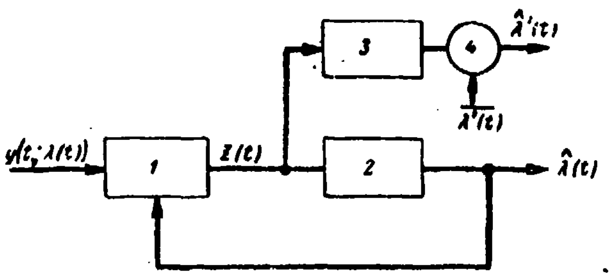

Fig. 9.39. Optimum meter for measuring speed as the derivative of the coordinate: 1 — discriminator; 2 — smoothing circuits; 3 — filter estimating speed; 4 — adder.

Of greatest interest is synthesis of the optimum filter for estimating speed, since remaining operations of the meter either are already known from synthesis of the meter of coordinate $\lambda(t)$, or are trivial. Pulse response $h(t, \tau)$ of the filter estimating speed can be found simultaneously with the pulse response of the filter of the tracking meter of the coordinate $g(t, \tau)$ from solution of equations (6.6.106), (6.6.108), (6.6.31), and (6.6.32), if we consider that function $F(t, s)$ in this case is defined as

$$F(t, s)=\delta'(t-s), \tag{9.8.36}$$

where the stroke is the sign of differentiation.

Then these equations have the form

$$h(t, \tau)=b(t, \tau)+K_{\text{опт}}\int_{\tau}^{t} b(t, s)\, g(s, \tau)\, ds, \tag{9.8.37}$$

$$b(t, \tau)+K_{\text{опт}}\int_{t_0}^{t} b(t, s)\, R(s, \tau)\, ds=\frac{dR(t, \tau)}{dt}$$
$$(b(t, \tau)=0 \text{ when } t<\tau), \tag{9.8.38}$$

$$g(t, \tau)=c(t, \tau)+K_{\text{опт}}\int_{\tau}^{t} c(t, s)\, g(s, \tau)\, ds, \tag{9.8.39}$$

$$c(t, \tau)+K_{\text{опт}}\int_{t_0}^{t} c(t, s)\, R(s, \tau)\, ds=R(t, \tau)$$
$$(c(t, \tau)=0 \text{ when } t<\tau), \tag{9.8.40}$$

where $K_{\text{опт}}$ – gain factor of the optimum discriminator for measurement of coordinate $\lambda(t)$;

$R(t, \tau)$ – correlation function of coordinate $\lambda(t)$.

Equations (9.8.39) and (9.8.40) determine the optimum structure of smoothing circuits of the meter of $\lambda(t)$, and equations (9.8.37) and (9.8.38) determine the structure of additional circuits, needed for estimating the derivative $\lambda'(t)$.

Actually filters with pulse responses $g(t, \tau)$ and $h(t, \tau)$ usually have many common elements, which simplifies the circuit of the optimum meter, making, at the same time, somewhat conditional the diagram of Fig. 9.39. Furthermore, it is possible for production of the estimate of speed to carry out filtration, not of voltage from the discriminator output, but the output quantity $\hat{\lambda}(t)$ of the coordinate-tracking meter, but here there should be applied a filter with pulse response $\vartheta(t, \tau)$ determined from equation

$$h(t, \tau)=\int_{\tau}^{t} \vartheta(t, s)\, g(s, \tau)\, ds. \tag{9.8.41}$$

In accordance with (6.6.109) variance of error of measurement of speed by the optimum system is defined as

$$\sigma_{\lambda'}^{2}=\left.\frac{db(t, \tau)}{d\tau}\right|_{\tau=t}. \tag{9.8.42}$$

After giving these general results we consider certain examples of optimum speed meters. We assume that the form and basic characteristic $K_{\text{опт}} = 1/S_{\text{опт}}$ of the optimum discriminator are known. Let us calculate here the pulse responses of smoothing filters and error of measurement of speed.

1. Let us assume that $\lambda(t)$ be a stationary random process or a process with stationary increments and that we are interested in moments of time very far removed from the beginning of observation t_0. Then in accordance with results of Paragraph 6.8.5 solutions of equations (9.8.38) and (9.8.40) can be presented by their Fourier transforms

$$C(i\omega)=\frac{1}{\Psi(i\omega)}\left[\frac{S(\omega)}{\Psi(-i\omega)}\right]_{+},$$
$$c(t-\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} C(i\omega)e^{i\omega(t-\tau)}d\omega, \tag{9.8.43}$$

$$B(i\omega)=\frac{1}{\Psi(i\omega)}\left[\frac{i\omega S(\omega)}{\Psi(-i\omega)}\right]_{+};$$

$$b(t-\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}B(i\omega)e^{i\omega(t-\tau)}d\omega, \tag{9.8.44}$$

where $S(\omega)$ – spectral density of process $\lambda(t)$;

$$\Psi(i\omega)\Psi(-i\omega)=1+K_{\text{опт}}S(\omega),$$

where $\Psi(i\omega)$ has zeroes and poles in the upper half-plane of complex variable ω, and $\Psi(-i\omega)$ has them in the lower. Operation $[\ ]_+$ signifies singling out of that part of the expression in parentheses which has poles in the upper half-plane of ω. A solution of the problem of measurement of the derivative exists if $S(\omega)$ at infinity seeks zero not slower than $1/\omega^4$. Functions $g(t-\tau)$ and $h(t-\tau)$ are determined according to (9.8.39) and (9.8.37) by their Fourier transforms with the help of expressions

$$G(i\omega)=\frac{C(i\omega)}{1-K_{\text{опт}}C(i\omega)},\quad H(i\omega)=B(i\omega)\,[1+K_{\text{опт}}G(i\omega)]. \tag{9.8.45}$$

If, for instance, $\lambda(t)$ corresponds to motion of the target with uncorrelated random accelerations, caused by various random disturbances, where the mean square speed of the target developed in 1 sec due to the random disturbances is equal to B_0, then $\lambda(t)$ – double integral of white noise with spectral density B_0, and spectral density $S(\omega)=B_0/\omega^4$. Direct application of the written equalities here gives

$$\left.\begin{aligned}G(i\omega=p)=G(p)&=\frac{K(1+pT_K)}{p^2},\\ H(p)&=\frac{K}{p},\end{aligned}\right\} \tag{9.8.46}$$

where

$$K=\sqrt{\frac{B_0}{K_{\text{опт}}}},\quad T_K=\left(\frac{4}{B_0K_{\text{опт}}}\right)^{1/4}.$$

Thus, the optimum smoothing filter of the meter tracking coordinate $\lambda(t)$ is a double integrator with correction, which was noted for the case of the $\lambda(t)$- double integral of white noise and earlier, and the optimum filter for estimating speed is an integrator with gain K. As a result the signal which is the estimate of speed $\lambda'(t)$ can be taken from the output of the first integrator of the filter of the tracking system, or be formed by means of transmission of the output of the coordinate-tracking meter $\hat{\lambda}(t)$ through a differentiating RC-circuit with transfer function $\Theta(p)=p/1+pT_K$. Consequently, the system analyzed in the preceding

paragraph, with $T = T_K$ is optimum for the case when the target moves with uncorrelated accelerations. Parameters of the optimum system K and T_K are not arbitrary, but according to (9.8.46) are functions of the intensity of accelerations B_0, characterizing maneuverability of target, and of the intensity of equivalent noises at the input of the linearized discriminator $S_{\text{опт}} = 1/K_{\text{опт}}$.

Using the fact that variance of measurement of coordinate $\sigma_\lambda^2 = c(t, t)$, while variance of measurement of speed is determined by expression (9.8.42), we obtain

$$\sigma_\lambda^2 = \left(\frac{4B_0}{K_{\text{опт}}^3}\right)^{1/4}, \quad \sigma_{\lambda'}^2 = \left(\frac{4B_0^3}{K_{\text{опт}}}\right)^{1/4}. \tag{9.8.47}$$

From formulas (9.8.47) it is clear that error of measurement of speed depends more on maneuverability of the target (B_0) and less on the level of noises $1/K_{\text{опт}}$ than error of measurement of the coordinate. For illustration we calculated error of measurement of distance (σ_d) and of radial speed ($\sigma_{d'}$) of an optimum system for two values of equivalent spectral density $S_{\text{опт}}$ and several values of B_0.

Results of the calculation are given in Table 9.6. In this table there are given values of the gain factor of the open loop, the time constant of the correcting circuit and the effective bandwidth of the system.

Table 9.6

$S_{\text{опт}} = \frac{1}{K_{\text{опт}}} \left[\frac{м^2}{\text{cps}}\right]$	10^2					10^4				
$B_0 \left[\frac{м^2}{\sec^4}\right]$	1	10	10^2	10^3	10^4	1	10	10^2	10^3	10^4
$K_{\text{опт}}K = \sqrt{B_0 K_{\text{опт}}} \left[\frac{1}{\sec^2}\right]$	0,1	0,316	1	3,16	10	0,01	0,0316	0,1	0,316	1
T_K [sec]	4,5	2,5	1,41	0,79	0,45	14,1	7,9	4,5	2,5	1,41
$\Delta f_{эф}$ [cps]	0,17	0,3	0,53	0,95	1,7	0,053	0,095	0,17	0,3	0,53
σ_d [м]	6,7	9	12	16	21	37,5	50	67	90	118
$\sigma_{d'} \left[\frac{м}{\sec}\right]$	1,6	3,8	9	21,4	50	2,9	6,8	16,2	38,5	90

Errors given in the table characterize both fluctuation and dynamic components; moreover the latter is understood in the statistical sense. From the table it follows that change of the magnitude of $S_{опт}$ two orders leads to change of error of measurement of distance by a factor of 5.6 and of error of measurement of speed by a factor of 1.8. Even with high maneuverability of the target accuracy of measurement of speed is relatively high.

The obtained results give the possibility of comparing potential accuracy of measurement of radial speed by Doppler frequency and by the derivative of distance in the case of coherent radiation. In the considered example speed is the integral of white noise, and according to (9.5.34) variance of error of its measurement by the Doppler frequency shift is equal to

$$\sigma_v^2 = \sqrt{\frac{B_v}{K_v}}, \tag{9.8.48}$$

where $1/K_v = S_{v\,опт}$ – equivalent spectral density for an optimum frequency discriminator. For large signal-to-noise ratios, according to (9.5.29)

$$\frac{1}{K_v} = S_{v\,опт} = \frac{\lambda^2 \Delta f_c}{8\pi^2}. \tag{9.8.49}$$

Here λ – wavelength; Δf_c – effective width of the spectrum of fluctuations of the signal.

Variance of error of measurement of speed as the derivative of distance is determined by (9.8.47) and depends on $K_{опт}$. For large signal-to-noise ratios h according to (7.2.15) and (7.2.16)

$$K_{опт} = \frac{1}{S_{опт}} = \frac{8\beta^2 h \Delta f_c}{c^2}, \tag{9.8.50}$$

where c – speed of light;

β – mean square width of the spectrum of modulation of the signal utilized for measurement of distance.

Thus, the ratio of errors is

$$\frac{\sigma_{d'}}{\sigma_v} = \left(\frac{4B_v K_v^2}{K_{опт}}\right)^{1/8} = \left(\frac{\pi^2 B_v}{\Delta f_c^3 \lambda^2} \frac{8\omega_0^2}{\beta^2 h}\right)^{1/8}, \tag{9.8.51}$$

where ω_0 – carrier frequency of the emitted signal.

In order to determine the usual order of magnitudes of ratios of errors, we consider that product $\Delta f_c\,\lambda$ weakly depends on the wavelength and frequently is a magnitude of the order of 1 m/sec. The ratio of the carrier frequency to the width of the spectrum of modulation is usually very great, so that even for large

signal-to-noise ratios $\omega_0^2/h\beta^2 > 10^5$ to 10^6. Then, considering that usually the width of the spectrum of the reflected signal has an order of units or tens of cycles per second, we find that even with small maneuverability of targets ($B_0 \sim$ ~ 1 m^2/sec^3) ratio $\sigma_{d'}/\sigma_v > 3$ to 5. Thus, measurement of radial speed by Doppler frequency shift of the signal ensures great accuracy where, with increase of maneuverability of the target, the gain increases. It is necessary to note that ratio $\sigma_{d'}/\sigma_v$ weakly depends on its determining parameters; therefore even with rather strong change of them the relationships of accuracies of measurement of speed by the two discussed methods changes little.

2. Let us assume that the law of change of coordinate $\lambda(t)$ is known with an accuracy of certain random coefficients $\mu_k (k = 0, 1, 2, \ldots, n)$. Thus are matters, for instance, when a certain body moves in a ballistic curve. Coefficients μ_k here are random due to the randomness of the initial conditions. As we already have repeatedly indicated, frequently there are observed conditions where presentation of $\lambda(t)$ in this form is valid:

$$\lambda(t)=\sum_{k=0}^{n}\mu_k f_k(t), \tag{9.8.52}$$

where $f_k(t)$ — assigned functions.

If, for instance, $f_k(t) = (t - t_0)^k$, coefficients μ_k constitute values of $\lambda(t)$ and n its first derivative at the initial moment $t = t_0$. We assume that $\overline{\mu}_k = 0$. This does not disrupt generality, since the known mathematical expectation can always be accounted for during measurement. Then the correlation function of process $\lambda(t)$ is

$$R(t,\tau)=\sum_{i,k=0}^{n}M_{ik}f_i(t)f_k(\tau)=\mathbf{f}^+(t)\mathbf{M}\mathbf{f}(\tau), \tag{9.8.53}$$

where $M_{ik} = \overline{\mu_i\mu_k}$;

$\mathbf{M}$ — symmetric matrix with elements M_{ik};

$\mathbf{f}(t) = [f_0(t), f_1(t), \ldots, f_n(t)]$ — column vectors, and $^+$ signifies transposition ($\mathbf{f}^+(t)$ — line vector).

Equations (9.8.37)-(9.8.40) here are equations with a degenerate nucleus and their solutions are easy. They have form:

$$c(t,\tau)=\mathbf{f}^+(t)[\mathbf{M}^{-1}+\mathbf{U}(t)]^{-1}\mathbf{f}(\tau)=\sum_{i,k=0}^{n}A_{ik}(t)f_i(t)f_k(\tau), \tag{9.8.54}$$

$$g(t,\tau)=\mathbf{f}^+(t)[\mathbf{M}^{-1}+\mathbf{U}(\tau)]^{-1}\mathbf{f}(\tau)=\sum_{i,k=0}^{n}A_{ik}(\tau)f_i(t)f_k(\tau), \tag{9.8.55}$$

$$b(t,\tau)=\left[\frac{d}{dt}\mathbf{f}(t)\right]^{+}[\mathbf{M}^{-1}+\mathbf{U}(t)]^{-1}\mathbf{f}(\tau)=$$

$$=\sum_{i,k=0}^{n} A_{ik}(t)\frac{df_i(t)}{dt}f_k(\tau), \qquad (9.8.56)$$

$$h(t,\tau)=\left[\frac{d}{dt}\mathbf{f}(t)\right]^{+}[\mathbf{M}^{-1}+\mathbf{U}(\tau)]^{-1}\mathbf{f}(\tau)=$$

$$=\sum_{i,k=0}^{n} A_{ik}(\tau)\frac{df_i(t)}{dt}f_k(\tau), \qquad (9.8.57)$$

where matrix $A(t) = [M^{-1} + U(t)]^{-1}$, and matrix $U(t) = \|U_{ik}(t)\|$ is determined by expression

$$U_{ik}(t)=K_{\text{опт}}\int_{t_0}^{t} f_i(s)f_k(s)\,ds. \qquad (9.8.58)$$

The block diagram of an optimum meter of coordinate $\lambda(t)$ and the rate of its change $\lambda'(t)$, corresponding to the obtained expressions, is presented in Fig. 9.40.

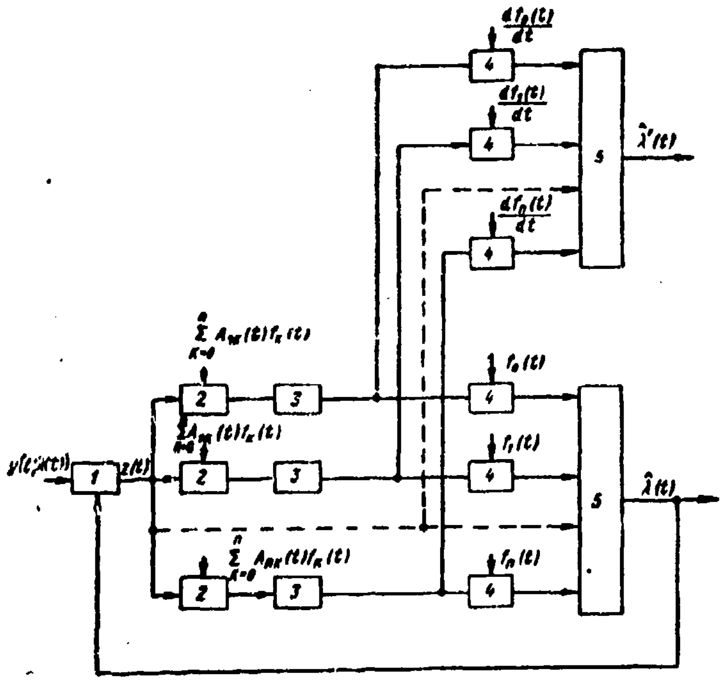

Fig. 9.40. Optimum speed meter with quasi-regular change of coordinate $\lambda(t)$: 1 – discriminator; 2 – variable-gain amplifier; 3 – integrator; 4 – variable-gain amplifier; 5 – adder.

To the input of the discriminator of the meter there is fed a mixture of the signal with noise $y(t, \lambda(t))$. From the output of the optimum discriminator voltage $z(t)$ is fed to $n + 1$ amplifiers with variable gain factors.

The law of change of the gain factor of the i-th amplifier is determined by expression $\sum_{k=0}^{n} A_{ik}(t) f_k(t)$. Output voltages of amplifiers are integrated, as a result of which there are formed estimates of coefficients μ_i. Then the voltages representing the estimates are passed through two groups of amplifiers with gain factors $f_i(t)$ and $df_i(t)/dt$ respectively. The formed voltages are added by groups. As a result of summation of voltages from outputs of the amplifiers with gain factors $f_i(t)$ there will be formed the estimate of coordinate $\hat{\lambda}(t)$, which controls tuning of the discriminator, and from those with gain factors $df_i(t)/dt$ we get the estimate of speed $\hat{\lambda}'(t)$.

Let us consider with what accuracy speed is measured by such a system. Variance of error of measurement of speed according to (9.8.42) has the form

$$\sigma^2_{\lambda'} = \left[\frac{d}{dt}\,\mathbf{f}(t)\right]^+ [\mathbf{M}^{-1} + \mathbf{U}(t)]^{-1} \frac{d\mathbf{f}(t)}{dt} = \\ = \sum_{i,k=0}^{n} A_{ik}(t) \frac{df_i(t)}{dt} \frac{df_k(t)}{dt}. \tag{9.8.59}$$

Let us turn to the particular case $n = 0$, $f_0(t) = f(t)$. Then, designating $M_{00} = \overline{\mu_0^2} = \sigma_0^2$, we obtain

$$\left.\begin{aligned} A(t) &= \frac{1}{\frac{1}{\sigma_0^2} + K_{\text{опт}} \int_{t_0}^{t} f^2(s)\, ds}, \\ \sigma^2_{\lambda'} &= \frac{\left[\frac{df(t)}{dt}\right]^2}{\frac{1}{\sigma_0^2} + K_{\text{опт}} \int_{t_0}^{t} f^2(s)\, ds}. \end{aligned}\right\} \tag{9.8.60}$$

For large t

$$\sigma^2_{\lambda'} \approx \frac{\left[\frac{df(t)}{dt}\right]^2}{K_{\text{опт}} \int_{t_0}^{t} f^2(s)\, ds}. \tag{9.8.61}$$

Usually the asymptotic value of (9.8.61) is reached rather fast. From the obtained expressions it is clear that for functions $f(t)$ growing with unlimited increase of t no faster than any finite power of t error of measurement, as $t \to \infty$, approaches zero. If, for instance, $f(t) = t^n$ and $t_0 = 0$, then $\sigma^2_{\lambda'} = n^2(2n + 1)/K_{\text{опт}} t^3$. In general, when variance $\sigma^2_{\lambda'}$ is determined by formula (9.8.59), for large times of observation of matrix

$$\mathbf{A}=[\mathbf{M}^{-1}+\mathbf{U}(t)]^{-1}\cong\mathbf{U}^{-1}(t)$$

and the expression for variance of error of measurement is simplified.

Thus, for instance, if $\lambda(t)$ is a polynomial of the n-th degree and $t_0 = 0$, i.e., $f_i(t) = t^i$, calculating by the given formulas, we have

$$\sigma^2_{\lambda'}=\frac{a_n}{K_{\text{опт}}t^3}=\frac{a_n}{t^3}S_{\text{опт}}, \tag{9.8.62}$$

where values of a_n for various n are given in Table 9.7.

Table 9.7

n	1	2	3	4	5	6	7
a_n	12	192	1 200	4 800	14 700	37 600	83 800

As $t \to \infty$ error in the considered cases approaches zero. Therefore, with a quasi-regular law of change of $\lambda(t)$ by the optimum system of measurement of speed there is attained substantially higher accuracy of measurement than with a purely random law of change of $\lambda(t)$, if only the time of measurement can be sufficiently large. Here, however, more strongly than for purely random change of $\lambda(t)$, error is influenced by the equivalent intensity of noise $S_{\text{опт}}$. Error turns out to be proportional to $\sqrt{S_{\text{опт}}}$. In spite of this, for large times of measurement error is small and the given method of measurement can exceed all others in accuracy, including measurement of Doppler frequency shift. Thus, for instance, in the case of change of distance according to the law $d(t) = \mu t^n$ (μ — random coefficient) for large times of observation variance of measurement of speed, as the derivative of distance, was just now found, and variance of measurement of speed by Doppler frequency shift is determined by formula (9.5.42), and, if we consider that $V(t) = n\mu t^{n-1}$, it is equal to

$$\sigma^2_v=\frac{2n-1}{K_{v\,\text{опт}}t}. \tag{9.8.63}$$

The ratio of variances is

$$\frac{\sigma^2_{d'}}{\sigma^2_v}=\frac{n^2(2n+1)K_{v\,\text{опт}}}{(2n-1)K_{d\,\text{опт}}}\frac{1}{t^2} \tag{9.8.64}$$

and as $t \to \infty$ it approaches zero. In the last expressions $K_{V\,\text{опт}}$ — the gain factor of a optimum discriminator for measurement of speed by Doppler frequency shift, $K_{d\,\text{опт}}$ — the gain factor of the optimum range discriminator. It should be noted that such an effect occurs only with quasi-regular change of range.

§ 9.9. Speed Meters Under the Action of Interferences

Speed meters, just as other devices, in a radar, can experience not only natural noises of the system, but also all the possible interferences which were mentioned in Chapter I. In the presence of interferences accuracy of measurement of speed decreases or even the process of measurement may be disturbed [70]. This is connected with different nonlinear phenomena in elements of the meter, in particular with the phenomenon of breakoff of tracking in tracking speed meters discussed above. Disruption of the process of measurement of speed may also take place in meters using differentiation of distance or angles. Only in these forms of meters it occurs due to breakoff of tracking of range finders or goniometers. Let us consider briefly questions of the influence of interferences on speed meters.

In most cases speed meters are coherent and use during processing of signals narrow-band filters up to detection. Thus stands the matter with respect to Doppler speed meters, since, as was shown above, for an incoherent pulse signal these meters do not ensure the usually acceptable accuracy. Meters of speed as the derivative of a coordinate also in many cases are based on coherent processing of the signal. In connection with this circumstance many forms of stochastic interferences, having a spectrum width much larger than the passband of filters in the meter, are normalized during passage through filters and their action turns out to be equivalent to the action of Gaussian white noise. Here, the action of interference reduces to equivalent increase of the level of noises at the input of the radar receiver or to decrease of the signal-to-noise ratio. Therefore, in the case of action on speed meters by such interferences we can use all the preceding results for accuracy of measurement and for estimating failures of tracking speed meters. The only difference as compared to the preceding formulas is the fact that the signal-to-noise ratio h should be replaced by quantity

$$h_{\pi ш} = \frac{P_c}{2\Delta f_0 (N_0 + N_\pi)} = \frac{h}{1 + \frac{N_\pi}{N_0}}, \qquad (9.9.1)$$

where $h_{\pi ш}$ — signal-to-noise ratio in the presence of noise and interference;

N_π — spectral density of white noise, equivalent in its action to interference.

Further calculation of noise immunity of speed meters thus reduces to finding quantity N_π, which was already done in Chapter VII during the analysis of the influence of interferences on coherent range finders. We shall give the basic formula obtained there. For noise interference, according to (7.14.3),

$$\frac{h_{пш}}{h}=\frac{1}{1+\frac{P_п G_{пп}}{2N_0 \Delta f_п}\frac{G_{пр}\lambda^2}{(4\pi)^2 d_п^2}}, \tag{9.9.2}$$

where λ — wavelength;

$P_П$ — power of the jamming transmitter;

$\Delta f_П$ — width of the spectrum of interfernce;

$G_{ПП}$ — antenna gain of the station of interferences;

$G_{Пр}$ — gain of the receiving antenna of the radar in the direction to the source of interference;

$d_П$ — distance of the source of interferences.

Magnitudes of lowering of the signal-to-noise ratio with noise interference are illustrated by graphs of Fig. 7.51.

During calculation of noise immunity of radar it is necessary to determine the critical strength of interference, and consequently also quantity $h_{ПШ} = h_{Кр}$, at which one or another undersirable change of properties of system sets in. In this case such a change may be error of the speed meter reaching a certain assigned impermissible magnitude or breakoff of tracking, determined by one of the above-indicated methods. Critical intensity of noise interference $(P_П G_{ПП}/\Delta f_П)_{Кр}$ [w/cps] can be found from (9.9.2) if we consider that $N_0 = P_c/2\Delta f_c\ h$ and according to the radar formula

$$P_c=\frac{P_{пер}G^2\sigma_ц\lambda^2}{(4\pi)^3 d_ц^4}, \tag{9.9.3}$$

where $\sigma_Ц$ — reflecting surface of the target;

G — antenna gain of radar in the direction to the target;

$P_{пер}$ — mean power of the transmitter of the radar;

$d_Ц$ — distance to the target.

Under the condition of spatial coincidence of the source of interferences with the target and with the condition normally observed in conditions of measurements in the presence of only natural noise $h \gg h_{Кр}$ it is easy to find that

$$\left(\frac{P_п G_{пп}}{\Delta f_п}\right)_{кр}\approx\frac{1}{h_{кр}}\frac{P_{пер}G\sigma_ц}{4\pi\Delta f_c d_ц^2}, \tag{9.9.4}$$

where $h_{Кр}$ is found either by formulas of §§ 9.5 and 9.8 for accuracy of measurement of speed (if we assign permissible accuracy), or by formulas of § 9.6 if we apply criteria connected with breakoff of tracking.

For illustration we give the following example.

Let us assume that the radar is characterized by the following data: $P_{\text{пер}} = 100$ w, $G = 9\cdot10^{-3}$, $\lambda = 10$ cm, $\Delta f_c = 30$ cps and should work for a target where $\sigma_{\text{ц}} = 10\ m^2$ at distance $d_{\text{ц}} = 100$ km. Let us assume that according to one of the criteria there is obtained the critical value of the signal-to-interference ratio $h_{\text{кр}} = 1$. Then from (9.9.4), $(P_{\text{п}}G_{\text{пп}}/\Delta f_{\text{п}})_{\text{кр}} = 2.37$ w/Mc, i.e., for the given sufficiently typical values of parameters it composes a fully attainable quantity in contemporary conditions. If, however, we take $G = 25\cdot10^3$ (increasing the diameter of the antenna from 3 to 5 m), raise the power of the transmitter to $P_{\text{пер}} = 1$ kw, and assume the presence of a target with $\sigma_{\text{ц}} = 50\ m^2$, then $(P_{\text{п}}G_{\text{пп}}/\Delta f_{\text{п}})_{\text{кр}} = 340$ w/Mc, i.e., composes a normally unattainable magnitude.

For pulse chaotic interference with the assumptions made in Paragraph 7.14.2

$$\frac{h_{\text{вх}}}{h} = \frac{1}{1+\frac{\nu\tau_{\text{п}}^2 P_{\text{п}} G_{\text{пп}}}{2N_0\sqrt{1+\Delta f_{m\,\text{эф}}^2\tau_{\text{п}}^2}}\,\frac{G_{\text{пр}}\lambda^2}{(4\pi)^2 d_{\text{п}}^2}}, \tag{9.9.5}$$

where $P_{\text{п}}$ — power in a pulse of the jamming transmitter;

$\tau_{\text{п}}$ — pulse duration of interferences;

ν — average frequency of pulses of interference;

$\Delta f_{m\,\text{эф}}$ — effective width of spectrum of modulation of the signal; the remaining designations are as before.

If we again assume spatial coincidence of the source of interference with the target and fulfillment of inequality $h \gg h_{\text{кр}}$, the critical value of the magnitude of intensity of interference $\nu\tau_{\text{п}}^2 P_{\text{п}} G_{\text{пп}}$ [w/Mc] according to (9.9.3) and (9.9.5) is defined as

$$(\nu\tau_{\text{п}}^2 P_{\text{п}} G_{\text{пп}})_{\text{кр}} = \frac{1}{h_{\text{кр}}}\,\frac{P_{\text{пер}} G\sigma_{\text{ц}}}{4\pi\Delta f_c d_{\text{ц}}^2}\sqrt{1+\Delta f_{m\,\text{эф}}^2\tau_{\text{п}}^2}, \tag{9.9.6}$$

i.e., composes a magnitude $\sqrt{1 + \Delta f_{m\,\text{эф}}^2\tau_{\text{п}}^2}$-times larger than the critical intensity of noise interference, which shows the lesser effectiveness of pulse chaotic interference in the conditions formulated above. If, for instance, duration of pulses of interference and of the signal coincide, then $\Delta f_{m\,\text{эф}}\tau_{\text{п}}$ is a quantity of the order of unity. When $\Delta f_{m\,\text{эф}}\tau_{\text{п}} = 1$, for creation of the same effect as from noise interference it is necessary to have $\sqrt{2}$ more power of pulse chaotic interference.

For passive interference, broad in band as compared to the signal, according to (7.14.13)

$$\frac{h_{\text{вх}}}{h} = \frac{1}{1+h\,\frac{\sigma_{\text{п}}}{\sigma_{\text{ц}}}\,f\,(\Delta\omega_{\text{д}})}, \tag{9.9.7}$$

where $\sigma_Ц$ — reflecting surface of target;

$\sigma_П$ — reflecting surface of interference in the resolution cavity of the radar;

$\Delta\omega_Д$ — difference of Doppler frequencies of signals reflected from the target and interference;

$f(\omega)$ — function describing the form of the spectrum of the signal reflected from interference ($f(0) = 1$); and it is assumed that there is selected a sufficiently large frequency of repetition F_r.

From (9.9.7) for $h \gg h_{кр}$ we obtain

$$(\sigma_П)_{кр} \approx \frac{1}{h_{кр}} \frac{\sigma_Ц}{f(\Delta\omega_Д)}, \qquad (9.9.8)$$

from which it is easy to calculate the density of passive interference necessary to "swamp" radar (to knock out the speed meter). It is clear that with great detuning of Doppler frequencies of signals reflected from target and from interference $f(\Delta\omega_Д) \ll 1$, noise immunity with respect to passive interference is high; in particular, when $h_{кр} \approx 1$ it turns out that $(\sigma_П)_{кр} \gg \sigma_Ц$. If however $\Delta\omega_Д \approx 0$, $f(\Delta\omega_Д) \approx 1$ and for $h_{кр} \approx 1$ we have relationship $(\sigma_П)_{кр} \approx \sigma_Ц$, which signifies for attainable densities of interferences absolutely unsatisfactory noise immunity.

Further conclusions, ensuing from lowering of the signal-to-interference ratio, according to formulas (9.9.2), (9.9.5), and (9.9.7), do not differ at all from those given in § 7.14; therefore we will not repeat them here.

Intermittent active interferences act on coherent speed meters just as on coherent range meters. In the presence of rapid intermittent (as compared to inertia of the speed meter) interferences and during application of high-speed AGC systems interferences act just as continuous interferences of the same mean power.

With a slow AGC system we start to see nonlinear phenomena, which appear in various ways for different schemes of processing the signal and for their quantitative estimating require carrying out of corresponding investigations of the separate forms of circuits.

Slowly intermittent interferences lead to parameters of the system of measurement of speed intermittently changing in connection with the fact that there exist intervals of time in which the signal-to-noise ratio is equal to h, and intervals in which this ratio is equal to $h_{пш}$. Besides the fact that in intervals of action of interference there can occur the same phenomenon as during continuous interference, there also occurs parametric influence, which may cause parametric excitation of the tracking system. These questions require further study; their formulation does not differ at all from that given in § 7.14.

As we already noted during the analysis of frequency discriminators, there exists a type of discriminator — with commutation switching of reference voltages — which is especially susceptible to the influence of intermittent interferences with frequencies of switching close to the frequency of the reference voltage or to one of the frequencies which is a multiple of it. As a result of appearing beats between the reference voltage and interference there can appear low-frequency oscillations, passing to the output of the tracking system and leading to the appearance of large errors or even to breakoff of tracking.

It is necessary to make several remarks about the influence on coherent speed meters of return interference. This interference is applied mainly during pulse radiation of the radar and, as was indicated in Chapter I, can be created both by starting by pulses of the radar of a certain transmitter, and also by amplification and reradiation of the radar pulses themselves. In the first case there appear difficulties in ensuring coherence of return interference, but the incoherent pulse interference for coherent radar is not very effective. In the second case coherence is ensured automatically. However, if there is absent some additional modulation of the pulses of return interference, being radiated from the irradiated target, it permits us more exactly than by the reflected signal to measure parameters of motion of the target, in particular speed.

Such a phenomenon takes place due to the absence of fluctuations and to the great power of pulses of interference as compared to the reflected signal. Therefore, for a system of measurement of speed by Doppler frequency return interference is only effective with additional frequency modulation, where changes of frequency of the interference should be so slow that they are tracked by the tracking system. In this respect we have complete analogy with the influence of return interference on range finders with this difference only, that variable delay of modulation, necessary for misleading the range finder, is replaced by a variable carrier frequency of pulses of interference for misleading the speed meter. When the jamming transmitter is not on the target, just as during breeding of pulses of interference, all remarks made when examining the influence of return interference on coherent range finders remain valid.

Considering the influence of interferences on meters of speed as the derivative of a coordinate, one should turn first of all to noise immunity of range finders and goniometers. Short duration failure of meters tracking coordinates leads to

cessation of measurement of speed. In cases when there is no short duration failure and increase of errors of measurement of coordinates is determined only by change of signal-to-noise ratio ($h_{\text{П Ш}}/h$ – in coherent case and $q_{\text{П Ш}}/q$ – in the incoherent case), increased error of measurement of speed also is determined by the formulas obtained above, if in them we replace h by $h_{\text{П Ш}}$ (or q by $q_{\text{П Ш}}$).

§ 9.10. Conclusion

Resuming the basic results of theoretical research of questions of measurement of speed in the given chapter we first of all should emphasize that the basic method of measurement of the radial component of speed should be considered measurement by Doppler frequency shift. This method for coherent radiation ensures in the vast majority of cases greater accuracy than the other considered method, based on differentiation of the distance to the target.

Being interested in the most wide-spread tracking Doppler speed meters, we with not very limiting assumptions found an optimum frequency discriminator and optimum smoothing circuits of such meters. The optimum frequency discriminator cannot be realized exactly; however two of the considered practically realizable schemes are: a discriminator with a tuned circuit and phase shifter and a discriminator with two mixers and differentiation, very close to the optimum discriminator in its properties. Here as the criterion of optimality we, as in preceding chapters, used accuracy of measurement.

When using a tracking meter for tracking purely random components of change of speed optimum smoothing circuits frequently are linear with constant parameters and in a number of cases coincide in form with practically applied filters. In a case when speed is a quasi-regular process, optimum filters possess variable parameters and frequently are characterized by gradual cutoff of the discriminator and by accuracy of measurement unlimitedly increasing in time. It is necessary to note that in practice speed will always have a certain purely random component. Therefore, very small errors are not obtained, even if we ignore instrument errors, disregard of which is justified only with sufficiently large fluctuation and dynamic errors.

With intense noises and interferences tracking meters are subject to failure (breakoff of tracking). The formulas obtained for quantitative estimate of this phenomenon can be, unfortunately, used only for rough estimation of the order of magnitudes of intensities of noises and interferences at which failure begins. One should make such a conclusion, first, in connection with the fact that analysis of

breakoff of tracking could be conducted only for smoothing circuits of very simple form and, secondly, the obtained results rather strongly differ for various criteria of breakoff and for various methods of approximation of the discrimination characteristic.

Investigation of potentialities of nontracking Doppler speed meters leads to the conclusion that in principle these meters can ensure the same accuracy as tracking ones if we consider measurements with sufficiently fast signal fluctuations.

During attempts to measure speed by Doppler frequency shift for an incoherent pulse signal we usually obtain very low accuracy of measurement, due to which with this form of radiation measurement of speed as the derivative of distance is preferable. The tangential component of speed can be measured as the derivative of an angular coordinate.

Accuracies of measurement of speed as the derivative of a coordinate found in a number of cases have acceptable magnitudes for accelerations which are not too high; with high accelerations accuracies of measurement of speed are low. This also pertains to optimum meters in the case of purely random change of the coordinate. With quasi-regular change of it potential error of measurement of speed approaches zero as $t \to \infty$, for radial speed in certain cases even faster than error of measurement by Doppler shift of the signal frequency. Consequently, in these cases instrument error predominates.

Quantitative estimation of the influence of interferences on speed meters is determined by the fact that these meters in most cases are coherent. Therefore, the influence of the most wide-spread interferences reduces simply to the influence of equivalent noise and, consequently, to change of the signal-to-noise ratio.

Thus, we have conducted a rather detailed analysis of speed meters from positions of accuracy of measurement, including investigation of the question of cessation of these measurements under the influence of intense noises and interferences. In order of formulating still unsolved problems, connected with measurement of speed, we should first of all pay attention to investigation of the resolution capability of speed meters, including synthesis of meters optimum from this point of view. Currently urgent questions of resolution of targets still have not found sufficiently complete theoretical solution and need careful consideration.

It is desirable, besides frequency, to investigate phase discriminators, which

can be applied with success in Doppler speed meters, giving, possibly, no worse results than frequency discriminators. The number of analyzed types of frequency discriminators also should be increased.

Very urgent, for meters of other parameters of motion of the targets, too, is more detailed study of questions of breakoff of tracking. In particular, it should be interesting to find solution of these questions with application of smoothing circuits of high order. Apparently, here we will not manage to obtain all necessary regularities purely theoretically, and it is necessary to utilize appropriately formulated experiment.

In these investigations we assumed Gaussianness both of the reflected signal, and also the process of change of speed. Although this assumption corresponds to a very large number of practical cases, it nonetheless is of interest to investigate accuracy of speed meters with other assumptions. It would also be very interesting to conduct a whole complex of investigations for slow fluctuations of the signal, the speed of which is commensurate with the speed of processes of set-up in the systems. The questions found in this chapter are only a very superficial reflection. Questions of the influence on speed meters of various forms of interferences, including synthesis of meters optimum in the presence of thse interferences, need more careful consideration.

CHAPTER X

MEASUREMENT OF ANGULAR DATA WITH A COHERENT SIGNAL

§ 10.1 Introductory Remarks

Measurement of angular data is one of the most important functions executed by radar sets of various assignment. Continuously increasing requirements on accuracy of measurement of angles has led to a need for thorough theoretical research of possibilities of radar measurement of angular data. Especially important is the problem of finding potential accuracy of goniometry in conditions of a fluctuating radar signal received against a background of internal noises, and also the problem of synthesis of circuits of optimum goniometers realizing this potential accuracy. However, in contemporary radar literature there exists a comparatively small number of works on these problems.

The first works in which there are attempts at optimization of radar goniometers are [10 and 11]. In them there is considered the problem of optimum measurement of angular data by a surveillance radar. Here there is introduced a series of very limiting assumptions. Circuits of goniometers are assumed assigned up to and including the amplitude detector, and only post-detector processing is optimized. The target is assumed either not fluctuating at all, or fluctuating so that its reflecting surface does not change during the time of passage over the target of the directional pattern of the antenna, but takes independent values in different periods of scanning. For solution of problems of optimization in these works there is used the theory of statistical maximum likelihood estimates.

We note further the work [45], in which, in point of fact, the radio channel of a tracking radar goniometer is optimized. We consider the method of direction finding with a scanning directional pattern. The circuit of the radio channel again

is assumed assigned up to and including the amplitude detector. The signal is assumed to be fluctuating so that its amplitude is constant within the duration of pulses, but is independent from pulse to pulse. For solution of problems of optimization there is used the method of inverse probability.

The very limiting initial assumptions used in [10, 11, 45] lower the value of the results obtained in them. However, already in these works there is shown the connection between the considered problems and mathematical statistics, although its powerful techniques are evidently used insufficiently.

Recently there appeared works [12, 46] where there is solved the problem of estimating angular data on the basis of the observed realization of the signal. The signal is assumed fluctuating, but rigidly correlated during the time of observation. Consideration was conducted, for example, for methods of instantaneous comparison of signals, both in amplitude and phase. In [12, 46] they used estimates of maximum a posteriori probability. The essential fact is that in these works with respect to the circuit of the goniometer there were introduced no preliminary assumptions, and it was completely optimized.

The assumption of rigid correlatedness of the signal for the time of observation in these works is rather limiting. More common in practice is the case when the time of correlation of fluctuations of the signal is comparable to or considerably less than the time of observation.

In the present and subsequent chapters we shall give a detailed and systematic study of the problem of radar measurement of angular data relying on this last case. The account will be conducted basically in the same plan as in all the preceding chapters: first we synthesize and study optimum circuits; then we analyze circuits close to optimum. During the synthesis of optimum goniometers we will make no preliminary assumptions about the possible circuit of the goniometer. However, we shall assume assigned the method of direction finding (i.e., the structure of the antenna system) and the method of coding angular data in the radar signal, determined by this. Here we shall systematically consider existing or possible methods of direction finding.

Synthesis of radar goniometers without assignment of the method of direction finding is a difficult problem which can be solved sufficiently accurately only for antennas of the phased array type. This question is the subject of § 10.13.

In the present chapter we shall study the problem of radar measurement of angular data with a coherent signal. The material here is divided into three basic

groups of problems: goniometers with a tracking antenna, goniometers in which tracking is carried out electronically (not by the antenna), and, finally, nontracking goniometers (not containing a feedback circuit).

The main part of the chapter is devoted to goniometers with a tracking antenna, since this class of goniometers is the broadest. First we synthesize the circuit of an optimum radio channel for these goniometers with very general assumptions about the method of direction finding utilized. Then this circuit is specified for the most important methods of direction finding utilized in practice: the method of pattern scanning, the method of scanning with compensation, of instantaneous amplitude comparison of signals, the method of phase center scanning and of instantaneous phase comparison of signals. The most attention is paid to the question of the influence on accuracy of synthesized goniometer circuits of various deviations from idealness in them, inevitable during practical realization of the circuits.

Then we study radar goniometers with antennas of phased array type. For these goniometers together with the optimum radio channel we also synthesize the optimum method of direction finding (optimum processing of the field in the aperture of the antenna system).

Then we investigated total error of a closed-loop goniometer tracking system in linear conditions (with small mistuning) and touch briefly on question of nonlinear phenomena in tracking goniometers, occurring with high noise levels (breakoff of tracking). Approximately in the same aspect we consider goniometers in which tracking is carried out electronically. Here we study the so-called method of two-dimensional scanning of the directional pattern.

At the end of the chapter we study nontracking goniometers. Here basic attention is paid to synthesis of the "estimator unit" (or "unit of primary processing"), which issues the maximum likelihood estimate of an angular coordinate from the realization of a signal of finite duration, within limits of which the angular coordinate of the target does not change. We study methods of instantaneous amplitude and phase comparison of signals most frequently used in nontracking goniometers.

The last section is devoted to study of the influence of certain forms of interferences on radar goniometers.

§ 10.2. Radar Methods of Direction Finding

In radar goniometers the measured parameters (angular coordinates of the target) are coded in the received radar signal. On angular target coordinates (on current angular mismatches in goniometers with a tracking antenna) there may depend both the

amplitude and the phase of the received signal. The form of this dependence is determined by the structure of the antenna system or, so to speak, the method of direction finding applied in the goniometer. Obviously, the structure of optimum goniometer systems will essentially depend on the method of direction finding; therefore, before solving the problem of synthesis of optimum goniometers we shall give a description of methods of direction finding. Our description will be rather short, inasmuch as radar methods of direction finding are well-known.

At present there have been proposed a large number of methods of direction finding, using different physical principles. The most well-known of the methods of direction finding is the so-called method of scanning the directional pattern of the receiving antenna [47]. Reception of the signal in this method is by an antenna whose directional pattern scans with respect to the axis of scanning in an angular sector, narrow as compared to the width of the pattern. On the received signal here there is superimposed a signal, modulated in amplitude according to a law which depends on the angular divergence of the target from the axis of scanning.

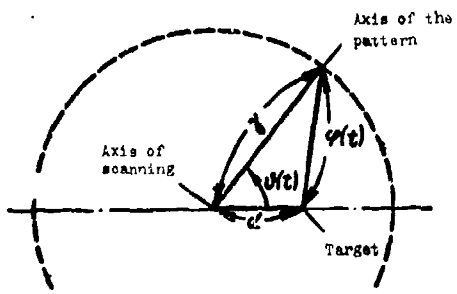

Fig. 10.1. The method of scanning of the directional pattern of a receiving antenna.

We shall find the form of amplitude modulation of the received signal introduced by the scanning pattern. Here and subsequently we shall limit ourselves to the case of measurement of one angular coordinate of the target, considering tracking of the other coordinate exact. The grounds for such consideration will be given in Chapter XII. devoted to multi-dimensional meters.

Consider Fig. 10.1, in which there is depicted a section of a sphere, passed from an antenna through a target (with large distances to the target this section can be considered two-dimensional). If we designate the measured angular divergence from the axis of scanning by α, the angle between the axis of the directional pattern and the axis of scanning by γ, and the polar coordinate of the current position of the axis of the directional pattern by $\vartheta(t)$, then, as it is easy to see from Fig. 10.1, the angle between the direction to the target and the axis of the directional pattern will be equal to

$$\varphi(t)=\sqrt{\gamma^2+\alpha^2+2\gamma\alpha\cos\vartheta(t)}.$$

Let us assume that the directional pattern (for voltage) has axisymmetric

form $Gg(\varphi)$. In this case the received signal will be amplitude-modulated according to the law

$$U_a(t,\ \alpha)=g\{\sqrt{\alpha^2+\gamma^2+2\alpha\gamma\cos\vartheta(t)}\}. \tag{10.2.1}$$

Here we assume that target is irradiated by a certain nonscanning pattern. For definitiveness we shall consider this assumption carried out if nothing further is said. More complicated cases can easily be considered if desired, and the difference of them will be quite immaterial.

In expression (10.2.1) function $g(\varphi)$ in a definite way is normalized. Its normalization will be selected later. Coefficient G accounts for the gain of the directional pattern.

For small α expression (10.2.1) can be rewritten in the following form:

$$U_a(t,\ \alpha)=g(\gamma)[1+\mu_a\alpha\cos\vartheta(t)], \tag{10.2.2}$$

where

$$\mu_a=g'(\gamma)/g(\gamma) \tag{10.2.3}$$

is the gain factor of the directional pattern along the axis of scanning.

Obviously, when the target is located on the scan axis, amplitude modulation of the received signal disappears; therefore, the direction determined by the scan axis is also frequently called the equisignal direction.

The law of change of angle $\vartheta(t)$ may differ. When $\vartheta(t) = \Omega_{CK}t + \vartheta_0$ we obtain so-called conical scanning of the directional pattern (with angular frequency Ω_{CK}); if $\vartheta(t)$ takes values ϑ_0, $\vartheta_0 + \pi/2$, $\vartheta_0 + \pi$, $\vartheta_0 + 3\pi/2$, etc., changing over intervals of time $T_{CK}/4$, we obtain so-called quadrant scanning (with period T_{CK}).

The next method of direction finding which we shall consider is the method of scanning with two directional patterns, often called the method of scanning with compensation. (As we shall see later, this method provides compensation of the harmful influence of fluctuations of the signal.) In the given method reception of the reflected signal is effected by two directional patterns scanning with respect to one axis. Angles of inclination of axes of the patterns to the scan axis may be different. In particular, one of the angles may be equal to 0, which corresponds to one fixed pattern.

The mutual location of the target and the patterns in the method of scanning with compensation is depicted in Fig. 10.2, where by α we again designate angular divergence of the target from the scan axis; by γ_1 — the angle between the scan

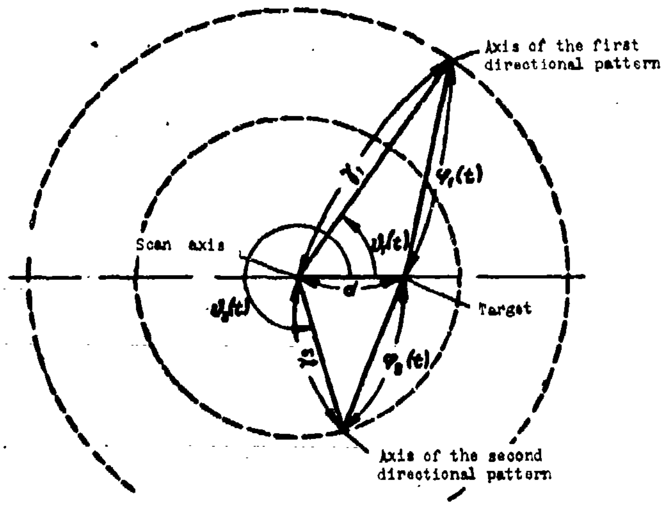

Fig. 10.2. Method of scanning with compensation.

axis and the axis of the i-th pattern (i = 1, 2); by $\vartheta_i(t)$ – the current value of the polar coordinate of the i-th pattern. From this figure it is easy to see that the angle between the direction to the target and the axis of the i-th pattern is equal to

$$\varphi_i(t)=\sqrt{\alpha^2+\gamma_i^2+2\alpha\gamma_i\cos\vartheta_i(t)}.$$

If we now designate by $G_i g_i(\varphi)$ the form of the i-th pattern (for voltage), we find that the signal received in this pattern will be amplitude-modulated according to the law

$$U_{ai}(t,\ \alpha)=g_i\{\sqrt{\alpha^2+\gamma^2_i+2\alpha\gamma_i\cos\vartheta_i(t)}\}. \tag{10.2.4}$$

From (10.2.4) it is clear that if the target is located on the scan axis modulation of signals received in both patterns disappears, i.e., the scan axis determines the equisignal direction. We note also that if one of the directional patterns is fixed, i.e., for it $\gamma_i = 0$, the signal received in this pattern will have amplitude $U_{ai}(t,\ \alpha) = g_i(\alpha)$, depending on the angular divergence of the target from the sca axis, but not depending on time.

For small α, (10.2.4) can be recorded in the form

$$U_{ai}(t,\ \alpha)=g_i(\gamma_i)[1+\mu_{ai}\alpha\cos\vartheta_i(t)], \tag{10.2.5}$$

where

$$\mu_{ai}=g'_i(\gamma_i)/g_i(\gamma_i) \tag{10.2.6}$$

is the gain factor of the directional pattern.

Methods of scanning of a directional pattern without compensation and with compensation are usually used in goniometers in which tracking is carried out directly by the antenna. In these goniometers the measured value of angular divergence of the target from the scan axis is used to control the antenna, which turns in such a way that the scan axis passes through the target. Consequently, the angular position of the scan axis, obviously, is also the measured value of the angular coordinate of the target, and angular divergence of the target from the scan axis is the difference

between the true value of the angular coordinate and its measured value.

Let us consider now the method of direction finding called the method of instantaneous amplitude comparison of signals (IAC) [27, 48, 49]. In this method, as we know, during measurement of angle α reception of the reflected signal is conducted by two patterns, the intersection of which by the plane of angle α is depicted in Fig. 10.3. If we designate by γ_i the angle between the axis of the i-th pattern (i = 1, 2) and the equisignal direction; by α – the angular divergence of the target from the equisignal axis; and by $Gg_1(\varphi)$ – the form of the i-th pattern in the considered plane (voltage), then it is clear that amplitude of signal received by the 1st and 2nd patterns will be equal, respectively, to,

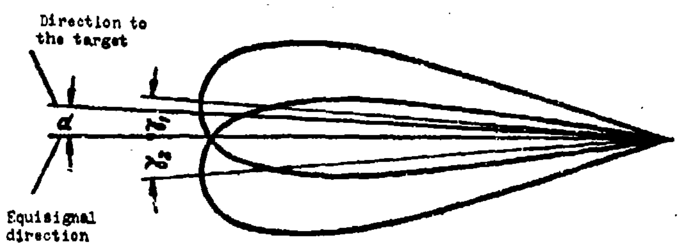

Fig. 10.3. Method of instantaneous amplitude comparison of signals.

$$
\begin{aligned}
U_{a_1}(t, \alpha) &= g_1(\gamma_1 - \alpha),\\
U_{a_2}(t, \alpha) &= g_2(\gamma_2 + \alpha).
\end{aligned}
\qquad (10.2.7)
$$

When $\alpha = 0$, i.e., when the target is located on the equisignal direction, amplitudes of the received signals become identical, since $g_1(\gamma_1) = g_2(\gamma_2)$ (Fig. 10.3).

Let us note that from the formal point of view the IAC method can be considered a particular case of the method of scanning with compensation, in which the directional patterns are motionless and have coordinates $\vartheta_1(t) = 0$ and $\vartheta_2(t) = \pi$.

For small α expressions (10.2.7) can be rewritten in the form

$$
U_{a_1}(t, \alpha) = g_1(\gamma_1)(1 - \mu_{a_1}\alpha), \quad U_{a_2}(t, \alpha) = g_2(\gamma_2)(1 + \mu_{a_2}\alpha), \qquad (10.2.8)
$$

where gain factors μ_{a_i} are determined analogously to (10.2.6).

The IAC method is used both in goniometers with a tracking antenna and in nontracking goniometers. In the last case the directional patterns are selected sufficiently wide so that the sector in which they intersect covers the sector of possible values of the angular coordinate of the target. Here the patterns are motionless, and the angular coordinate of the target is measured by the difference of amplitudes of the received signals.

Next we consider the so-called method of linear or two-dimensional scanning [[illegible]]. In this method a rather narrow directional pattern periodically passes over a certain

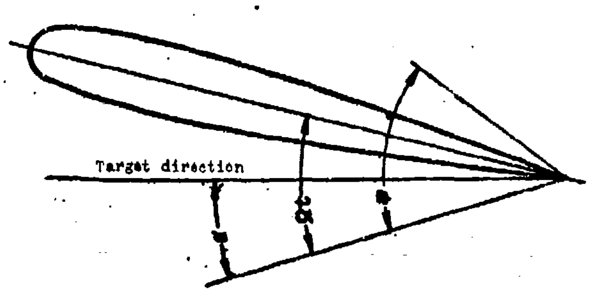

Fig. 10.4. Method of two-dimensional scanning.

angular sector, covering the sector of possible values of the angular target coordinate. If the angular target coordinate in this sector is α (Fig. 10.4), the intersection of the pattern with the considered plane has with respect to power the form Gg(φ) (we assume that irradiation of the target is by this pattern, and we disregard the propagation time of the radar signal to the target and back), and Ω — angular velocity of motion of the directional pattern over the sector, then at the input of the receiver there will be received a periodic sequence of signal packs (pulses), having form Gg(Ωt) and delayed relative to the beginning of the period time $\tau_3 = \alpha/\Omega$. In other words, the received signal will be amplitude-modulated according to the law

$$U_{a}(t,\alpha)=\sum_{n}g[\Omega(t-\tau_{3}+nT_{\text{п}})], \tag{10.2.9}$$

where T_{Π} — period of survey of the sector by the directional pattern.

Here it is assumed for simplicity that the pattern has no reverse movement over the sector. If, upon reaching the boundary of the sector the pattern instantly jumps to its beginning, obviously,

$$T_{\text{п}}=\Phi/\Omega,$$

where Φ — angular dimension of the sector.

Goniometers using the method of two-dimensional scanning are usually tracking-type. However, tracking here is carried out by an electronic circuit, which tracks the signal pulses. Therefore, the method of two-dimensional scanning is often called the method of pulse tracking. This method we shall consider subsequently.

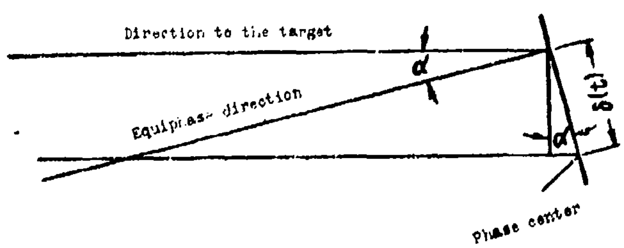

Fig. 10.5. Method of phase center scanning.

Let us now turn to consideration of the so-called method of phase center scanning. In this method reception of the reflected signal is by an antenna whose phase center shifts (scans) in the aperture plane of the antenna. Then the received signal becomes phase-modulated according to a law which depends on the acceptance angle of the radar signal, i.e., on the angular position of the target.

Let us find the form of this modulation. We designate by α the angle between the target direction and the normal to the aperture plane of the antenna (which determines the equiphase direction), and by $\delta(t)$ — the coordinate of the projection of the phase center on the considered plane (Fig. 10.5). Then, obviously, the phase of the received signal is time-dependent and is equal to

$$\Phi(t, \alpha) = \varphi_0 + \frac{2\pi\delta(t)}{\lambda}\sin\alpha, \tag{10.2.10}$$

where φ_0 — phase of the received signal when the target is on the equiphase direction;

λ — wavelength of the radar signal.

For small α, $\sin\alpha$ in (10.2.10) can be replaced by α. The method of phase center scanning is usually used in goniometers with a tracking antenna. The angle α here is the difference between the true and measured values of the angular target coordinate.

A considerably more wide-spread method of direction finding is the method of instantaneous phase comparison of signals (IPC) [27, 48, 49]. Here during measurement of angle α reception of the reflected signal is by two antennas whose phase centers lie in the plane of angle α and are separated distance d, called the base of the antenna system. By α we shall understand the angle between the target direction and the normal to the plane in which the phase centers lie (Fig. 10.6). It is easy to see that phases of signals received by the 1st and 2nd antennas are equal, respectively, to

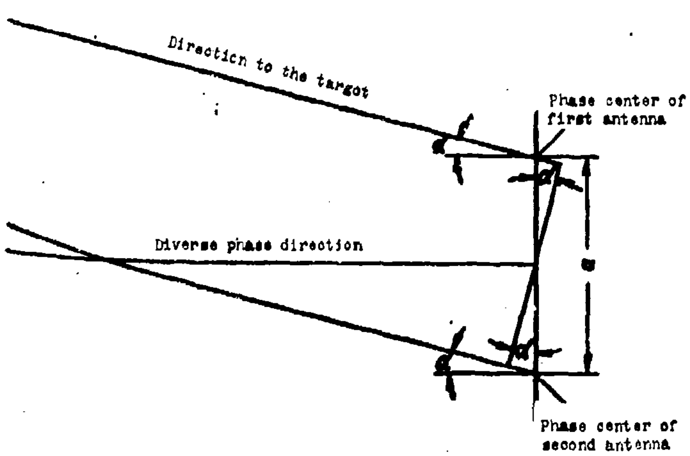

Fig. 10.6. Method of instantaneous phase comparison of signals.

$$\Phi_1(t, \alpha) = \varphi_0 - \frac{\pi d}{\lambda}\sin\alpha; \quad \Phi_2(t, \alpha) = \varphi_0 + \frac{\pi d}{\lambda}\sin\alpha, \tag{10.2.11}$$

where all designations are the same as in (10.2.10).

The IPC method is used both in goniometers with a tracking antenna, and also with a nontracking antenna. In nontracking goniometers the antenna system is motionless, and its base d is not too great for angular sector $|\alpha| < \arcsin \lambda/2d$ [in which phases $\Phi_1(t, \alpha)$ and $\Phi_2(t, \alpha)$ (10.2.11) unambiguously determine α]

to cover the sector of possible values of angular coordinates of the target. Angular target coordinate α is determined in this case by the difference of phases of the received signals.

Thus, we have characterized all basic methods of direction finding applied at present in radar. We note the possibility of combined use of various methods of direction finding. Generalizations in this direction are obvious. For instance, it is possible to use a method with simultaneous scanning of the directional pattern and phase center scanning.

The received signal in this case becomes modulated in amplitude and phase with laws of modulation which depend on the angular position of the target, i.e., becomes, as it were, more informative, which should lead, ultimately, to increase of accuracy of measurement of angular data. Also there can be used the method of instantaneous amplitude-phase comparison of signals, consisting of the reception of the reflected signal in two antennas with separated phase centers and directional patterns opened a certain angle to one another. The received signals in this method will differ both in amplitudes and in phases, the difference of which depends on the acceptance angle of the radar signal. Continuing generalizations in this direction, we arrive at a radar method of direction finding which consists of reception of a radar signal in an arbitrary number of antennas, directional patterns and phase centers of which scan by certain laws. Any of the methods of direction finding is a particular case of such a generalized method and is obtained upon specifying the number of receiving antennas and laws of scanning of their patterns and phase centers.

Subsequently we shall consider, as far as possible, goniometers with the generalized method of direction finding, specifying, where it is necessary, final results. In particular, this approach is applicable to goniometers with a tracking antenna.

§ 10.3. Optimum Radio for Goniometers with a Tracking Antenna

In the following sections we will study the most important class of radar goniometers — goniometers with a tracking antenna. The present section is devoted to synthesis of an optimum radio channel for such goniometers.

As we already said, among goniometers with a tracking antenna we include goniometers for which target tracking is produced directly by the antenna array. Goniometers of this type are very widely used, inasmuch as they possess best the power characteristics (power is always radiated in the direction of the target).

In the case of goniometers with a tracking antenna the optimum radio channel

can be synthesized for the generalized method of direction finding described in the preceding paragraph, when reception of the reflected signal is conducted in any number of antennas, directional patterns and phase centers of which scan by arbitrary laws. Here we shall assume that there exists a single equisignal direction: if the target is located on this direction, the amplitude and phase modulations of the received signals introduced during scanning of the directional patterns and phase centers of the antennas disappear. This assumption is very significant subsequently.

Most methods of direction finding applied in practice — scanning of the directional pattern without compensation and with compensation, IAC, phase center scanning, IPC — are applicable to this case.

Thus, we assume that the goniometer has n antennas, and the received signal at the output of the i-th antenna due to scanning of its directional pattern and phase center is modulated in amplitude and phase by laws $U_{ai}(t, \alpha)$ and $\Phi_i(t, \alpha)$, depending on the angular divergence α of the target from the equisignal axis. Obviously, $U_{ai}(t, 0)$ = const and $\Phi_i(t, 0)$ = const, since by condition coordinate $\alpha = 0$ determines the equisignal axis. We assume that there is carried out normalization

$$U_{ai}(t, 0) = 1. \tag{10.3.1}$$

We introduce the designation

$$U_i(t, \alpha) = U_{ai}(t, \alpha)\, e^{i\Phi_i(t, \alpha)}. \tag{10.3.2}$$

Quantity $U_i(t, \alpha)$ we shall call the complex directional pattern of the i-th antenna of the goniometer.

The received signal at the output of the i-th antenna can be recorded in the form

$$\begin{aligned} y_i(t) = {} & \sqrt{P_{ci}}\, U_{ai}(t, \alpha)\, u_a(t) \{a(t) \cos[\omega_0 t + \psi(t) + \\ & + \Phi_i(t, \alpha)] + b(t) \sin[\omega_0 t + \psi(t) + \Phi_i(t, \alpha)]\} + \\ & + \sqrt{N_{0i}}\, n_i(t) = \sqrt{P_{ci}}\, \mathrm{Re}\, U_i(t, \alpha) E(t) u(t) e^{i\omega_0 t} + \sqrt{N_{0i}}\, n_i(t). \end{aligned} \tag{10.3.3}$$

Here $u_a(t)$ and $\psi(t)$ are laws of amplitude and phase modulation of the sounding signal; function $u_a(t)$ is normalized

$$\frac{1}{T_r} \int_0^{T_r} u_a(t)^2 dt = 1, \tag{10.3.4}$$

where T_r — the period of repetition of the sounding signal:

$$u(t) = u_a(t)\, e^{i\psi(t)}; \tag{10.3.5}$$

$a(t)$ and $b(t)$ – independent stationary random processes with zero mean value and with correlation function $\overline{a(t)a(s)} = \overline{b(t)b(s)} = \rho(t - s)$, normalized so that

$$\rho(0)=1;$$
$$E(t)=a(t)+ib(t); \tag{10.3.6}$$

$n_i(t)$ – white noise with unit spectral density; noises $n_i(t)$, $i = 1, 2, \ldots, n$ are independent;

N_{0i} – spectral density of noise added to the signal at the input of the i-th antenna;

$P_{c\,i}$ – mean power of this signal.

We assume that the time position and carrier frequency of signal (10.3.3) are known exactly. This means that we know exactly the delay of the received signal (distance to the target) and Doppler shift of frequency of the reflected signal (speed of the target). Being interested exclusively in the problem of measurement of angular data, we can consider these assumptions realized. The influence of inaccurate knowledge of the delay and frequency of the received signal on measurement of angles will be considered subsequently during the analysis of goniometer circuits.

The set of signals (10.3.3) will form an n-dimensional normal process, characterized by a matrix of correlation functions $\mathbf{R}(t_1, t_2, \alpha) = \|R_{ij}(t_1, t_2, \alpha)\|$. From (10.3.3) we can easily find that

$$R_{ij}(t_1, t_2, \alpha)=\sqrt{P_{ci}P_{cj}}\,\rho(t_1-t_2)\,\mathrm{Re}\,U_i(t_1, \alpha)\,U^*_j(t_2, \alpha)\times$$
$$\times u(t_1)\,u^*(t_2)\,e^{i\omega_0(t_1-t_2)}+\delta_{ij}\sqrt{N_{0i}N_{0j}}\,\delta(t_1-t_2), \tag{10.3.7}$$

where δ_{ij} – Kronecker delta, and by z^* there is designated the complex conjugate of z.

To finding the operation executed by the optimum radio channel we need to know the joint functional of the probability density of the set of signals (10.3.3). Such a functional was already given in § 6.7. In this case it is equal to

$$P[y_1(\tau),\ y_2(\tau), \ldots,\ y_n(\tau),\ \alpha]=K\exp\left\{-\frac{1}{2}\int_0^T\int_0^T \mathbf{y}^+(t_1)\mathbf{W}(t_1, t_2)\mathbf{y}(t_2)\,dt_1dt_2\right\}, \tag{10.3.8}$$

where

$(0, T)$ – interval observation of the process;

$\mathbf{Y}(t)$ – column-vector of signals (10.3.3);

$\mathbf{Y}^+(t)$ – row vector of these signals;

$\mathbf{W}(t_1, t_2) = \|W_{ij}(t_1, t_2)\|$ – matrix of so-called inverse-correlation functions, satisfying equation

$$\int_0^T \mathbf{W}(t_1, t_2)\mathbf{R}(t_2, t_3, \alpha)\,dt_2=\mathbf{I}\delta(t_1-t_3), \tag{10.3.9}$$

($\mathbf{I}$ – unit matrix).

In more expanded form matrix equation (10.3.9) can be written in the form of the following system of integral equations:

$$\sum_{j=1}^{n}\int_{0}^{T} W_{ij}(t_1, t_2) R_{jk}(t_2, t_3, \alpha)\, dt_2 = \delta_{ik}\delta(t_1 - t_3). \tag{10.3.10}$$

Coefficient K in (10.3.8) is determined by the relationship

$$\frac{\partial \ln K}{\partial \alpha} = -\frac{1}{2}\int_0^T\int_0^T \mathrm{S_p}\, \frac{\partial \mathbf{R}(t_1, t_2, \alpha)}{\partial \alpha}\, \mathbf{W}(t_1, t_2)\, dt_1 dt_2 =$$

$$= -\frac{1}{2}\sum_{i,j=1}^{n}\int_0^T\int_0^T \frac{\partial R_{ij}(t_1, t_2, \alpha)}{\partial \alpha} W_{ij}(t_1, t_2)\, dt_1 dt_2. \tag{10.3.11}$$

We introduce likelihood function $L(\alpha)$, which is simply the functional of the distribution of probabilities, considered as a function of parameter α.

In Chapter VI it was shown that operation $z(t)$ of an optimum radio channel of a meter of angle α is determined by relationship

$$\int_{t-\Delta t}^{t} z(\tau)\, d\tau = \frac{\partial \ln L(\alpha)}{\partial \alpha}\bigg|_{\alpha=0}. \tag{10.3.12}$$

Here likelihood function $L(\alpha)$ is taken in interval $(t, t-\Delta t)$, where Δt considerably exceeds the time of correlation of fluctuations of the signal, but during Δt the angular coordinate of the target α can be considered constant.

Substituting (10.3.8) in (10.3.12) and considering (10.3.11), we can obtain

$$\frac{\partial \ln L(\alpha)}{\partial \alpha}\bigg|_{\alpha=0} = -\frac{1}{2}\sum_{i,j=1}^{n}\int_{t-\Delta t}^{t}\int_{t-\Delta t}^{t}\left\{\frac{\partial R_{ij}(t_1, t_2, \alpha)}{\partial \alpha} W_{ij}(t_1, t_2)\bigg|_{\alpha=0} + \right.$$

$$\left. + \frac{\partial W_{ij}(t_1, t_2)}{\partial \alpha}\bigg|_{\alpha=0} y_i(t_1) y_j(t_2)\right\} dt_1 dt_2. \tag{10.3.13}$$

Thus, to find the operation of an optimum radio channel we need to find values of functions $W_{ij}(t_1, t_2)$ and their derivatives with respect to α when $\alpha = 0$.

We shall look for a solution of system of equations (10.3.9) in the form

$$W_{ij}(t_1, t_2) = \frac{\sqrt{P_{0i}P_{0j}}}{N_{0i}N_{0j}}\, v(t_1, t_2)\, \mathrm{Re}\, U_i(t_1, \alpha) U^*_j(t_2, \alpha) \times$$

$$\times\, u(t_1) u^*(t_2)\, e^{i\omega_0(t_1 - t_2)} + \frac{\delta_{ij}}{\sqrt{N_{0i}N_{0j}}}\, \delta(t_1, t_2), \tag{10.3.14}$$

where function $v(t_1, t_2)$ is slow in comparison with $e^{i\omega_0 t}$. Substituting expressions (10.3.7) and (10.3.14) in equations (10.3.10) and replacing under the signs of intervals rapidly oscillating functions by their time-averaged values, we obtain

for $v(t_1, t_2)$ the equation

$$\Delta f_c \int_{t-\Delta t}^{t} |u(t_2)|^2 \left\{ \sum_{i=1}^{n} h_i |U_i(t_2, \alpha)|^2 \right\} \rho(t_1 - t_2) v(t_2, t_3) dt_2 + \\ + \rho(t_1 - t_3) + v(t_1, t_3) = 0, \tag{10.3.15}$$

where Δf_c – effective width of the spectrum of signal fluctuations

$$\Delta f_c = \frac{1}{\int_{-\infty}^{\infty} \rho(t) dt}, \tag{10.3.15'}$$

$h_i = P_{c\,i}/2N_{0i}\Delta f_c$ – ratio of mean signal power at the output of the i-th antenna to the power of the corresponding noise in the band of signal fluctuations.

Equation (10.3.15) must be solved during synthesis of circuits of optimum goniometers. In the case of goniometers with a tracking antenna it can be solved. As we determined, we need to find only the values of function $v(t_1, t_2)$ and its first derivatives with respect to α when $\alpha = 0$. However, for goniometers with a tracking antenna $U_i(t, 0) = 1$ [by virtue of normalization of (10.3.1)]. Substituting $\alpha = 0$ in (10.3.15) we obtain

$$h_\Sigma \Delta f_c \int_{t-\Delta t}^{t} |u(t_2)|^2 \rho(t_1 - t_2) v(t_2, t_3)|_{\alpha=0} dt_2 + \\ + \rho(t_1 - t_3) + v(t_1, t_3)|_{\alpha=0} = 0, \tag{10.3.16}$$

where

$$h_\Sigma = h_1 + h_2 + \ldots + h_n. \tag{10.3.16'}$$

We introduce now the assumption of smallness of the period of repetition of the signal as compared to the time of correlation of its fluctuations, which was proven and widely used in the preceding chapters. With this assumption function $|u(t_2)|^2$ will be rapidly oscillating as compared to other functions under the sign of the integral in (10.3.16), and therefore it can be replaced there by its time-averaged value. Using normalization (10.3.4), instead of (10.3.16) we obtain equation

$$h_\Sigma \Delta f_c \int_{t-\Delta t}^{t} \rho(t_1 - t_2) v(t_2, t_3)|_{\alpha=0} dt_2 + \rho(t_1 - t_3) + v(t_1, t_3)|_{\alpha=0} = 0, \tag{10.3.17}$$

coinciding with equation (4.3.5) (with accuracy of a coefficient). Solution of this equation, as it was shown in Chapter IV, has the form

$$v(t_1, t_3)|_{\alpha=0} = -\frac{1}{2\pi\Delta f_c} \int_{-\infty}^{\infty} \frac{S_0(\omega)}{h_\Sigma S_0(\omega) + 1} e^{i\omega(t_1 - t_3)} d\omega, \tag{10.3.18}$$

where

$$S_0(\omega)=\Delta f_0 \int_{-\infty}^{\infty} \rho(t)\, e^{i\omega t} dt \tag{10.3.19}$$

is the spectrum of fluctuations of the signal, normalized in such a way that $S_0(0) = 1$.

We determine now function $v'(t_1, t_2)|_{\alpha=0}$. Differentiating equation (10.3.15) with respect to α and considering $\alpha = 0$, we obtain

$$\begin{aligned}\Delta f_0 \int_{t-\Delta t}^{t} |u(t_2)|^2 \left\{\sum_{i=1}^{n} h_i \frac{\partial}{\partial\alpha} |U_i(t_2, \alpha)|^2\right\}\Bigg|_{\alpha=0} \rho(t_1-t_2)\times \\ \times v(t_2, t_3)|_{\alpha=0}\, dt_2 + v'(t_1, t_3)|_{\alpha=0} + \\ + h_\Sigma \Delta f_0 \int_{t-\Delta t}^{t} |u(t_2)|^2 \rho(t_1-t_2)\, v'(t_2, t_3)|_{\alpha=0}\, dt_2 = 0.\end{aligned} \tag{10.3.20}$$

Not stopping to discuss methods of solution of this equation, we immediately give the final result

$$\begin{aligned}v'(t_1, t_3)\Big|_{\alpha=0} = \Delta f_0 \int_{t-\Delta t}^{t} |u(t_2)|^2 \left\{\sum_{i=1}^{n} h_i \frac{\partial}{\partial\alpha} |U_i(t_2, \alpha)|^2\right\}\times \\ \times v(t_1, t_2)\, v(t_2\, t_3)\, dt_2\Big|_{\alpha=0}.\end{aligned} \tag{10.3.21}$$

It is easy by simple substitution of this expression in (10.3.20) to prove that this really is its solution.

We introduce subsequently the assumption that $T_r \ll T_{CK}$ (T_{CK} — period of scanning of directional patterns and phase centers), which practically always is realized. Here $|u(t_2)|^2$ under the integral in (10.3.21) changes considerably more rapidly than the remaining functions, and can be replaced by its time-averaged value, equal to 1. The expression for $v'(t_1, t_2)|_{\alpha=0}$ then takes the simple form

$$\begin{aligned}v'(t_1, t_3)\Big|_{\alpha=0} = \Delta f_0 \int_{t-\Delta t}^{t} \left\{\sum_{i=1}^{n} h_i \frac{\partial}{\partial\alpha} |U_i(t_2, \alpha)|^2\right\}\times \\ \times v(t_1, t_2)\, v(t_2, t_3)\, dt_2\Big|_{\alpha=0}.\end{aligned} \tag{10.3.21'}$$

Thus, we have found all the basic characteristics necessary for construction of the operation of an optimum radio channel.

Using (10.3.21'), (10.3.8) and (10.3.11), it is possible to expand expression

(10.3.12) in the following way:

$$\int_{t-\Delta t}^{t} z(\tau)\,d\tau = -\sum_{l,k=1}^{n} \frac{\sqrt{P_{cl}P_{ck}}}{N_{0l}N_{0k}} \Bigg\{ \frac{\partial}{\partial \alpha} \mathrm{Re} \int_{t-\Delta t}^{t} \int_{t-\Delta t}^{t} U_l(t_1, \alpha) \times U^*_k(t_2, \alpha)\, u(t_1)\, u^*(t_2)\, v(t_1, t_2)\, y_l(t_1) \times$$
$$\times y_k(t_2)\, e^{i\omega_0(t_1-t_2)} dt_1 dt_2 \Big|_{\alpha=0} - \mathrm{Re} \int_{t-\Delta t}^{t} \int_{t-\Delta t}^{t} u(t_1)\, u^*(t_2) \times$$
$$\times v'(t_1, t_2)\Big|_{\alpha=0} y_l(t_1)\, y_k(t_2)\, e^{i\omega_0(t_1-t_2)} dt_1 dt_2 \Bigg\}. \qquad (10.3.22)$$

If, as this is done in Chapter IV, we introduce function $h_{1\,\text{опт}}(t)$, related to $v(t_1, t_2)$ by equation

$$\int_{-\infty}^{\infty} h_{1\,\text{опт}}(t_1 - t_2)\, h_{1\,\text{опт}}(t_2 - t_3)\, dt_2 = -h_\Sigma \Delta f_c v(t_1, t_3)\Big|_{\alpha=0}, \qquad (10.3.23)$$

and designate

$$h_{2\,\text{опт}}(t) = \sqrt{h_\Sigma}\, \Delta f_c v(t_1, t_1 + t)\Big|_{\alpha=0} \qquad (10.3.23')$$

[as can be seen from (10.3.18) $v(t_1, t_1 + t)|_{\alpha=0}$ depends only on t], then from (10.3.22) we can obtain an explicit expression for operation of an optimum radio channel:

$$z(\tau) = \frac{4\Delta f_c}{P_{c1}} \Bigg\{ \frac{\partial}{\partial \alpha} \Big| \int_{t-\Delta t}^{t} h_{1\,\text{опт}}(\tau - s) \Big[\sum_{l=1}^{n} \sqrt{\frac{P_{cl}}{P_{cl}}}\, h_l U_l(s, \alpha) \times$$
$$\times u(s)\, y_l(s) \Big] e^{i\omega_0 s} ds \Big|^2 - \sum_{l=1}^{n} h_l \frac{\partial}{\partial \alpha} |U_l(t_1, \alpha)|^2 \times$$
$$\times \Big| \int_{t-\Delta t}^{t} h_{2\,\text{опт}}(\tau - s) \Big[\sum_{l=1}^{n} \sqrt{\frac{P_{cl}}{P_{cl}}}\, h_l u(s)\, y_l(s) \Big] e^{i\omega_0 s} ds \Big|^2 \Bigg\}\Bigg|_{\alpha=0}. \qquad (10.3.24)$$

Expression (10.3.24) shows what operations the optimum radio channel of a radar goniometer should produce on the received signals. The most essential here are operations of optimum linear filtration. Pulse responses of the obtained optimum filters in a definite way are matched with the spectrum of signal fluctuations. The filter with pulse response $h_1(t)$, determined by relationship (10.3.23), already has been repeatedly encountered in preceding chapters. The square of its gain-frequency response has the form

$$|H_{1\,\text{опт}}(i\omega)|^2 = \frac{h_\Sigma S_0(\omega)}{1 + h_\Sigma S_0(\omega)}. \qquad (10.3.25)$$

Phase-frequency response of this filter is arbitrary. If we approximate $S_0(\omega)$ by function

$$S_0(\omega)=\frac{1}{1+\left(\frac{\omega}{2\Delta f_0}\right)^2}, \tag{10.3.26}$$

then for $|H_{1\text{опт}}(i\omega)|^2$ we obtain

$$|H_{1\text{опт}}(i\omega)|^2=\frac{h_\Sigma}{1+h_\Sigma}\,\frac{1}{1+\left(\frac{\omega}{2\Delta f_0\sqrt{1+h_\Sigma}}\right)^2}, \tag{10.3.27}$$

i.e., the form of the square of the gain-frequency response of this filter coincides with the form of the spectrum of signal fluctuations, and the width of its transmission band is larger than the width of the spectrum of fluctuations by a factor of $\sqrt{1+h_\Sigma}$. The physical meaning of this fact was discussed in Chapters IV and VII.

The second filter contained in (10.3.24) has pulse response $h_{2\text{опт}}(t)$ (10.3.23), i.e., frequency response

$$H_{2\text{опт}}(i\omega)=\frac{\sqrt{h_\Sigma}S_0(\omega)}{h_\Sigma S_0(\omega)+1}. \tag{10.3.28}$$

This filter, obviously, is not realizable physically. The question of construction of a physically realizable equivalent of this filter will be considered below in § 10.5.

For construction of a block diagram realizing operation (10.3.24) it is useful to rewrite the latter in real form. For this we introduce intermediate frequency $\omega_{\text{пр}}$, considerably exceeding the width of the frequency spectrum of signals $y_i(t)$ (10.3.3). Then operation (10.3.24) can be reduced to form

$$\begin{aligned}
z(\tau)=\frac{8\Delta f_0}{P_{ci}}\int_{t-\Delta t}^{t} h_{1\text{опт}}(\tau-s)\cos\omega_{\text{пр}}(\tau-s)\sum_{i=1}^{n}\sqrt{\frac{P_{c1}}{P_{ci}}}\times\\
\times\Big[h_i U'_{ai}(s,0)\,y_i(s)\,u_a(s)\cos[\omega_\Gamma s+\psi(s)]+\\
+\Phi_i(s,0)\,y_i(s)\,u_a(s)\sin[\omega_\Gamma s+\psi(s)]\Big]\,ds\Big\{\int_{t-\Delta t}^{t} h_{1\text{опт}}(\tau-s)\times\\
\times\cos\omega_{\text{пр}}(\tau-s)\sum_{i=1}^{n}\sqrt{\frac{P_{c1}}{P_{ci}}}\,h_i y_i(s)\,u_a(s)\cos[\omega_\Gamma s+\psi(s)]\,ds-\\
-\sum_{i=1}^{n}h_i U'_{ai}(s,0)\Big[\int_{t-\Delta t}^{t} h_{2\text{опт}}(\tau-s)\cos\omega_{\text{пр}}(\tau-s)\times\\
\times\sum_{i=1}^{n}\sqrt{\frac{P_{c1}}{P_{ci}}}\,h_i u_a(s)\,y_i(s)\cos[\omega_\Gamma s+\psi(s)]\,ds\Big]^2\Big\},
\end{aligned} \tag{10.3.29}$$

where $\omega_\Gamma=\omega_0-\omega_{\text{пр}}$ heterodyne frequency.

The block diagram of the unit realizing operation (10.3.29) (with an accuracy of immaterial coefficient $8\Delta f_{c}/P_{c\,1}$), is shown in Fig. 10.7. First, in the circuit there occurs normalization of the received signals in order to eliminate effects related to nonidentical gain of directional patterns and unequalness of noises acting in each channel (if this takes place). If the gain of the patterns and noises are identical, the shown normalization is absent.

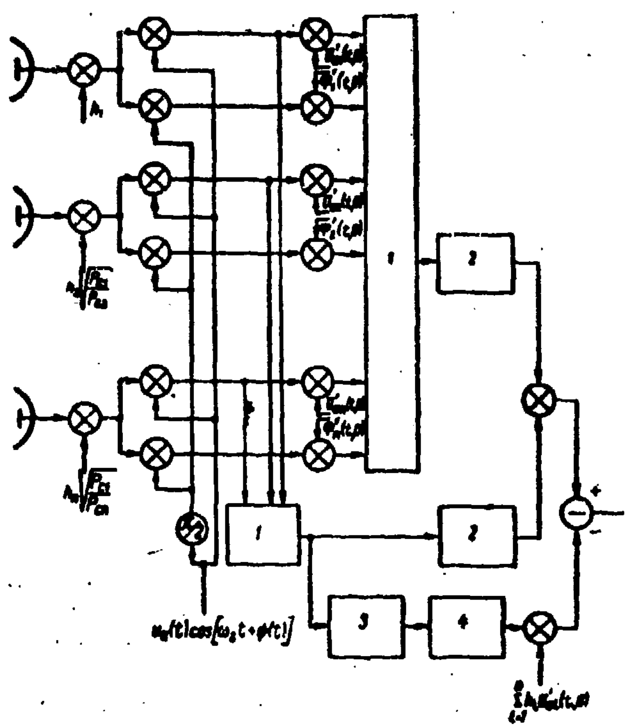

Fig. 10.7. Optimum circuit of the radio channel of a radar goniometer with a generalized method of direction finding: 1) adder; 2) optimum filters with frequency response $H_{1\,\text{опт}}(i\omega)$; 3) optimum filter with frequency response $H_{2\,\text{опт}}(i\omega)$; 4) square-law detector.

After normalization, the signal from the output of each antenna is separated into two channels, conditionally called subsequently the first and second, and in each channel it is heterodyned. Here signals of the heterodyne, directed towards the second channels (see Fig. 10.7), are shifted with respect to signals directed towards the first channels, by $\pi/2$. Heterodyne signals should coincide in form with the emitted radar signal.

After heterodyning, the signals in the first channels are multiplied by derivatives with respect to the measured angle in the equisignal direction of moduli of complex gains of corresponding directional patterns, and signals in the second channels are multiplied by the derivatives of the arguments of complex gains of these patterns. After that all signals are added, filtered by an optimum filter with characteristic (10.3.25) and are fed to one input of the multiplier (phase detector). As a reference signal for this phase detector there is used the sum of signals of the first channels (see Fig. 10.7), passed through precisely the same optimum filter. This same sum, furthermore, proceeds to an optimum filter with characteristic (10.3.27), is detected by a square-law detector and is multiplied by the sum of derivatives in the equisignal direction of moduli of complex gains of all directional patterns, (taken with weights equal to the signal-to-noise ratio in the corresponding channels). The obtained signal is subtracted from the output signal of the phase

detector.

Thus, all operations produced by the optimum circuit are sufficiently simple and are encountered in one form or another in radar goniometer technology. However, the sequence of operations and responses of the filters are new.

The physical meaning of transformations of the signal in the optimum circuit we shall discuss later, in examining concrete methods of direction finding.

Let us turn to the question of potential accuracy of measurement of angles in goniometers of the considered type. As it was shown in Chapter VI, the equivalent spectral density of the optimum radio channel is equal to

$$S_{\text{опт}} = \frac{\Delta t}{\left[\frac{\partial \ln L(\alpha)}{\partial \alpha}\right]^2\Big|_{\alpha=0}}, \tag{10.3.30}$$

where likelihood function $L(\alpha)$ is constructed as before in an interval of duration Δt, small as compared to the time of correlation of angular shifts, but sufficiently great as compared to the time of correlation of fluctuations of the reflected signal [in this case expression (10.3.30), as it is possible to show, does not depend on Δt].

Calculation by the formula (10.3.30) using expressions (10.3.8) and (10.3.11) gives

$$S_{\text{опт}} = \Delta t\left[-\frac{1}{2}\int_{t-\Delta t}^{t}\int_{t-\Delta t}^{t}\sum_{i,j=1}^{n} W'_{ij}(t_1, t_2)\times R'_{ij}(t_1, t_2, \alpha)\,dt_1 dt_2\right]^{-1}\Big|_{\alpha=0}. \tag{10.3.31}$$

This formula is very important subsequently.

Using expressions (10.3.7), (10.3.8) and (10.3.14), it is possible to reduce (0.3.31) to the following form:

$$\begin{aligned} S_{\text{опт}} = &\Big\{\frac{2\Delta f_c}{\Delta t}\int_{t-\Delta t}^{t}[1+v(t,t)|_{\alpha=0}]\,dt\times \\ &\times\frac{1}{\Delta t}\int_{t-\Delta t}^{t}\sum_{i=1}^{n} h_i|U'_i(t,0)|^2\,dt - \frac{2\Delta f_c^2}{\Delta t}\int_{t-\Delta t}^{t}\int_{t-\Delta t}^{t}\rho(t_1-t_2)\times \\ &\times v(t_1,t_2)|_{\alpha=0}\Big[\operatorname{Re}\sum_{i=1}^{n} h_i U'_i(t_1,0)\sum_{i=1}^{n} h_i U^{*\prime}_i(t_2,0)\Big]dt_1 dt_2 + \\ &+\frac{4\Delta f_c^2}{\Delta t}\int_{t-\Delta t}^{t}\int_{t-\Delta t}^{t}[\rho(t_1-t_2)+v(t_1,t_2)|_{\alpha=0}]\,v(t_1,t_2)|_{\alpha=0}\times \\ &\times\Big[\sum_{i=1}^{n} h_i|U_i(t,0)|^2\sum_{i=1}^{n} h_i|U_i(t_2,0)|^2\Big]dt_1 dt_2\Big\}^{-1}. \end{aligned} \tag{10.3.32}$$

Expression (10.3.32) is very cumbersome, since it contains uncalculable integrals. Calcula' integrals in this formula in such a general form, without specifying func... $U_1(t, \alpha)$, i.e., the method of direction finding, is difficult. These calculations, and also investigation of potential accuracy we shall postpone until our consideration of concrete methods of direction finding. Formula (10.3.32) will be the starting point for these considerations.

Thus, we succeeded in a very general form in synthesizing the circuit of an optimum radio channel of a radar goniometer with a tracking antenna. The results obtained here are basis for a systematic study of this class of goniometers from the point of view of their potential and real properties, which will be conducted in subsequent paragraphs.

§ 10.4. Method of Scanning of the Directional Pattern (Optimum Circuits)

The method of scanning of the directional pattern has been used in radar for a long time. Experience in the use of this method testifies to its low noise immunity to active interferences which are modulated by frequencies close to frequencies of scanning, and to decrease of accuracy of measurement of angles by this method in the presence of amplitude fluctuations of the reflected signal, the spectrum of which contains as components the frequencies of scanning. In particular, in the presence of such fluctuations of the signal in circuits of the method of scanning the pattern utilized in practice even with complete absence of noises there exists a final fluctuating error of target tracking. These factors have a simple physical explanation: when using the method of pattern scanning the useful information about the angular coordinates of the target is included in the amplitude of the received signals, and any distortion of their amplitude leads directly to distortion of the useful information and, ultimately, to error of measurement. In this connection study of optimum circuits of the method of pattern scanning which have, obviously, means of optimum suppression of the harmful influence of amplitude fluctuations of reflected signal is of great theoretical and practical interest.

10.4.1. Synthesis of Optimum Circuits for the Method of Scanning the Directional Pattern

The optimum circuit for the method of pattern scanning is easily obtained from the general circuit of Fig. 10.7. For this method, obviously, n = 1 (one antenna), $F(t, \alpha) = 0$ (the phase center of the antenna is motionless), and $U_a(t, \alpha)$ is expressed by formula (10.2.2). Using (10.3.29), we obtain the optimum circuit for measurement

of angle α in the form depicted in Fig. 10.8.

The circuit of Fig. 10.8 contains three channels to which the signal proceeds after heterodyning. In the first channel there first is produced multiplication of the signal by a function varying according to the law of its amplitude modulation, introduced by the scanning pattern, and then optimum filtration (10.3.25). The output signal of this channel enters the phase detector. Useful information about angular coordinates of the target is contained in the side spectral components of the received signal, and in the absence of the shown multiplication we would be forced to further insert a sufficiently broad-banded filter, which would not cut off the side spectral components of the signal. This would lead to unnecessary increase of noises at the filter output. The operation applied in the optimum circuit produces transfer of the side spectral components of the signal to the center frequency, after which there can be carried out filtration of the signal by a sufficiently narrow-band filter, passing the signal and noise in the optimum relationship.

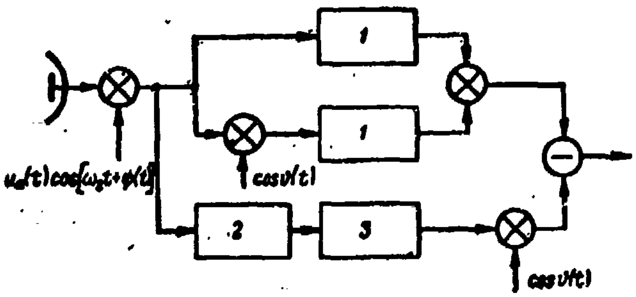

Fig. 10.8. Optimum circuit of the radio channel of a goniometer with a scanning pattern: 1 — optimum filters with frequency response $H_{1\,\text{опт}}(i\omega)$; 2 — optimum filter with frequency response $H_{2\,\text{опт}}(i\omega)$; 3 — square-law detector.

As the reference signal to the phase detector there proceed center spectral components of the received signal, separated by an optimum filter in the second channel. The filter here is the same as in the first channel.

At the output of the phase detector there will be formed, obviously, the basic component of the signal of error. The third channel, as it is easy to see, is intended exactly for compensation of the harmful influence of amplitude fluctuations of the reflected signal. At first the signal in this channel is filtered by an optimum filter with response (10.3.28), and then it is detected by a square-law detector, at the output of which there are obtained in pure form amplitude fluctuations of the signal. Multiplying them by a function which varies according to the law of pattern scanning, we separate the information about angular coordinates contained in the amplitude which is false. The false information is then extracted from the total information separated by the first two channels. Thus there occurs optimum compensation of the harmful influence of amplitude fluctuations of the signal.

Having considered in general form the work of the optimum circuit, let us turn to investigation of certain limiting cases of practical interest. The first of these cases is the case of a high frequency of scanning, considerably exceeding the width of the spectrum of signal fluctuations. Here, as it is easy to see, the third channel in the circuit of Fig. 10.8 disappears. The circuit of the optimum radio channel takes the form depicted in Fig. 10.9. The physically obtained simplification is understandable, since with high frequencies of scanning amplitude fluctuations of the signal which are low-frequency cease to affect accuracy of direction finding. The circuit of Fig. 10.9 is rather simple and is very attractive, but in the practice of radar goniometry it has not been encountered. More detailed study of this circuit is of interest and will be made in the following section.

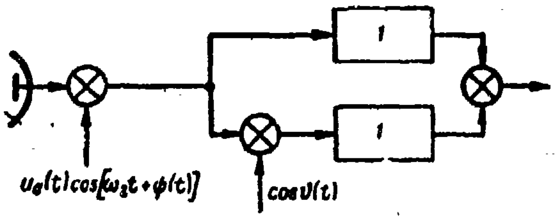

Fig. 10.9. Optimum circuit of the radio channel of a goniometer with a scanning pattern with high frequencies of scanning. 1 — optimum filters with frequency response $H_{1\,\text{опт}}(i\omega)$.

Let us turn now to another extreme case, when the frequency of scanning is considerably smaller than the transmission band of the filter (10.3.25).

Here multiplication by cos $\vartheta(t)$ in the first channel can be transferred to the output of the filter. The circuit of the optimum radio channel takes the form depicted in Fig. 10.10, where the filter has the square of the gain-frequency response

$$|H_{3\,\text{опт}}(i\omega)|^2 = \frac{4hS_0(\omega)}{[1+hS_0(\omega)]^2}. \qquad (10.4.1)$$

We already met a filter of such form in Chapter VII. As was shown there $|H_{3\,\text{опт}}(i\omega)|^2$ is a double-humped curve with maxima at frequencies ω_1 and ω_2, for which $hS_0(\omega) = 1$, and a dip to $4h/(1 + h)^2$ at zero frequency. Thanks to this form of the gain-frequency response of the filter, we in the very best manner suppress center components of the signal, emphasizing to the greatest degree its side components, bearing the useful information. Here we achieve the maximum possible decrease of the harmful influence of amplitude fluctuations of the signal on accuracy of direction finding. When $h < 1$ the response of the filter becomes single-humped. Physically this is explained by the fact that for high noise levels it becomes more profitable to cut off natural noises more strongly than to worry about decrease of the influence of signal fluctuations.

We note that the circuit of the method of pattern scanning usually applied in

practice coincides in structure with the circuit of Fig. 10.10. The results above show in what cases this circuit turns out to be optimum, i.e., realizes the potentialities of the method of pattern scanning. More detailed study of this circuit and comparison of it with the circuit of Fig. 10.9 will be given in the next section.

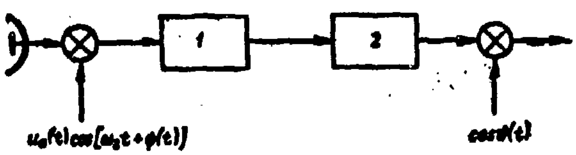

Fig. 10.10. Optimum circuit of the radio channel of a goniometer with a scanning pattern with low frequencies of scanning: 1 — optimum filter with frequency response $H_{3\ \text{опт}}(i\omega)$; 2 — square-law detector.

In conclusion let us note that optimum circuits for measurement of the angle in another plane will be precisely the same, only everywhere multiplication by $\cos \vartheta(t)$ will be replaced by multiplication by $\sin \vartheta(t)$.

10.4.2. Equivalent Spectral Density of the Optimum Circuit of the Method of Scanning the Directional Pattern

Let us turn to the question of the potential accuracy of the method of pattern scanning.

Considering n = 1 in (10.3.31), substituting there expression (10.2.2) and producing calculation of integrals, we obtain

$$S_{\text{опт}} = \frac{\pi}{\mu_a^2 h^2}\left\{c^2 \int_{-\infty}^{\infty} \frac{S_0(\omega)^2}{hS_0(\omega)+1}\, d\omega + \right.$$

$$\left. + \sum_{k=-\infty}^{\infty} |c_k|^2 \int_{-\infty}^{\infty} \frac{1-hS_0(\omega)}{1+hS_0(\omega)} \cdot \frac{S_0(\omega)\, S_0(\omega+k\Omega)}{1+hS_0(\omega+k\Omega)}\, d\omega \right\}^{-1}, \tag{10.4.2}$$

where

$$c^2 = \frac{1}{T_{\text{ск}}} \int_0^{T_{\text{ск}}} [\cos \vartheta(t)]^2 dt, \quad c_k = \frac{1}{T_{\text{ск}}} \int_0^{T_{\text{ск}}} \cos \vartheta(t)\, e^{ik\Omega t} dt; \tag{10.4.3}$$

$T_{\text{ск}}$ — period of scanning;

$\Omega = 2\pi/T_{\text{ск}}$ — angular frequency of scanning.

Let us note now that $S_{\text{опт}}$ for measurement of the angle in the other plane will have the same form as (10.4.2), only coefficients c^2 and c_k, should be replaced by s^2 and s_k, respectively:

$$s^2 = \frac{1}{T_{\text{ск}}} \int_0^{T_{\text{ск}}} [\sin \vartheta(t)]^2\, dt, \quad s_k = \frac{1}{T_{\text{ск}}} \int_0^{T_{\text{ск}}} \sin \vartheta(t)\, e^{ik\Omega t} dt. \tag{10.4.4}$$

The complicated form of formula (10.4.2) does not permit us to see directly from it the law governing change of $S_{\text{опт}}$ depending upon h and other quantities. Therefore we will consider the most interesting limiting cases, as we did in examining the optimum circuit.

In the case of a rather high frequency of scanning, when $S_0(\omega + \Omega) \approx 0$, the second component in parentheses in (10.4.2) vanishes and we obtain

$$S_{\text{опт}} = \frac{\pi}{\mu_a^2 h^2}\left\{c^2 \int_{-\infty}^{\infty} \frac{S_0(\omega)^2}{1+hS_0(\omega)}\,d\omega + c_0^2 \int_{-\infty}^{\infty} \frac{[1-hS_0(\omega)]\,S_0(\omega)^2}{[1+hS_0(\omega)]^2}\,d\omega\right\}^{-1}. \tag{10.4.5}$$

In the usually occurring range of variation of h the second integral in (10.4.5) is negative [in approximation (10.3.26) this occurs when $h > 1$]. Then $S_{\text{опт}}$ will be the smallest if

$$c_0 = \frac{1}{T_{\text{ск}}} \int_0^{T_{\text{ск}}} \cos\vartheta(t)\,dt = 0, \tag{10.4.6}$$

and $S_{\text{опт}}$ is equal to

$$S_{\text{опт}} = \frac{\pi}{\mu_a^2 h^2 c^2} \frac{1}{\int_{-\infty}^{\infty} \frac{S_0(\omega)^2}{1+hS_0(\omega)}\,d\omega}. \tag{10.4.7}$$

Accuracy in this case, obviously, does not depend on the form of the law of scanning and is determined only by c^2. If it was problem of measurement of only one angular coordinate, then the best scanning would be jumps of the directional pattern to two extreme positions in the plane of this angle. Here $\cos\vartheta(t) = \pm 1$, and $c^2 = 1$. However, such scanning is unsuitable if it is necessary to measure angular coordinates of a target in two planes. Roughly speaking c^2 determines that part of the total received power which goes to measurement of the angle in one plane. Coefficient s^2 (10.4.4) has analogous meaning (obviously, $c^2 + s^2 = 1$). In this connection the relationship of c^2 and s^2 should be selected based on the required relationship of accuracies of measurement of angles in the two planes. In particular, if these accuracies must be identical, which normally is the case then we naturally consider $c^2 = s^2 = 1/2$. This we shall assume subsequently.

Let us now turn to the limiting case of low frequencies of scanning. If the frequencies of scanning are so low that for those values of k for which $c_k \approx 0$, we still have $S_0(\omega + k\Omega) \approx S_0(\omega)$, then instead of (10.4.2) we obtain (considering, as agreed, $c^2 = 1/2$)

$$S_{\text{опт}} = \frac{\pi}{\mu_a^2 h^2} \frac{1}{\int_{-\infty}^{\infty} \left[\frac{S_0(\omega)}{1+hS_0(\omega)}\right]^2 d\omega}. \tag{10.4.8}$$

For comparison of accuracies at high and low frequencies of scanning let us note that during transition from low frequencies of scanning to high quantity $2\pi/\mu_a^2 h^2 S_{\text{опт}}$ changes by

$$\int_{-\infty}^{\infty} \frac{S_0(\omega)^2\,[1-hS_0(\omega)]}{[1+hS_0(\omega)]^2}\,d\omega. \tag{10.4.9}$$

This integral coincides with the second integral in (10.4.5). As we already established, for sufficiently large values of h [in approximation (10.3.26), when $h > 1$] this integral is positive, i.e., accuracy of high-frequency scanning is higher than accuracy of low-frequency scanning. For small h everything will be the opposite. This fact is curious. It is explained by the fact that for small h the basic influence on accuracy of direction finding is rendered by natural noises. At low frequencies of scanning the total transmission band of the filters must be smaller, which leads to decrease of natural noises.

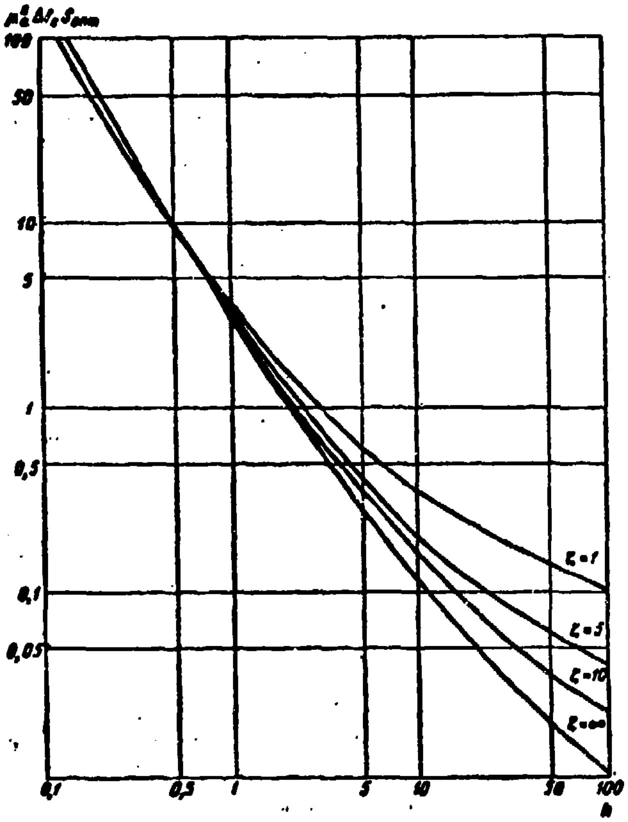

Fig. 10.11. Dependence of $S_{опт}$ on h for the method with a scanning pattern for different frequencies of scanning.

Let us now perform calculation by formula (10.4.2) with approximation of the spectrum (10.3.26). For simplicity we shall consider the case of conical scanning of the directional pattern, i.e., $c_{\pm 1} = 1/2$, and at $i \neq \pm 1$, $c_i = 0$. Performing the necessary calculations, we obtain

$$S_{опт} = \frac{1}{\mu_a^2 \Delta f_c} \cdot \frac{\sqrt{1+h}\,(1+\sqrt{1+h})}{h^2} \times \frac{\zeta^6 + 2(3h+4)\zeta^4 + (3h+4)^2\zeta^2 + 4h^2(h+1)}{\zeta^6 + (5h+10+2\sqrt{1+h})\zeta^4 + 4(h^2+4h+6+2\sqrt{1+h})\zeta^2 + 4h(1+h\sqrt{1+h})}, \quad (10.4.10)$$

where $\zeta = \Omega/2\Delta f_c$.

The graph of the dependence of $S_{опт}$ on h, calculated by formula (10.4.10), is shown in Fig. 10.11. All laws governing change of $S_{опт}$ are easily perceived from this graph: with growth of h, $S_{опт}$ rather sharply drops; for values of h normal in measuring systems (when $h > 1$) with decrease of the scan frequency $S_{опт}$ increases.

We note the very simple formulas for $S_{опт}$ ensuing from (10.4.10) for large and small values of the signal-to-noise ratio for high and low frequencies of scanning:

$$S_{опт} = \frac{1}{\mu_a^2 \Delta f_c h} \quad h \gg 1,\ \zeta \gg 1; \quad S_{опт} = \frac{2}{\mu_a^2 \Delta f_c h^2} \quad h \ll 1,\ \zeta \gg 1;$$

$$S_{опт} = \frac{1}{\mu_a^2 \Delta f_c \sqrt{h}} \quad h \gg 1,\ \zeta \ll 1; \quad S_{опт} = \frac{1}{\mu_a^2 \Delta f_c h} \quad h \ll 1,\ \zeta \ll 1. \quad (10.4.11)$$

The simplicity of these formulas permits us to recommend them as basic during appraisals of potential accuracy of the method of pattern scanning. From them it is clear that with low frequencies of scanning, the equivalent spectral density with growth of the signal-to-noise ratio increases considerably slower than with high frequencies of scanning. However, even with low frequencies of scanning as $h \rightarrow \infty$ quantity $S_{\text{ОПТ}} \rightarrow 0$. This contradicts the established opinion that with low frequencies of scanning even without noises accuracy will be limited thanks to the influence of amplitude fluctuations of the signal. Such an opinion developed only as a result of operations of nonoptimal circuits. With optimum processing of signals, as can be seen from what has been said, we succeed in sufficiently completely compensating the harmful influence of amplitude fluctuations of the signal. Let us note that formulas (10.4.11), although calculated for a rather particular approximation of the spectrum of signal fluctuations, give a sufficiently good approximation for $S_{\text{ОПТ}}$ in more general cases.

§ 10.5. Investigation of Synthesized Circuits for the Method of Scanning of Directional Patterns

10.5.1. Introductory Remarks

In the present section we shall pursue a study of a group of questions connected with practical realization of circuits, close to optimum. As a result of these investigations we should find how well it is possible to realize the potential accuracy of the method of pattern scanning. Furthermore, it is necessary to obtain sufficiently simple formulas for calculation of accuracy of real circuits.

First of all it is necessary to find what features distinguish real circuits from theoretical. Circuits synthesized in the preceding section in the form in which they are depicted in Figs. 10.8, 10.9 and 10.10 are, obviously, very simplified. They reflect only the basic, fundamental operations produced on the signal. In the practical realization these circuits will require additional technical components. The basic ones of these components, of course, should be considered during analysis of the real accuracy of the considered circuits. In itself appraisal of the complexity of technical construction of different circuits is also one of the important characteristics of real circuits.

In Fig. 10.11 as an example there is depicted a circuit of the type of Fig. 10.10 taking into account basic technical components, inevitable during construction of this circuit. The signal in this circuit first enters a mixer, carrying out identical transfer of signal to the intermediate frequency. This mixer is the basic

source of natural noises added to the signal. Then there follows an UPCh [i-f amplifier] whose passband is usually very great (considerably exceed the width of the spectrum of signal fluctuations). After the UPCh the signal enters a mixer with phase modulation. Here the signal is transferred to the second intermediate frequency with inversion of phase modulation. Then there is an amplitude modulator, carrying out multiplication of the signal by a function varying according to the law of amplitude modulation introduced by the antenna. The frequency of the signal of the heterodyne with phase modulation is continuously tuned by the system of speed measurement, and its phase modulation, as well as the modulating signal from the amplitude modulator, is tuned by the system of range measurement.

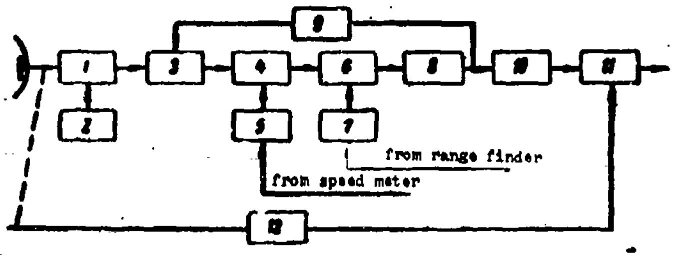

Fig. 10.12. Practical circuit of the radio channel of a goniometer with a scanning pattern: 1 — mixer of r-f amplifier; 2 — heterodyne oscillator; 3 — UPCh; 4 — mixer with phase modulation; 5 — controlled heterodyne with phase modulation; 6 — amplitude modulator; 7 — controlled generator of amplitude modulation; 8 — narrow-band filter; 9 — system of automatic gain control; 10 — amplitude detector; 11 — phase detector; 12 — GON [reference voltage generator].

After these operations the signal enters a narrow-band filter whose frequency response is matched as far as possible with the spectrum of fluctuations of the signal (usually, of course, matching concerns only the transmission bandwidth of the filter). In view of the limitedness of the range of linearity of amplifiers in practical circuits there always are used AGC systems, maintaining the necessary signal level in the circuit. From the output of the narrow-band filter the signal enters an amplitude detector and then a phase detector. Reference voltage proceeds to the phase detector from a generator of reference voltages (GON); the same voltage controls scanning of the directional pattern.

Analogous variants of circuits of Figs. 10.8 and 10.9 complicated by basic technical components we shall not give, since they are very similar to the circuit considered just now, from the example of which we clearly see the basic features of such complication.

Thus, practical circuits rather greatly differ from their theoretical prototype. Now it is easy to establish in what components of practical circuits difference of the operations produced on signals from idealized optimum operations is possible. First of all, due to the limited accuracy of measurement of speed there will be

produced insufficiently exact tuning of the frequency of the heterodyne with phase modulation, and after transformation the signal will have a frequency differing somewhat from that to which the narrow-band filter is tuned. Due to the limited accuracy of measurement of range the phase modulation of the heterodyne signal will lag somewhat relative to the phase modulation of the received signal (or outstrip it). The same pertains to the signal of the amplitude modulator.

In the case of pulse radiation to avoid losses of the useful signal pulses of the amplitude modulator (strobes) are made, as a rule, even several times longer than pulses of the received signal. Finally, the form of the spectrum of fluctuations of the reflected signal, and also the signal-to-noise ratio in the circuit, necessary for calculation of the frequency response of the narrow-band filter, are known by us very approximately. This will lead to uncontrollable divergence of the frequency response of the narrow-band filter from the calculated, which may be considerable.

Thus, to exactly realize an optimum circuit in practice is impossible. It is possible to exactly maintain the structure of the circuit, i.e., the sequence of basic operations produced on the received signal; however parameters of the circuit will differ to greater or lesser extent from their optimum values.

We shall make several remarks about comparative appraisal of circuits of Figs. 10.8, 10.9 and 10.10 from the point of view of complexity of their technical construction. It is clear that the circuit of Fig. 10.8 is the most complicated of them, and we shall subsequently allot to it the minimum attention.

Circuits of Figs. 10.9 and 10.10 can be considered identical in complexity; in any case, the difference between them is insignificant. The fact is that the most complicated components of these circuits (UPCh's, mixers with inversion of phase modulation, amplitude modulators, phase detectors) are contained in both circuits in identical number.

Subsequently the circuit of Fig. 10.8 (but with parameters, possibly differing from their optimum values) we shall call a quasi-optimum circuit; the circuit of Fig. 10.9 — a circuit with narrow-band filters; and the circuit of Fig. 10.10 — a circuit with a broad-band filter. These names correctly reflect the essence of the matter and will be used as basic terms.

10.5.2. Investigation of the Quasi-Optimum Circuit and the Circuit With Narrow-Band Filters

We start our investigation of circuits with the circuit of Fig. 10.8. Let us assume that the heterodyne signal has the form $v_a(t) \cos [\omega_\Gamma + \varphi(t)]$, where $v_a(t)$

differs, in general, from the amplitude modulation of the received signal $u_a(t)$, and $\varphi(t)$ differs from its phase modulation $\psi(t)$. Thus, the received signal is multipli... by

$$v_a(t)\cos[\omega_r t+\varphi(t)]=\operatorname{Re} v(t)e^{i\omega_r t},$$
$$v(t)=v_a(t)e^{i\varphi(t)}. \tag{10.5.1}$$

The pulse response of filters in the first two channels we designate $h_1(t)\cos[\omega_{\text{пр}}t+\theta_1(t)]$. These filters we shall consider identical, which substantially will simplify further calculations. The pulse response of the filter in the third channel we shall designate by $h_2(t)\cos[\omega_{\text{пр}}t+\theta_2(t)]$. We set

$$\dot h_i(t)=h_i(t)e^{i\theta_i(t)},\quad H_i(i\omega)=\int_0^\infty \dot h_i(t)e^{i\omega t}dt,\quad i=1,2, \tag{10.5.2}$$

i.e., $H_i(i\omega)$ – frequency response of the low-frequency equivalents of the considered filters.

For definitiveness during the analysis of circuits we shall limit ourselves only to consideration of uniform conical scanning of the directional pattern. Modulation of the received signal $U_a(t, \alpha)$ is expressed in this case by formula (10.2.2). Obviously, the basic laws obtained here are qualitatively preserved with other laws of scanning.

The signal at the output of the circuit of Fig. 10.8 with these assumptions can be recorded in the following rather bulky form:

$$\begin{aligned} z(t)=&\int_{-\infty}^{t}\operatorname{Re}\dot h_1(t-\tau)e^{i\omega_{\text{пр}}(t-\tau)}y(\tau)\operatorname{Re}v(\tau)e^{i\omega_r\tau}\cos\Omega\tau d\tau\times\\ &\times\int_{-\infty}^{t}\operatorname{Re}\dot h_1(t-\tau)e^{i\omega_{\text{пр}}(t-\tau)}y(\tau)\operatorname{Re}v(\tau)e^{i\omega_r\tau}d\tau-\\ &-\left[\int_{-\infty}^{t}\operatorname{Re}\dot h_2(t-\tau)e^{i\omega_{\text{пр}}(t-\tau)}y(\tau)\operatorname{Re}v(\tau)e^{i\omega_r\tau}d\tau\right]^2\cos\Omega t. \end{aligned} \tag{10.5.3}$$

Producing in reverse order the reasonings by which we performed transition from expression (10.3.24) to (10.3.29), it is possible to reduce (10.5.3) to the form most convenient for further calculations, namely:

$$\begin{aligned} z(t)=\operatorname{Re}&\int_{-\infty}^{t}\dot h^*_1(t-\tau)v(\tau)y(\tau)\cos\Omega\tau e^{i\omega_0\tau}d\tau\times\int_{-\infty}^{t}\dot h_1(t-\tau)v^*(\tau)y(\tau)e^{-i\omega_0\tau}d\tau-\\ &-\left|\int_{-\infty}^{t}\dot h^*_2(t-\tau)v(\tau)y(\tau)e^{i\omega_0\tau}d\tau\right|^2\cos\Omega t. \end{aligned} \tag{10.5.4}$$

The first stage of calculation of accuracy of the circuit is calculation of the slope of the discrimination characteristic (or transmission factor of the radio channel)

$$K_Д = \frac{d}{d\alpha} \lim_{T\to\infty} \frac{1}{T} \int_0^T \overline{z(t)}\, dt \Big|_{\alpha=0} \tag{10.5.5}$$

and of systematic error

$$\Delta = \frac{1}{K_Д} \lim_{T\to\infty} \frac{1}{T} \int_0^T z(t)\, dt \Big|_{\alpha=0}. \tag{10.5.6}$$

We shall calculate these quantities in detail, since the calculations produced here are typical for calculation of goniometer systems. Multiplying in (10.5.4) the integrals, averaging under the sign of integrals the rapidly oscillating terms, and producing also statistical averaging , we obtain

$$\begin{aligned}\overline{z(t)} = &\int_{-\infty}^{t}\int_{-\infty}^{t} P_c U_a(t_1, \alpha) U_a(t_2, \alpha) \cos \Omega t_1 u^*(t_1) u(t_2) \times \\ &\times \rho(t_1 - t_2) \dot{h}_1^*(t - t_1) \dot{h}_1(t - t_2) v(t_1) v^*(t_2)\, dt_1 dt_2 + \\ &+ N_0 \int_{-\infty}^{t} |\dot{h}_1(t - \tau)|^2 |v(\tau)|^2 \cos \Omega\tau d\tau - \\ &- \Big[\int_{-\infty}^{t}\int_{-\infty}^{t} P_c U_a(t_1, \alpha) U_a(t_2, \alpha) u^*(t_1) u(t_2) \rho(t_1 - t_2) \times \\ &\times \dot{h}_2^*(t - t_1) \dot{h}_2(t - t_2) v(t_1) v^*(t_2)\, dt_1 dt_2 - \\ &- N_0 \int_{-\infty}^{t} |\dot{h}_2(t - \tau)|^2 |v(\tau)|^2 d\tau\Big] \cos \Omega t.\end{aligned} \tag{10.5.7}$$

Then we allow for the circumstance that functions u(t) and v(t) vary considerably more rapidly than the remaining functions under the signs of integrals in (10.5.7). Averaging them, we have

$$\begin{aligned}\overline{z(t)} = &\varkappa_1 \int_{-\infty}^{t}\int_{-\infty}^{t} P_c U_a(t_1, \alpha) U_a(t_2, \alpha) \cos \Omega t_1 \rho(t_1 - t_2) \times \\ &\times \dot{h}_1^*(t - t_1) \dot{h}_1(t - t_2)\, dt_1 dt_2 + \varkappa_2 N_0 \int_{-\infty}^{t} |\dot{h}_1(t-\tau)|^2 \cos \Omega t d\tau - \\ &- \cos \Omega t \Big[\varkappa_1 \int_{-\infty}^{t}\int_{-\infty}^{t} P_c U_a(t_1, \alpha) U_a(t_2, \alpha) \rho(t_1 - t_2) \times \\ &\times \dot{h}_2^*(t - t_1) \dot{h}_2(t - t_2)\, dt_1 dt_2 - N_0 \varkappa_2 \int_{-\infty}^{t} |\dot{h}_2(t - \tau)|^2 d\tau\Big],\end{aligned} \tag{10.5.8}$$

where

$$\varkappa_1 = \left|\frac{1}{T_r}\int_0^{T_r} u(t) v^*(t)\, dt\right|^2, \quad \varkappa_2 = \frac{1}{T_r}\int_0^{T_r} |v(t)|^2 dt. \tag{10.5.9}$$

Finding the derivative of (10.5.8) with respect to α for $\alpha = 0$ and averaging it in time, we find

$$K_Д=\frac{P_c\mu_a\varkappa_1}{2}\left[\int_0^\infty\int_0^\infty \dot{h}_1^*(t_1)\dot{h}_1(t_2)\rho(t_1-t_2)(1+\cos\Omega(t_1-t_2))\times\right.$$
$$\left.\times dt_1dt_2-2\,\mathrm{Re}\int_0^\infty\int_0^\infty \dot{h}_2^*(t_1)\dot{h}_2(t_2)\rho(t_1-t_2)\cos\Omega t_1 dt_1 dt_2\right]=$$
$$=\frac{P_c\mu_a\varkappa_1}{2\pi\Delta f_c}\int_{-\infty}^{\infty}[|H_1(i\omega)|^2+|H_1(i\omega+i\Omega)|^2-$$
$$-2\,\mathrm{Re}\,H_2(i\omega)H_2^*(i\omega+i\Omega)]S_0(\omega)\,d\omega. \quad (10.5.10)$$

Considering $\alpha = 0$ directly in expression (10.5.8), we obtain $\Delta = 0$, i.e., systematic error is absent.

Now we shall study calculation of the basic accuracy characteristic of a circuit — the equivalent spectral density. This quantity is defined, as it is known, in the following way:

$$S_{экв}=\frac{1}{K_Д^2}\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\int_{-\infty}^{\infty}\overline{z(t)z(\tau)}\,d\tau\Big|_{\alpha=0}. \quad (10.5.11)$$

Omitting the calculations, which are very similar to those just now made, we obtain a result in the following form:

$$S_{экв}=\frac{2\pi}{\mu_a^2h'^2}\int_{-\infty}^{\infty}[|H_1(i\omega)|^4+|H_1(i\omega)|^2|H_1(i\omega+i\Omega)|^2+$$
$$+2|H_2(i\omega)|^2|H_2(i\omega+i\Omega)|^2-4H_1(i\omega+i\Omega)\times$$
$$\times\mathrm{Re}\,H_2(i\omega)H_2^*(i\omega+i\Omega)][1+h'S_0(\omega+\Omega)]\times$$
$$\times[1+h'S_0(\omega)]\,d\omega\left\{\int_{-\infty}^{\infty}[|H_1(i\omega)|^2+|H_1(i\omega+i\Omega)|^2-\right.$$
$$\left.-2\,\mathrm{Re}\,H_2(i\omega)H_2^*(i\omega+i\Omega)]S_0(\omega)\,d\omega\right\}^{-2}, \quad (10.5.12)$$

where

$$h'=h\frac{\left|\frac{1}{T_r}\int_0^{T_r}u(t)v^*(t)\,dt\right|^2}{\frac{1}{T_r}\int_0^{T_r}|v(t)|^2\,dt}=h\frac{\varkappa_1}{\varkappa_2}. \quad (10.5.13)$$

Expression (10.5.12) is rather bulky. However, from it we can draw certain general conclusions with respect to the influence on $S_{экв}$ of various factors. In particular, from (10.5.12) it is clear that imperfectnesses of heterodyning lead to equivalent change of the signal-to-noise ratio according to the formula (10.5.13). It is easy to see that

$$\frac{\left|\frac{1}{T_r}\int\limits_0^{T_r} u(t)\,v^*(t)\,dt\right|^2}{\frac{1}{T_r}\int\limits_0^{T_r} |v(t)|^2 dt} \leqslant 1,$$

where the sign of equality is attained only in case v(t) = u(t). For instance, when $u_a(t)$ are pulses of duration $\tau_{\text{и}}$, $v_a(t)$ — pulses of duration $\tau_{\text{с}}$, and there is no phase modulation of the signal (or it is exactly inverted), we obtain:

$$\frac{\left|\frac{1}{T_r}\int\limits_0^{T_r} u(t)\,v^*(t)\,dt\right|^2}{\frac{1}{T_r}\int\limits_0^{T_r} |v(t)|^2 dt} = \min\left(\frac{\tau_{\text{и}}}{\tau_{\text{с}}}, \frac{\tau_{\text{с}}}{\tau_{\text{и}}}\right). \tag{10.5.14}$$

Thus, any imperfectness of heterodyning leads to an effect, equivalent to increase of natural noises of the receiver.

This phenomenon has a very general character. In the preceding chapters it was shown that it occurs, for instance, in range and frequency metering radar circuits. Imperfectness of heterodyning in all cases led to equivalent decrease of the signal-to-noise ratio, where this decrease is expressed by a single formula (10.5.13). It is obvious that this phenomenon will take place in all circuits of radar goniometers. Therefore, subsequently, during investigation of goniometer circuits we for simplicity shall consider heterodyning ideal. Imperfectness of heterodyning if desired can easily be accounted for by means of replacement of signal-to-noise ratio h by h'.

Let us consider now in greater detail expression (10.5.12) for $S_{\text{экв}}$. It contains three components: one which does not depend on signal-to-noise ratio h, a second which is proportional to 1/h and, a third which is proportional to $1/h^2$. The first component arises as a result of nonlinear transformation of the useful signal, carried out in the circuit. This component is the result of incorrect selection of characteristics of filters. In the optimum circuit, as we noted in the preceding section, such a component of error is absent. The other two components in (10.5.8) are explained, respectively, by interaction of the useful signal with noise and nonlinear transformation of noise. These components of error cannot be completely eliminated, and it is possible only to decrease them by proper selection of characteristics of filters.

By direct variation of expression (10.5.12) with respect to frequency responses of filters $H_1(i\omega)$ it is easy to prove that $S_{\text{экв}}$ reaches a minimum, when $|H_1(i\omega)|^2$ is determined by formula (10.3.25), and $H_2(i\omega)$ is determined by formula (10.3.28), where h_Σ will be replaced, of course, by h. Then $S_{\text{экв}}$ will become equal to $S_{\text{опт}}$. This confirms the correctness of the obtained results.

Let us return now to the question of the physically realizable equivalent of an optimum filter with frequency response (10.3.28), left unsolved in § 10.3. From formula (10.5.12) it is clear that frequency response $H_2(i\omega)$ is contained in it in the form of expressions Re $H_2(i\omega)H_2(i\omega + i\Omega)$. This means that $S_{\text{экв}}$ will not vary if instead of a filter with characteristic $H_{2\,\text{опт}}(i\omega)$ we use a filter with characteristic $H_{2\,\text{опт}}(i\omega)e^{-ik\omega T_{\text{ск}}}$, where k is any integer. Selecting k so large that $h_{2\,\text{опт}}(kT_{\text{ск}}) = \sqrt{h_\Sigma}\,\Delta f_c\, v(t, t + kT_{\text{ск}})|_{\alpha=0} \approx 0$, it is possible to make a filter with characteristic $H_{2\,\text{опт}}(i\omega)e^{-ik\omega T_{\text{ск}}}$ physically realizable.

We turn to consideration of the circuit of Fig. 10.9. It, as it was established, is optimum for high frequencies of scanning. Therefore, besides allowance for non-idealnesses of processing in this circuit it is of interest to estimate also the worsening of accuracy as compared to the optimum which this circuit gives for other face values of the frequency of scanning. If filters in the circuit have frequency response $H_1(i\omega)$, and the heterodyne signal has the form (10.5.1), the expression for equivalent spectral density is obtained immediately from (10.5.12), if there we set $H_2(i\omega) \equiv 0$. Then

$$S_{\text{экв}} = \frac{\pi}{2\mu_2^2 h^4} \int_{-\infty}^{\infty} [|H_1(i\omega)|^4 + |H_1(i\omega)|^2 |H_1(i\omega + i\Omega)|^2] \times$$
$$\times [1 + hS_0(\omega)][1 + hS_0(\omega + \Omega)]\, d\omega \times$$
$$\times \left\{ \int_{-\infty}^{\infty} [|H_1(i\omega)|^2 + |H_1(i\omega + i\Omega)|^2] S_0(\omega)\, d\omega \right\}^{-2}. \tag{10.5.13}$$

In spite of the considerable simplification of this formula as compared to (10.5.12), it is difficult to perceive the dependence of accuracy of measurement on the form of the frequency response of the filter or the magnitude of the frequency of scanning directly from it. For investigation of these dependences it is necessary to produce calculation of integrals in (10.5.13), using suitable approximations of

the frequency response of the filters and of the spectrum of fluctuation.

Let us produce this calculation, approximating frequency response of the filter $H_1(i\omega)$ by expression

$$H_1(i\omega)=\frac{1}{1+\frac{i\omega}{2\Delta f_{\text{ф}}}}, \tag{10.5.16}$$

and $S_0(\omega)$ by expression (10.3.26). Here we obtain

$$\begin{aligned}S_{\text{экв}}=&\frac{\zeta^2+(1+x)^2}{\mu_a^2\Delta f_{\text{ф}}[\zeta^2+2(1+x)^2]^2}\Bigg\{\Bigg[\frac{(1+x)[\zeta^2+(1+x)^2(1+2x)]}{2(1-x)x[\zeta^2+(1+x)^2]}+\\&+\frac{2x(1+x)[\zeta^4+5\zeta^2(1+x^2)+4(x-1)^2(1+x)(1+2x)}{[\zeta^2+(x-1)^2](\zeta^2+4)(\zeta^2+4x^2)}-\\&-\frac{x^2[\zeta^4+\zeta^2(5x^2+4x+3)+2(x-1)^2(x+1)(2x+1)}{(1-x)[\zeta^2+(x-1)^2](\zeta^2+4x^2)}\Bigg]+\\&+\frac{1}{h}\Bigg[\frac{2x(1+x)[\zeta^4+\zeta^2(5x^2+4x+3)+2(x-1)^2(x+1)(2x+1)]}{[\zeta^2+(x-1)^2](\zeta^2+4x^2)}+\\&+\frac{\zeta^4(1+2x)+\zeta^2(1+x^2)(4x+3)+2(x+1)^4(1+2x)}{2x[\zeta^2+(1+x)^2]}\Bigg]+\\&+\frac{1}{h^2}\frac{(x+1)^2[\zeta^2+(x+1)^2](\zeta^2+8x^2)}{2x(\zeta^2+4x^2)}\Bigg\},\end{aligned} \tag{10.5.17}$$

where $\zeta=\Omega/2\Delta f_c$; $x=\Delta f_{\text{ф}}/\Delta f_c$.

Let us investigate formula (10.5.17). Although it is obtained with rather particular approximations of the frequency response of the filters and of the spectrum of fluctuations of the signal, the laws governing change of $S_{\text{экв}}$ expressed by this formula are preserved, too, in more general cases. First of all we shall consider limiting cases when the laws of variation of $S_{\text{экв}}$ are considerably simplified. For small x instead of formula (10.5.17) we obtain

$$S_{\text{экв}}=\frac{1}{2\mu_a^2\Delta f_c x}\frac{\zeta^2+1}{(\zeta^2+2)^2}\left(1+\frac{\zeta^2+2}{h}+\frac{\zeta^2+1}{h^2}\right). \tag{10.5.18}$$

Thus, the spectral density of error with decrease of x, i.e., with narrowing of the passband of the filter, grows inversely proportionally to x. For small x the component of error from nonlinear transformation of the useful signal, monotonically drops with growth of the frequency of scanning. The remaining components of error almost do not depend on ζ.

For large x, i.e., with expansion of the passband of the filters, formula (10.5.17) takes the very simple form

$$S_{\text{экв}}=\frac{x}{2\mu_a^2\Delta f_c h^2}. \tag{10.5.19}$$

From this it is clear that for sufficiently broad-banded filters the spectral density of error of measurement is caused basically by nonlinear transformation of the signal and grows proportionally to x. Error in this case does not depend on the frequency of scanning.

For large ζ equivalent spectral density is equal to

$$S_{\text{экв}}=\frac{1}{2\mu_2^2\Delta f_{\text{в}}}\left(\frac{2+1/x}{h}+\frac{x+2+1/x}{h^2}\right). \qquad (10.5.20)$$

From (10.5.20) one may see that $S_{\text{ЭКВ}}$ reaches its minimum at $x = \sqrt{1 + h}$. This is understandable, since the considered circuit for large ζ is optimum. Decrease or increase of x in comparison with $\sqrt{1 + h}$ leads to increase of the spectral density.

For small ζ we also obtain a very simple formula:

$$S_{\text{экв}}=\frac{1}{4\mu_2^2\Delta f_{\text{в}}}\left(2+1/x+2\,\frac{2+1/x}{h}+\frac{x+2+1/x}{h^2}\right). \qquad (10.5.21)$$

In this case the optimum for x is attained when $x = 1 + h$ ($S_{\text{ЭКВ}}$ reaches its minimum value), and not when $x = \sqrt{1 + h}$, since for small ζ this circuit is not optimum, and the laws will be different here.

In order to judge the relative change of accuracy of the considered circuit as compared to the optimum, it is of interest to investigate ratio $S_{\text{ЭКВ}}/S_{\text{ОПТ}}$. Spectral density $S_{\text{ОПТ}}$ with approximation of the spectrum of fluctuations by formula (10.3.26) was calculated in the preceding section and was presented by formula (10.4.16). The graph of the dependence of ratio $S_{\text{ЭКВ}}/S_{\text{ОПТ}}$ on h for different values of x and ζ is shown in Fig. 10.13. From this figure it is clear that at high frequencies of scanning ($\zeta \approx 100$) accuracy of the considered circuit is close to optimum in a very broad range of variation of x (for x = 2-10 ratio $S_{\text{ЭКВ}}/S_{\text{ОПТ}}$ = 1-1.5). For low frequencies of scanning accuracy of the considered circuit rather strongly differs from the optimum; the difference is even greater, the larger h. This is explained by the fact that at low frequencies of scanning the equivalent spectral density of the considered circuit as $h \to \infty$ seeks a finite magnitude (since there is a component of $S_{\text{ЭКВ}}$ caused by nonlinear transformation of the useful signal); for an optimum circuit equivalent spectral density as $h \to \infty$ tends to zero.

We shall now discuss the spectral density of parametric fluctuations. As it was shown in Chapter VI, it has the form

$$S_{\text{пар}}=\frac{1}{2K_{\text{Д}}^2}\frac{d^2}{d\alpha^2}\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\int_{-\infty}^{\infty}\overline{z(t)z(s)}\,ds\Big|_{\alpha=0}. \qquad (10.5.22)$$

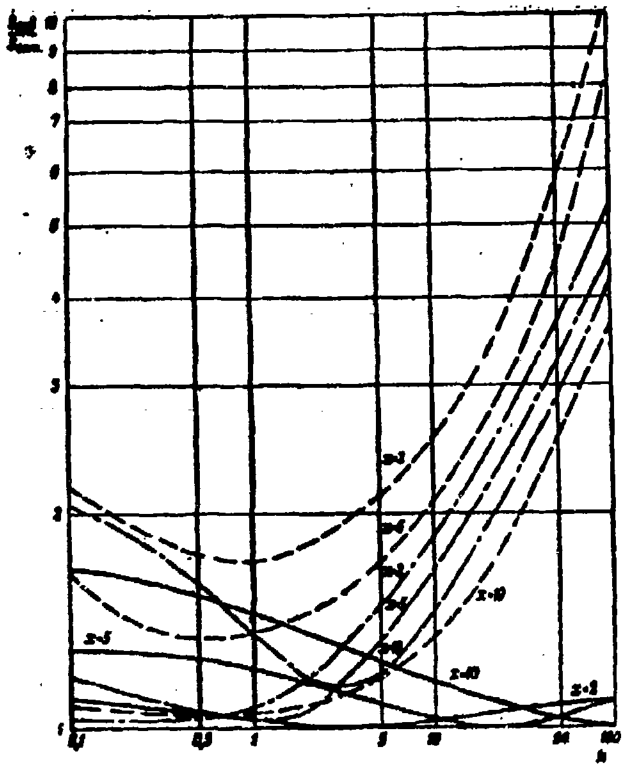

Fig. 10.13. The dependence of $S_{экв}/S_{опт}$ on h for a circuit with narrow-band filters of the radio channel of a goniometer with a scanning pattern: —— ζ = 100; - - - ζ = 10; -. -. -. ζ = 1.

We shall calculate $S_{пар}$ for the case of high frequencies of scanning, when $S_{экв}$ is already small, and a considerable share of total error is caused by parametric fluctuations. Calculations in finding $S_{пар}$ are very similar to those by which we calculated $K_д$ and $S_{экв}$. Therefore, we shall immediately give the final result

$$S_{пар} = 2\pi \frac{\int\limits_{-\infty}^{\infty} |H_1(i\omega)|^4 S_\varphi(\omega)^2 \, d\omega}{\left[\int\limits_{-\infty}^{\infty} |H_1(i\omega)|^2 S_\varphi(\omega) \, d\omega\right]^2}. \qquad (10.5.23)$$

Thus, $S_{пар}$ does not depend on the signal-to-noise ratio and frequency of scanning. Calculation by (10.5.23) with approximation (10.5.16) and (10.3.26) gives

$$S_{пар} = \frac{1}{\Delta f_c} \cdot \frac{1 + 3x + x^2}{x(1 + x)}. \qquad (10.5.24)$$

For large x quantity $S_{пар} \to 1/\Delta f_c$; for small x spectral density $S_{пар} \to 1/x\Delta f_c = 1/\Delta f_ф$. Laws of this kind are preserved qualitatively also for other forms of frequency response and of the spectrum of fluctuations.

10.5.3. Investigation of a Circuit with a Broad-Band Filter

Let us consider now the circuit of Fig. 10.10. If we designate the frequency response of the filter in this circuit $H(i\omega)$, the expression for the equivalent spectral density of such a circuit could be obtained from (10.5.8), if we set $H_1(i\omega) \equiv 0$, and $H_2(i\omega) = H(i\omega)$.

However let us note that for not too low frequencies of scanning modulation of received signal introduced by scanning pattern will be delayed during passage through

the filter. This delay should be, of course, introduced in the reference signal from the GON in order to ensure a maximum value of the transmission factor of the radio channel. Calculations of characteristics of a circuit in these conditions have to be redone.

The signal at the output of the circuit of Fig. 10.10 has the form

$$z(t)=\cos\Omega(t-\tau_з)\left|\int_{-\infty}^{t}\dot{h}(t-\tau)\,y(\tau)\,u(\tau)\,e^{i\omega_0\tau}\,d\tau\right|^2, \tag{10.5.25}$$

where $\dot{h}(t)$ — complex amplitude of the pulse response of the filter in this circuit [determined as in (10.5.20)], and delay $\tau_з$ should be selected from the condition of maximum $K_Д$. Using formula (10.5.5), we can obtain (calculations are similar to those already made, and therefore we omit them)

$$\Delta=0,$$

$$\max_{\tau_з}K_Д=\frac{P_c\mu_з}{2\pi\Delta f_c}\left|\int_{-\infty}^{\infty}H(i\omega)\,H^*(i\omega+i\Omega)\,S_0(\omega)\,d\omega\right|. \tag{10.5.26}$$

In an analogous way from formula (10.5.11) we have

$$\begin{gathered}S_{экв}=\frac{\pi}{\mu_з^2}\int_{-\infty}^{\infty}|H(i\omega)|^2\,|H(i\omega+i\Omega)|^2\times\\ \times\left\{S_0(\omega)\,S_0(\omega+\Omega)+\frac{1}{h}[S_0(\omega)+S_0(\omega+\Omega)]+\frac{1}{h^2}\right\}d\omega\times\\ \times\left[\int_{-\infty}^{\infty}|H(i\omega)\,H^*(i\omega+i\Omega)/S_0(\omega)\,d\omega\right]^{-2}.\end{gathered} \tag{10.5.27}$$

We shall perform our calculation from this formula, using approximation (10.?.1?) for $|H(i\omega)|^2$ and (10.3.26) for $S_0(\omega)$. As a result of calculation of integrals (10.5.27) we obtain

$$\begin{gathered}S_{экв}=\frac{x+1}{\mu_з^2\Delta f_c x}\left\{\frac{1}{h^2}\,\frac{[\zeta^2+(1+x)^2](1+x)}{\zeta^2+4(1+x)^2}+\right.\\ +\frac{1}{h}\,\frac{\zeta^4+\zeta^2(5x^2+4x+3)+2(x-1)^2(x+1)(2x+1)}{[\zeta^2+(x-1)^2][\zeta^2+4(x+1)^2]}+\\ \left.+\frac{\zeta^6+5\zeta^4(1+x)^2+4(x-1)^2(x^2+3x+1)}{[\zeta^2+(x-1)^2](\zeta^2+4)[\zeta^2+4(x+1)^2]}\right\},\end{gathered} \tag{10.5.28}$$

where as before $\zeta=\Omega/2\Delta f_c$; $x=\Delta f_ф/\Delta f_c$.

Let us consider different limiting cases for ζ and x. For small x from (10.5.28)

we have

$$S_{\text{экв}}=\frac{1}{\mu_a^2\Delta f_c x}\,\frac{1}{\zeta^2+4}\left(1+\frac{\zeta^2+2}{h}+\frac{\zeta^2+1}{h^2}\right). \qquad (10.5.29)$$

From this it is clear that the equivalent spectral density for small x is inversely proportional to x and monotonically drops with increase of ζ (the drop is basically due to the component of error caused by nonlinear transformation of the useful signal). Comparison of (10.5.29) with (10.5.18) shows that for small x the circuit of Fig. 10.10 has spectral density less by a factor of $(\zeta^2 + 1)\ (\zeta^2 + 4)/(\zeta^2 + 2)^2$ than the circuit of Fig. 10.9. However, let us note that

$$1<\frac{(\zeta^2+1)(\zeta^2+4)}{(\zeta^2+2)^2}<1{,}1,$$

so that both circuits for small x are practically identical.

For large x from (10.5.28) we obtain

$$S_{\text{экв}}=\frac{x}{\mu_a^2\Delta f_c h^2}. \qquad (10.5.30)$$

Comparison of (10.5.30) and (10.5.19) shows that for large x the circuit of Fig. 10.9 has a spectral density half as large as the circuit of Fig. 10.10.

We now consider the case of large ζ. From (10.5.28) we obtain

$$S_{\text{экв}}=\frac{1}{\mu_a^2\Delta f_c}\left(\frac{1+1/x}{h}+\frac{x+2+1/x}{h^2}\right). \qquad (10.5.31)$$

Here, obviously, there is an optimum at $x = \sqrt{1 + h}$. Comparison of (10.5.31) and (10.5.20) shows that for large ζ the circuit of Fig. 10.9 has the best accuracy $\left(\text{for small h, twice as good, for large h — better by a factor of } \frac{1}{2}\,\frac{2x + 1}{x + 1}\right)$.

Finally, for low frequencies of scanning (small ζ) from (10.5.28) we obtain

$$S_{\text{экв}}=\frac{1}{4\mu_a^2\Delta f_c}\left(2+1/x+2\,\frac{2+1/x}{h}+\frac{x+2+1/x}{h^2}\right), \qquad (10.5.32)$$

which completely coincides with (10.5.21). Thus, comparison of circuits of Fig. 10.9 and Fig. 10.10 shows that these circuits for identical parameters (x and ζ) have approximately identical accuracy; the circuit of Fig. 10.9 is less critical to widening of the pass band of the filters and ensures for wide filters twice the accuracy (with respect to equivalent spectral density) of the circuit of Fig. 10.10.

For more detailed study of accuracy of the circuit of Fig. 10.10 in Fig. 10.14 there is constructed the graph of the dependence of $S_{\text{экв}}/S_{\text{опт}}$ on h for this circuit, calculated with the same approximations as in the preceding case. From this graph it is clear that the considered circuit, in general, realizes the potential accuracy

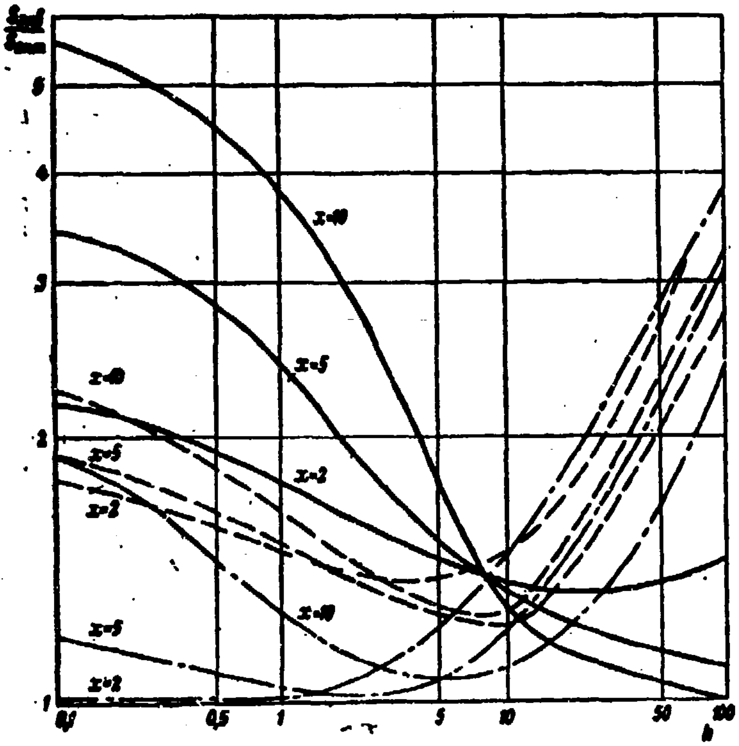

Fig. 10.14. The dependence of $S_{ЭКВ}/S_{ОПТ}$ on h for the circuit with a broad-band filter of the radio channel of a goniometer with a scanning pattern: —— ζ = 100; - - - ζ = 10; -. -. -. ζ = 1.

of the method of pattern scanning worse than the preceding circuit. For low frequencies of scanning ($\zeta < 10$) only in the range h = 1-10 does it ensure accuracy worse than the potential by a factor of 1-1.5 (with respect to equivalent spectral density). With further growth of h ratio $S_{ЭКВ}/S_{ОПТ}$ rather sharply increases (due to the component of error caused by nonlinear transformation of the signal). For high frequencies of scanning ($\zeta \approx 100$) accuracy of the circuit nears the optimum ($S_{ЭКВ}/S_{ОПТ}$ = 1-1.5) when h > 10.

We note that ratio $S_{ЭКВ}/S_{ОПТ}$ for both the considered circuits in a wide range of change of their parameters does not exeed 4.

§ 10.6. Method of Scanning with Compensation and Instantaneous Amplitude Comparison of Signals (Optimum Circuits)

Besides the method of scanning the directional pattern of the receiving antenna in goniometrical practice there are widely applied the method of scanning with compensation and the method of instantaneous amplitude comparison of signals (IAC). In the use of these methods there are simultaneously observed several signals, received

by different directional patterns. Inasmuch as fluctuations of amplitudes of the signals received by the various patterns will be identical, this is the prerequisite for their elimination or, at least, partial compensation. Therefore from these methods there is anticipated heightened, as compared to method of pattern scanning, accuracy of measurement during a fluctuating signal, and also increased noise immunity against amplitude-modulated active interferences.

Finally, a very important consideration in favor of the method of scanning with compensation and IAC is connected with the efficiency of use of the aperture of the antenna system. The fact is that during use of the method of pattern scanning one may approximately assume that there is used power from only part of the antenna aperture. The section from which power is selected shifts over the aperture of the antenna in accordance with a certain periodic law. In the presence of angular divergence of the target from the equisignal direction the distribution of the field in the aperture of the antenna is asymmetric, and, consequently, there occurs periodic change of amplitude of the received signal. If all the power from the aperture of the antenna system was used, the received signal would have constant amplitude and would not contain information about angular coordinates of the target. Thus, during use of the method of pattern scanning the received power comprises only a certain share of the power reaching the aperture of the antenna system (usually half). This circumstance is fundamental.

During use of scanning with compensation and IAC one may approximately assume that in one channel there is used power from one part of the antenna aperture, and in other — power from the remaining part, so that the total received power is equal to all the power entering the aperture of the antenna system. In the case of asymmetric distribution of the field in the aperture (with angular divergence of the target from the equisignal direction) the powers entering each of the channels are unequal and their difference depends on the angular coordinates of the target. Thus, the compensation method and IAC ensure more efficient use of the aperture of the antenna system than the method of pattern scanning: total power received during use of these methods is approximately twice the average power received with the method of scanning the directional pattern of the antenna system.

In the present section we shall study potentialities of the method of scanning with compensation and IAC and their optimum circuits. We combine consideration of these two methods in view of their very great similarity physically and mathematically.

From the technical point of view these methods, of course, rather greatly differ, which is not decisive in this case.

10.6.1. Synthesis of Optimum Circuits for Methods of Scanning with Compensation and IAC

The optimum circuit for the method of scanning with compensation can easily be obtained from the general circuit of Fig. 10.7, where it is necessary to set n = 2 (two antennas), $\Phi_1(t, \alpha) = 0$ (phase centers of antennas coincide and are fixed), and $U_{1a}(t, \alpha)$ is determined by formula (10.2.4). Here, using in (10.2.4) different concrete values for laws of scanning of patterns $\vartheta_1(t)$, it is possible to consider a large number of particular cases of the method of scanning with compensation. However we immediately will separate methods of scanning in which this method has the best potential properties and subsequently will limit our consideration to only such cases.

From consideration of the general circuit of Fig. 10.7 it is clear that the optimum circuit of the method of scanning with compensation, besides the two basic channels, contains, in general, an additional, third channel which, as was shown in § 10.3, carries out singling out of false information from the amplitude fluctuations of the signal, which permits us subsequently to partially compensate the harmful influence of these fluctuations. It is easy to see that with fulfillment of condition

$$h_1\mu_{a_1}\cos\vartheta_1(t)+h_2\mu_{a_2}\cos\vartheta_2(t)=0 \qquad (10.6.1)$$

in the circuit measuring angle α the third channel disappears. Consequently, with fulfillment of condition (10.6.1) in this circuit there occurs automatic compensation of amplitude fluctuations of signals, i.e., there is realized that concept which was pursued during development of the method of scanning with compensation.

Let us consider conditions (10.6.1) in greater detail. They signify a definite symmetry of receiving channels. If $h_1 = h_2$ and the directional patterns of the antennas are identical, these conditions will be satisfied when

$$\vartheta_1(t)=\vartheta(t)=\vartheta_2(t)+\pi. \qquad (10.6.2)$$

Expression (10.6.2) means that the directional patterns occupy the extreme possible positions and scan without changing their mutual location. The directional patterns forming such a figure are sometimes called "waltzing."

Thus, only with fulfillment of condition (10.6.2) does the method of scanning with compensation ensure complete compensation of amplitude fluctuations of the

signal. Any other forms of scanning one should recognize as unsatisfactory, since they do not ensure realization of the potential of this method. In particular, the widely known variant of the method of scanning with compensation with one fixed and one scanning pattern should be recognized as a failure. Henceforth we shall limit ourselves only to consideration of this method with "waltzing" patterns, considering here spectral densities of noises in the channels to be identical.

We shall make a series of remarks about the optimum circuit of scanning with compensation. Making the proper simplifications in formula (10.3.29), we can record the optimum operation of the radio channel for this method in the form

$$\begin{aligned} z(\tau) = \int_{t-\Delta t}^{t} h_{1\text{опт}}(\tau - s) \cos \omega_{\text{пр}}(\tau - s)\,[y_1(s) - y_2(s)] \times \\ \times \cos\vartheta(s)\, u_a(s) \cos[\omega_r s + \psi(s)]\, ds \int_{t-\Delta t}^{t} h_{1\text{опт}}(\tau - s) \times \\ \times \cos \omega_{\text{пр}}(\tau - s)\,[y_1(s) + y_2(s)]\, u_a(s) \cos[\omega_r s + \psi(s)]\, ds, \end{aligned} \tag{10.6.3}$$

where $y_i(s)$ – received signals, expressed by formulas (10.3.3).

Let us designate the useful components of these signals by $\tilde{y}_i(s)$, i.e., $\tilde{y}_i(s) = y_i(s) - \sqrt{N_0}n_i(s)$. Then

$$\begin{aligned} y_1(s) - y_2(s) = \tilde{y}_1(s) - \tilde{y}_2(s) + \sqrt{N_0}n_1(s) - \sqrt{N_0}n_2(s) = \\ = \tilde{y}_1(s) - \tilde{y}_2(s) + \sqrt{N_0}n_-(s),\ n_-(s) = n_1(s) - n_2(s), \\ y_1(s) + y_2(s) = y_1(s) + \tilde{y}_2(s) + \sqrt{N_0}n_1(s) + \sqrt{N_0}n_2(s) = \\ = \tilde{y}_1(s) + \tilde{y}_2(s) + \sqrt{N_0}n_+(s),\ n_+(s) = n_1(s) + n_2(s). \end{aligned} \tag{10.6.4}$$

Inasmuch as $n_1(s)$ and $n_2(s)$ are independent, independent, too, will be noises $n_+(s)$ and $n_-(s)$. It follows from this that from the point of view of statistical characteristics it is possible to form the sum and difference of signals after their mixing with the two independent noises or it is possible to add independent noises already to the "sum" and "difference" signal. This remark is very significant, since technically it is often more convenient to form the sum and difference signals directly at the outputs of the antenna system. Using this remark, it is possible to present the circuit of the optimum radio channel in the compensation method in the form depicted in Fig. 10.15.

Let us discuss in greater detail the operation of the circuit of Fig. 10.15. In this circuit signals from outputs of the antenna system first enter a unit which forms their sum and difference (hybrid ring junction, T-junction, etc). The sum and difference signals are heterodyned. Then the difference signal is multiplied

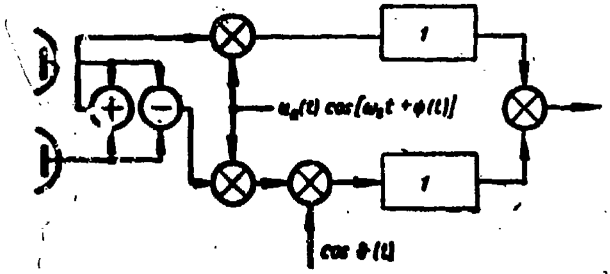

Fig. 10.15. Optimum circuit of the radio channel of a goniometer using scanning with compensation: 1) optimum filters with frequency response $H_{1\ \text{опт}}(i\omega)$.

by a function varying according to the law of scanning of the directional patterns. After these transformations the signals are filtered by optimum filters with response (10.3.25) and are multiplied. The optimum circuit of Fig. 10.15 is not more complicated than known circuits of the method of scanning with compensation and therefore deserves the attentive study which will be conducted in the next section. When the frequency of scanning of the directional patterns is sufficiently small as compared to the width of the passband of the filters in the circuit of Fig. 10.15, multiplication of the difference signal by a function varying according to the law of scanning can be carried out after filtration. Here the circuit can be transformed to the form depicted in Fig. 10.16. Here signals directly from the output of the antenna system are heterodyned, are filtered by optimum filters with response (10.3.25), are detected by square-law detectors, and are subtracted; the obtained difference is multiplied by a function varying according to the law of scanning. This circuit, well-known in goniometer practice, is close to the optimum only for low-frequency scanning of the directional patterns.

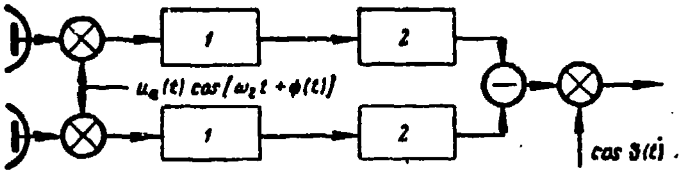

Fig. 10.16. Optimum circuit of the radio channel of a goniometer using scanning with compensation for low frequencies of scanning: 1) optimum filters with frequency response $H_{1\ \text{опт}}(i\omega)$; 2) square-law detectors.

The optimum circuits for the IAC method are very similar to the circuits of scanning with compensation. If we take as initial prerequisites identity of directional patterns of the antennas in the goniometer with IAC and identity of noises added to the signals, the optimum circuit for IAC will be obtained from the circuit synthesized just now for the method of scanning with compensation (Fig. 10.15) if we there set $\vartheta(t) = 0$. The optimum circuit for IAC is shown in Fig. 10.17. Obviously it can be converted identically to the form in Fig. 10.18. Both variants of circuits for IAC are known and have been described in the literature [49].

Let us note that with unequalness of directional patterns of the antenna system and unequalness of noises in optimum circuits for IAC besides the two basic channels there would appear, just as for the method of scanning with compensation, an additional channel intended for compensation of harmful effects caused by the shown

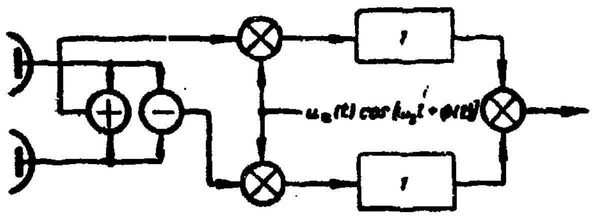

Fig. 10.17. Variant of an optimum circuit of the radio channel of a goniometer with IAC (circuit with multiplication of signals). 1) optimum filters with frequency response curve $H_{1\ \text{опт}}(i\omega)$.

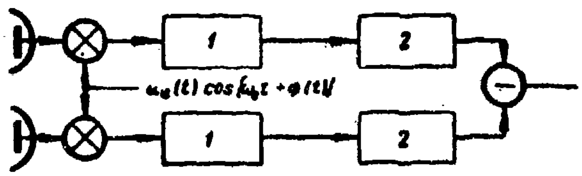

Fig. 10.18. Variant of an optimum circuit of the radio channel of a goniometer with IAC (circuit with subtraction of signals): 1) optimum filters with frequency response curve $H_{1\ \text{опт}}(i\omega)$; 2) square-law detectors.

disymmetry. However, consideration of these cases does not have practical value.

In conclusion we note that for measurement of an angular coordinate of a target in another plane the optimum circuits of the method of scanning with compensation, obviously, do not change, only multiplication by cos $\vartheta(t)$ must/~~be~~ replaced by multiplication by sin $\vartheta(t)$. Optimum circuits of IAC, in general, do not depend on what angular coordinate of the target is measured: IAC circuits will measure one or the other angular coordinate of the target depending on the pair of outputs of the antenna to which it is joined.

10.6.2. Equivalent Spectral Densities of Optimum Circuits for Methods of Scanning with Compensation and IAC

Let us turn to the question of potential accuracy of the method of scanning with compensation and IAC. We first consider the compensation method. Considering in (10.3.31) $n = 2$, $\Phi_1(t, \alpha) = 0$, introducing $U_{1a}(t, \alpha)$ according to expression (10.2.4), considering here all assumptions of symmetry made in the beginning of this section, in particular (10.6.2), and then calculating integrals in (10.3.31), for equivalent spectral density of the optimum circuit we obtain the following expression

$$S_{\text{опт}} = \left[\frac{h_\Sigma^2 \mu_a^2 c^2}{\pi} \int_{-\infty}^{\infty} \frac{S_0(\omega)^2}{1 + h_\Sigma S_0(\omega)} d\omega\right]^{-1}, \qquad (10.6.5)$$

where $h_\Sigma = h_1 + h_2$ (i.e., h_Σ — ratio of total signal power received by both directional patterns, to the power of noise of one channel in the band of signal fluctuations), and coefficient c^2 is the same as in formulas of accuracy for the method of pattern scanning (10.4.3). Equivalent spectral density during measurement of an angular coordinate in another plane is expressed by the same formula as (10.6.5), where it is necessary to replace c^2 by s^2 (10.4.4). As also in the method of pattern scanning, coefficients c^2 and s^2, the sum of which is equal to 1, characterize,

roughly speaking, the distribution of received power for measurement of angles in various planes.

Usually there is required identical accuracy of measurement of both angles; therefore we naturally select the law of scanning so that

$$c^2 = s^2 = \frac{1}{2}. \tag{10.6.6}$$

This we shall assume subsequently. However, note that if we need to measure only one of the angles, best accuracy will be rendered by switching of the directional patterns in the plane of this angle or even measurement with fixed patterns, located in the given plane; in these cases $\cos \vartheta(t) = \pm 1$ and $c^2 = 1$. For motionless patterns we obtain nothing other than a two-dimensional variant of the IAC method.

It is easy to show that potential accuracy of IAC when there is required measurement of angles in two planes with identical accuracy is expressed by formula (10.6.5), where $c^2 = 1/2$. From this it follows that potential accuracy of IAC and of the method of scanning with compensation are identical.

Comparison of formulas (10.6.5) and (10.4.9) shows also that potential accuracy of IAC (or of the method of scanning with compensation) and of the method of pattern scanning (with use in the latter of high frequencies of scanning) are identical if we consider identical the total powers received in each case. However, as was already shown, the power received in the method of pattern scanning is approximately half the total power received in the method of scanning with compensation or with IAC (for an antenna with the same total aperture area). Thus, if we fix the aperture area of the antenna potential accuracy of the method of pattern scanning, even with high frequencies of scanning, will be less than the accuracy of IAC (of the method of scanning with compensation). However, we note that this is connected with the worst use of the aperture of the antenna system during the method of pattern scanning.

Let us consider formula (10.6.5) in greater detail. For small noises, i.e., when $h_\Sigma >> 1$, from (10.6.5) we obtain the very simple expression

$$S_{\text{экв}} = \frac{1}{\mu_a^2 h_\Sigma \Delta f_c} \tag{10.6.7}$$

(remember that we set $c^2 = 1/2$).

Inasmuch as $h_\Sigma = (P_{c\,1} + P_{c\,2})/2N_0\Delta f_c$, equivalent spectral density in this case does not depend on statistical characteristics of fluctuations of the signal.

In general we make our calculation by formula (10.6.5), using approximation (10.5.20) of the spectrum of fluctuations. Here, we obtain

$$S_{\text{опт}} = \frac{1}{\mu_a^2 \Delta f_0} \frac{\sqrt{1+h_\Sigma}\,(1+\sqrt{1+h_\Sigma})}{h_\Sigma^2}. \tag{10.6.8}$$

We already have the graph of function (10.6.8) in Fig. 10.11 (case $\zeta = \infty$).

We note also the formula for $S_{\text{опт}}$ for small values of h_Σ, easily obtained from (10.6.8):

$$S_{\text{опт}} = \frac{2}{\mu_a^2 \Delta f_0 h_\Sigma^2}. \tag{10.6.9}$$

Subsequently for brevity we shall call the circuits of Figs. 10.15 and 10.17 circuits with multiplication, and the circuits of Figs. 10.16 and 10.18 — circuits with subtraction of signals.

§ 10.7. Investigation of Synthesized Circuits for Methods of Scanning with Compensation and for IAC

We shall investigate real accuracy of synthesized circuits for the method of scanning with compensation and, correspondingly, for IAC. The meaning of this investigation is to account for the influence on accuracy of different deviations from optimality in parameters of these circuits, inevitable during their practical realization.

Peculiarities appearing during practical realization of theoretically synthesized circuits were described in sufficient detail in § 10.5. All the facts mentioned there, obviously, directly apply to the considered circuits as well. However here there are certain additional peculiarities which must be considered during calculation of real accuracy. Basically they reduce to the fact that in each channel of these circuits there are mixers and UPCh's , and these devices are complicated and difficult to make identical; from this there follows nonidentity of the frequency responses of the channels inevitable in practical realization of such circuits, the influence of which on accuracy we must consider.

Thus, we shall assume subsequently that filters in channels have different pulse responses

$$h_1(t)\cos[\omega_{\text{пр}}t + \theta_1(t)] \text{ and } h_2(t)\cos[\omega_{\text{пр}}t + \theta_2(t)].$$

For frequency responses we preserve designations of (10.5.2).

10.7.1. Investigations of Circuits with Multiplication of Signals

The output signal from the circuit of Fig. 10.15 can, obviously, be recorded in the form

$$z(t) = \mathrm{Re} \int_{-\infty}^{t} \dot{h}_1(t-\tau)[y_1(\tau)+y_2(\tau)]\, u(\tau)\, e^{i\omega_0\tau}\, d\tau \times$$
$$\times \int_{-\infty}^{t} \dot{h}^*_2(t-\tau)[y_1(\tau)-y_2(\tau)]\, u^*(\tau)\, e^{-i\omega_0\tau} \cos\theta(\tau)\, d\tau, \tag{10.7.1}$$

where $y_i(t)$, (for $i = 1, 2$) are determined by formula (10.3.3), in which, in turn, $U_{ai}(t, \alpha)$ are expressed by formula (10.2.4).

The output signal from the circuit of Fig. 10.17 has a simpler form:

$$z(t) = \mathrm{Re} \int_{-\infty}^{t} \dot{h}_1(t-\tau)[y_1(\tau)+y_2(\tau)]\, u(\tau)\, e^{i\omega_0\tau}\, d\tau \times \tag{10.7.2}$$
$$\times \int_{-\infty}^{t} \dot{h}^*_2(t-\tau)[y_1(\tau)-y_2(\tau)]\, u^*(\tau)\, e^{-i\omega_0\tau}\, d\tau,$$

where $y_i(t)$ is also given by formula (10.3.3), in which $U_{ai}(t, \alpha)$ are expressed by formulas (10.2.5).

Then substituting the obtained expressions in (10.5.5) and (10.5.6) and acting by the rules already described, it is easy to find that in both cases $\Delta = 0$, and $K_Д$ will be expressed by the same formula

$$K_Д = \frac{P_{cz}}{2\pi\Delta f_e} \int_{-\infty}^{\infty} \mathrm{Re}\, H_1(i\omega)\, H^*_2(i\omega)\, S_e(\omega)\, d\omega. \tag{10.7.3}$$

Using formula (10.5.11), we obtain in both cases the following expressions:

$$S_{экв} = \frac{\int_{-\infty}^{\infty} |H_1(i\omega)|^2 |H_2(i\omega)|^2 [1 + h_z S_e(\omega)]\, d\omega}{\frac{\mu_z^2 h_z^2}{2\pi} \int_{-\infty}^{\infty} \mathrm{Re}\, H_1(i\omega)\, H^*_2(i\omega)\, S_e(\omega)\, d\omega}. \tag{10.7.4}$$

From (10.7.4) it follows that the minimum of $S_{ЭКВ}$ (which it is easy to prove by simple variation) takes place when $H_1(i\omega) = H_2(i\omega)$ and $|H_i(i\omega)|^2$ are expressed by formula (10.3.25). This minimum is equal simply to $S_{опт}$. Furthermore, from (10.7.3) and (10.7.4) it is easy to perceive the influence of nonidentity of frequency responses of filters. Obviously, nonidentity of the gain-frequency responses affects $S_{экв}$ little. Nonidentity of phase-frequency responses lead to essential increase of $S_{экв}$ (as a result of decrease of $K_Д$). Actually, with difference of the phase-frequency responses $\Delta\varphi(\omega)$

$$K_Д = \frac{P_{cz}\mu_z}{2\pi\Delta f_e} \int_{-\infty}^{\infty} |H_1(i\omega)|^2 |H_2(i\omega)|^2 S_e(\omega) \cos\Delta\varphi(\omega)\, d\omega, \tag{10.7.5}$$

from which we clearly see the cause of decrease of $K_Д$. In particular, if $\Delta\varphi(\omega)$ is sufficiently great (reaches limits $\pi/2$ still in the vicinity of the maximum of

$|H_1(i\omega)|$), $K_Д$ may be zero or even change sign. The circuit will not work here. Thus, of basic importance for the considered circuits is making the phase-frequency responses of filters identical.

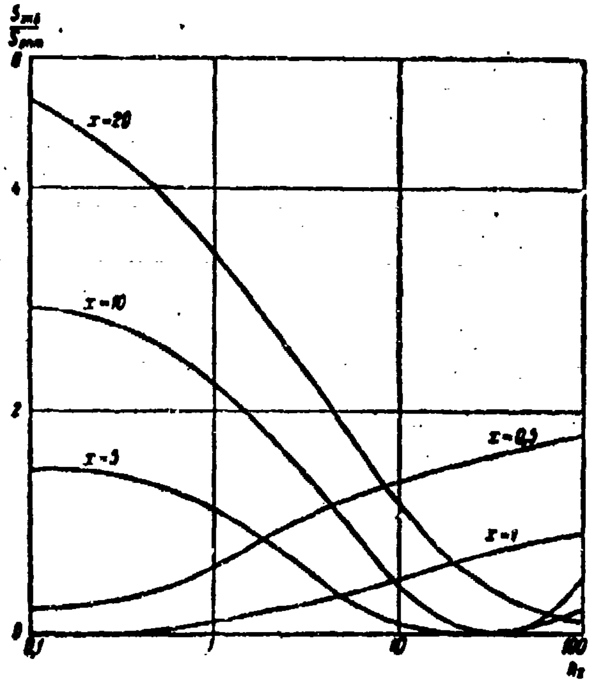

Fig. 10.19. The dependence $S_{ЭКВ}/S_{ОПТ}$ on $h_Σ$ for circuits with multiplication of signals in goniometers using scanning with compensation or IAC.

Let us calculate $S_{ЭКВ}$ by formula (10.7.4), considering filters identical and approximating their gain-frequency response by formula (10.5.16). Here we obtain

$$S_{экв}=\frac{1}{2\mu_A^2\Delta f_c}\left(\frac{2+1/x}{h_\Sigma}+\frac{x+2+1/x}{h_\Sigma^2}\right), \qquad (10.7.\ldots)$$

where as before $x = \Delta f_Ф / \Delta f_C$.

This formula is very simple, and from it, it is easy to perceive all the laws governing variation of $S_{ЭКВ}$. Curves of the dependence of $S_{ЭКВ}/S_{ОПТ}$ on $h_Σ$ are shown in Fig. 10.19. From this figure it is clear that change of the filter in very wide limits ($x = 0.5$-10) for $h_Σ > 1$ leads to increase of spectral density as compared to the optimum by not more than 2 times.

The spectral density of parametric fluctuations for the considered circuits with identical filters is easily calculated by formula (10.5.22) and turns out to accurately coincide with expression (10.5.23). Analysis of this expression was made in § 10.5.

10.7.3. Investigation of the Circuit with Subtraction of Signals for the Method of Scanning with Compensation

Let us turn to consideration of the circuit for the method of scanning with compensation (Fig. 10.16). As we established, this circuit is close to the optimum only with low frequencies of scanning. However, a circuit of this type is used in practice, and it is of interest to analyze its accuracy for face values of the frequency of scanning which are not too low.

The output signal of the circuit of Fig. 10.16 has the form

$$z(t) = \cos\theta\,(t-\tau_3)\left\{\left|\int_{-\infty}^{t} h_1(t-\tau)\,y_1(\tau)\,u(\tau)\,e^{i\omega_0\tau}d\tau\right|^2 - \right.$$

$$\left. -\left|\int_{-\infty}^{t} h_2(t-\tau)\,y_2(\tau)\,u(\tau)\,e^{i\omega_0\tau}d\tau\right|^2\right\}, \qquad (10.7.6)$$

where $y_1(\tau)$ and $y_2(\tau)$ are the same as in (10.7.1), and τ_3 — delay of the reference signal, selected from the condition of maximum $K_Д$ (see § 10.5).

This condition exceptionally hampers calculation in general, and we subsequently shall limit our consideration to the case of conical scanning of directional patterns. In this case all calculations are considerably simplified, and after simple calculations we obtain

$$\max_{\tau_3} K_Д = \frac{P_{c\Sigma}\mu_a}{2r\Delta f_c}\left|\int_{-\infty}^{\infty}[H_1(i\omega)\,H^*_1(i\omega+i\Omega) + \right.$$

$$\left. + H_2(i\omega)\,H^*_2(i\omega+i\Omega)]\,S_0(\omega)\,d\omega\right|, \qquad (10.7.7)$$

$$S_{экв} = \frac{4\pi}{\mu_a^2 h_\Sigma^2}\int_{-\infty}^{\infty}\Big\{[|H_1(i\omega)|^2\,|H_1(i\omega+i\Omega)|^2 +$$

$$+ |H_2(i\omega)|^2\,|H_2(i\omega+i\Omega)|^2]\left[1+\frac{h_\Sigma}{2}\,S_0(\omega+\Omega)\right]\times$$

$$\times\left[1+\frac{h_\Sigma}{2}\,S_0(\omega)\right] - \frac{h_\Sigma}{2}\,|H_1(i\omega)|^2\,|H_2(i\omega+i\Omega)|^2\times$$

$$\times S_0(\omega)\,S_0(\omega+\Omega)\Big\}\,d\omega\,\times$$

$$\times\left|\int_{-\infty}^{\infty}[H_1(i\omega)\,H^*_1(i\omega+i\Omega) + H_2(i\omega)\,H^*_2(i\omega+i\Omega)]\,S_0(\omega)d\omega\right|^{-2}, \qquad (10.7.8)$$

where h_Σ — ratio of the total signal power to the power of noise of one channel in the band of signal fluctuations. Systematic error Δ in the considered case is also equal to 0.

Let us consider formula (10.7.8) in greater detail. From it we can easily perceive that in general for different gain-frequency responses of filters there exists a component of equivalent spectral density caused by nonlinear transformation of the useful signal. This component in the given circuit causes error of measurement even with complete elimination of noises. With identical gain-frequency characteristics of filters such a component of equivalent spectral density disappears. The influence of nonidentity of filters on accuracy of the considered circuit leads basically to appearance of this component of error.

We shall conduct further investigation of formula (10.7.8), considering filters identical and using usual approximations (10.3.26) and (10.5.10). Here we obtain

$$S_{экв} = \frac{1+x}{\mu_a^2 \Delta f_c x} \times \left\{ \frac{1}{h_\Sigma} \frac{\zeta^4 + \zeta^2(5x^2+4x+3) + 2(x-1)^2(x+1)(2x+1)}{[\zeta^2+(x-1)^2][\zeta^2+4(x+1)^2]} + \right.$$
$$\left. + \frac{2}{h_\Sigma^2} \frac{(1+x)[\zeta^2+(1+x)^2]}{\zeta^2+4(x+1)^2} \right\}, \qquad (10.7.9)$$

where as before $\zeta = \Omega/2\Delta f_c$; $x = \Delta f_ф / \Delta f_c$.

For high frequencies of scanning, i.e., for large ζ from (10.7.9) we obtain

$$S_{экв} = \frac{1}{\mu_a^2 \Delta f_c} \left(\frac{1+1/x}{h_\Sigma} + \frac{x+2+1/x}{h_\Sigma^2} \right). \qquad (10.7.10)$$

Comparison of this formula with (10.7.5) shows that for high frequencies of scanning the considered circuit gives accuracy identical to the accuracy of the circuit of Fig. 10.15 only when $x \gg 1$ and $h_\Sigma^2 \gg 1$. In the remaining cases accuracy of the considered circuit is approximately half the accuracy of the circuit of Fig. 10.15 (in equivalent spectral density).

For low frequencies of scanning from (10.7.9) we obtain for $S_{экв}$ an expression exactly coinciding with (10.7.5), i.e., the accuracy of the considered circuit in this case is the same as accuracy of the circuit of Fig. 10.15.

Curves of the dependence of $S_{экв}/S_{опт}$ on h_Σ, calculated with approximations (10.3.26) and (10.5.16), for different ζ are shown in Fig. 10.20. From this figure it is clear that in the range of changes ζ = 1-3 the considered circuit sufficiently well realizes the potential of the method of scanning with compensation, giving for x = 1-5 equivalent spectral density exceeding $S_{опт}$ by not more then in 2.5 times

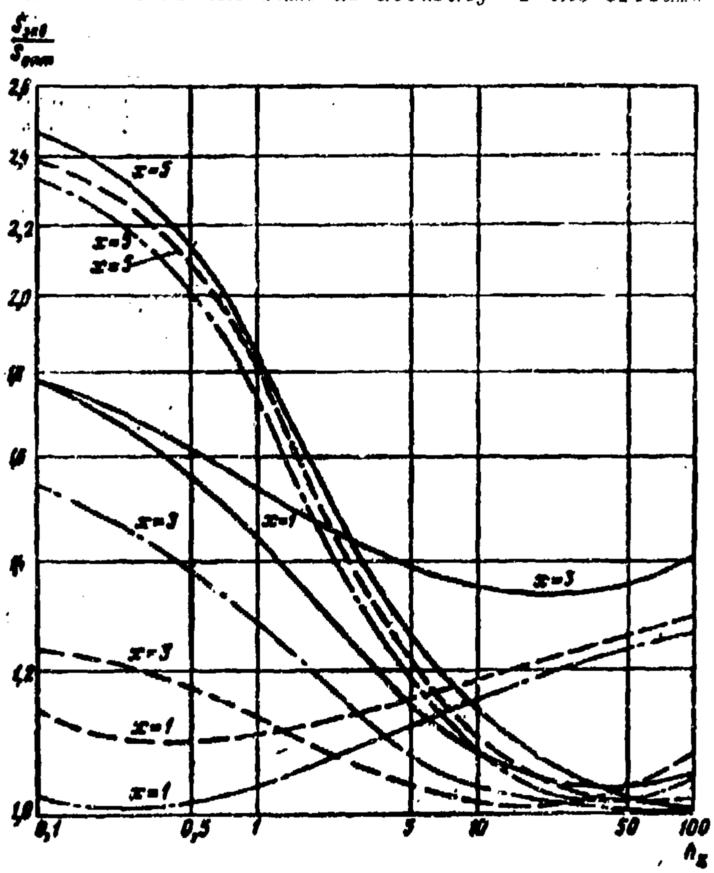

Fig. 10.20. The dependence of $S_{экв}/S_{опт}$ on h_Σ for a circuit with subtraction of signals of a goniometer using scanning with compensation: —— ζ = 3; - - - ζ = 1; -· -· -· ζ = 0.01.

(and for $h_\Sigma > 1$, $S_{экв}$ exceeds $S_{опт}$ by not more than 1.8 times). For higher frequencies of scanning properties of this circuit worsen.

10.7.3. Investigation of a Circuit with Subtraction of Signals for IAC

Let us analyze, finally, the last circuit of Fig. 10.18 for IAC. The output signal of this circuit is equal to

$$z(t)=\left|\int_{-\infty}^{t}\dot{h}_1(t-\tau)\,y_1(\tau)\,u(\tau)\,e^{i\omega_0\tau}d\tau\right|^2-$$
$$-\left|\int_{-\infty}^{t}\dot{h}_2(t-\tau)\,y_2(\tau)\,u(\tau)\,e^{i\omega_0\tau}d\tau\right|^2, \qquad (10.7.11)$$

where $y_i(t)$ are the same as in (10.7.2). Substituting this expression in (10.5.8) and (10.5.11), we obtain, respectively,

$$K_д=\frac{P_{c\Sigma}\mu_д}{2\pi\Delta f_c}\int_{-\infty}^{\infty}[|H_1(i\omega)|^2+|H_2(i\omega)|^2]\,S_0(\omega)\,d\omega, \qquad (10.7.12)$$

$$S_{экв}=\frac{4\pi}{\mu_д^2h_\Sigma^2}\int_{-\infty}^{\infty}\left\{[|H_1(i\omega)|^4+|H_2(i\omega)|^4]\left[1+\frac{h_\Sigma}{2}S_0(\omega)\right]^2-\frac{h_\Sigma^2}{2}|H_1(i\omega)|^2|H_2(i\omega)|^2S_0^2(\omega)\right\}d\omega\times$$
$$\times\left\{\int_{-\infty}^{\infty}[|H_1(i\omega)|^2+|H_2(i\omega)|^2]\,S_0(\omega)\,d\omega\right\}^{-2}. \qquad (10.7.13)$$

It is easy to see that (10.7.7) and (10.7.8) for small Ω are transformed into (10.7.12) and (10.7.13), respectively, i.e., accuracy of the circuit of Fig. 10.18 for IAC and accuracy of the circuit of Fig. 10.16 for the method of scanning with compensation for low frequencies of scanning are identical. From (10.7.13) it is clear that nonidentity of the gain-frequency responses of filters leads in the given circuit to the same effect as in the circuit of Fig. 10.16 of the method of scanning with compensation: the appearance of a very undesirable component of error, caused by nonlinear transformation of the signal. Furthermore, in the considered circuit for different gain-frequency responses there will exist certain systematic error differing from zero, calculable by formula (10.5.6):

$$\Delta = \frac{1}{2\mu_a h_\Sigma} \frac{\int_{-\infty}^{\infty} [|H_1(i\omega)|^2 - |H_2(i\omega)|^2] \left[1 + \frac{h_\Sigma}{2} S_0(\omega)\right] d\omega}{\int_{-\infty}^{\infty} [|H_1(i\omega)|^2 + |H_2(i\omega)|^2] S_0(\omega)\, d\omega}. \tag{10.7.14}$$

In particular, when $|H_1(i\omega)|^2 = K|H_2(i\omega)|^2$ (i.e., if only gains of the channels differ) and with approximation (10.5.12) we will have a quite simple expression

$$\Delta = \frac{1}{2\mu_a} \frac{1-K}{1+K} \left(\frac{1}{2} + \frac{x+1}{h_\Sigma}\right). \tag{10.7.15}$$

Caculation of $S_{\text{ЭКВ}}$ for approximations (10.3.26) and (10.5.16) will lead, obviously, to formula (10.7.5), since $S_{\text{ЭКВ}}$, as we already noted, coincides with (10.7._) when $\zeta \ll 1$, which in turn reduces to (10.7.5).

The spectral density of parametric fluctuations we shall consider again for identical filters (when systematic error is equal to 0, equivalent spectral density of error is also small, and parametric fluctuations give an appreciable contribution to total error of measurement). In this case it completely coincides with (10.5.23).

We sum up our investigation. First we see that with identical channels the circuit of Fig. 10.15 for the method of scanning with compensation and the circuit for the method of IAC (Figs 10.17 and 10.18) have absolutely identical accuracy, rather close to the potential accuracy of these methods. The circuit of Fig. 10.16 for the method of scanning with compensation is close in accuracy to these circuits only at low frequencies of scanning. With increase of the frequency of scanning accuracy of this circuit monotonically worsens, attaining a limit of approximately half its magnitude at low frequencies of scanning.

Nonidentity of channels influences, in general, different circuits differently. In the circuits of Figs. 10.15 and 10.17 with nonidentical channels accuracy remains absolutely identical. In these circuits the basic influence on accuracy is rendered only by nonidentity of the phase-frequency responses of the filters, leading to decrease of the transmission factor of the radio channel and through it to increase of equivalent spectral density.

In the circuits of Figs. 10.16 and 10.18 the basic influence on accuracy comes from nonidentity of the gain-frequency responses of the filters. In the circuit of Fig. 10.16 for the method of scanning with compensation this nonidentity leads to the appearance of a component of error caused by nonlinear transformation of the signal, and in the circuit for the IAC method (Fig. 10.18), furthermore, it leads

to the appearance of systematic error.

Now we introduce considerations about the technical complexity of the considered circuits.

As we already said in § 10.5, the most complicated components of goniometer circuits are the UPCh's, mixers with inversion of phase modulation, and amplitude modulators. It is easy to see that circuits with compensation contain the least number of such components: each of these circuits has 2 channels with such components and provides measurement of angular target data in two planes. It is easy to see that the circuit of the IAC method (Fig. 10.17) for measurement of angular coordinates in two planes will already require 3 such channels (since here there should be 1 channel with the sum signal and 2 channels with the difference signal), and the circuit of Fig. 10.18 even requires 4 channels (2 channel for measurement of the angle in one plane and 2 in the other). Thus, the circuits of the method of scanning with compensation are the simplest, the circuit of the IAC method of Fig. 10.17 is more complicated, and the circuit of Fig. 10.18 is the most complicated.

If we assume that making both the phase-frequency responses of filters and their gain-frequency responses identical is a difficult task, then among the considered set of circuits the circuit of Fig. 10.15 of the method of scanning with compensation will have obvious advantages over the other circuits: having the greatest accuracy and simplicity, it is least critical to nonidentity of channels. However, in many cases it is easier to ensure identity of gain-frequency response of filters. In this case the advantage, for the same reasons, will lie on the side of the circuit (Fig. 10.16) of the method of scanning with compensation with use in it of low frequencies of scanning. However, in view of the small difference of accuracies of the different circuits these recommendations one should consider very relative, and the best solution can be realized only by taking into account the concrete situation.

§ 10.8. Method of Phase Center Scanning

Let us turn to investigation of phase methods of direction finding, using for direction finding of a target effects connected with the dependence (under certain conditions) of phases of arriving radar signals on angular coordinates of the target. The simplest of the phase methods of direction finding is the method of phase center scanning. The given method, if it is possible to say so, is the "phase analog" of the method of pattern scanning.

The deficiency of this method is lowering of the accuracy of direction finding with low frequencies of phase center scanning, since useful information [illegible]

included in the law of measurement of the phase of the arriving signal, and at low frequencies of scanning this law will be distorted by phase fluctuations. Here intuition does not suggest any measures of compensating phase fluctuations. Study of the potential of this method, therefore, is of special interest.

10.8.1. Optimum Circuit for the Method of Phase Center Scanning

The optimum circuit for the method of phase center scanning is easily obtained from the general circuit of Fig. 10.7 if we there set $n = 1$ (one receiving antenna), $U_a(t, \alpha) = 1$ (i.e., the signal during reception does not obtain amplitude modulation), and $\Phi(t, \alpha)$ is expressed by formula (10.2.7). Substituting these date in (10.3.22), we obtain the optimum circuit of the method of phase center scanning in the form depicted in Fig. 10.21. The circuit for measurement of the angle in the other plane will be obviously, the same, only multiplication by $\delta(t)$ must be replaced by multiplication by a function varying according to the law of motion of the projection of the phase center in the other plane. The output signal of the antenna system here immediately enters two channels, in which there is realized heterodyning. A peculiarity of heterodyning in the considered case is the circumstance that the signal of the heterodyne proceeds to one of the channels through a phase shifter shifting the phase of this signal $\pi/2$. Then the signal in this channel is multiplied by a function varying according to the law of scanning of the phase center (more exactly, by the projection of this function in the plane of the measured angle). After such transformations signals of both channels are filtered by an optimum filter with response (10.3.25) and are multiplied.

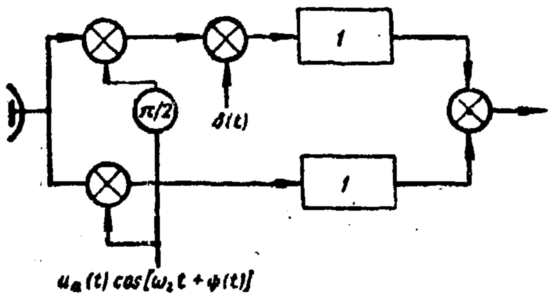

Fig. 10.21. Optimum circuit of the radio channel of a goniometer using phase center scanning. 1) optimum filters with frequency response $H_{1\,opt}(i\omega)$.

The physical meaning of these transformations is approximately the same as in the optimum circuit of the method of pattern scanning. Multiplying the received signal by a function which varies by the law of phase center scanning, we transfered the side spectral components of the signal, containing useful information, to the center frequency, after which it is possible to filter the signal by a sufficiently narrow-band filter, passing a minimum of noises without loss of useful information of the signal. It is curious that there is not any compensation of phase fluctuations

of the signal. Inasmuch as this circuit is optimum, then from this there follows the conclusion about the impossibility of compensation of phase fluctuations of the signal. In order to grasp the peculiarities distinguishing phase from amplitude fluctuations, partial compensation of which turns out to be possible, we note the following: addition of a constant to amplitude will change its law of distribution of probabilities, but addition of a constant to phase will not change its distribution of probabilities. This graphically shows that fluctuations of phase lead to more significant distortion of information included in the phase than occurs for amplitudes.

10.8.2. Equivalent Spectral Density of the Optimum Circuit for the Method of Phase Center Scanning

Let us consider potential accuracy of measurement of angles by the method of phase center scanning. Considering in formula (10.3.31) n = 1, substituting there expression (10.2.7) and producing the necessary calculations, we obtain

$$S_{\text{опт}} = \frac{\pi}{\left(\frac{\pi d}{\lambda}\right)^2 h^2} \left\{ c_{\Phi}^2 \int_{-\infty}^{\infty} \frac{S_0(\omega)^2}{1 + hS_0(\omega)} d\omega - \right.$$

$$\left. - \sum_{k=-\infty}^{\infty} |c_{k\Phi}|^2 \int_{-\infty}^{\infty} \frac{S_0(\omega) S_0(\omega + k\Omega)}{1 + hS_0(\omega)} d\omega \right\}^{-1}, \qquad (10.8.1)$$

where

$$c_{\Phi}^2 = \frac{1}{T_{\text{ск}} d^2} \int_0^{T_{\text{ск}}} \delta(t)^2 \, dt; \quad c_{k\Phi} = \frac{1}{T_{\text{ск}} d} \int_0^{T_{\text{ск}}} \delta(t) e^{ik\Omega t} dt; \qquad (10.8.2)$$

$T_{\text{ск}}$ – period of phase center scanning;

$\Omega = 2\pi/T_{\text{ск}}$ – angular frequency of scanning;

λ – wavelength;

d – maximum deviation of the phase center from its center positon.

In the other plane $S_{\text{опт}}$ will have the same form, only c_{Φ}^2 and $|c_{k\Phi}|^2$ must be replaced by s_{Φ}^2 and $|s_{k\Phi}|^2$, expressed analogously to (10.8.2) through the law of motion of the projection of the phase center in this plane.

Let us consider certain limiting cases. With high frequencies of scanning we obtain

$$S_{\text{опт}} = \frac{\pi}{\left(\frac{\pi d}{\lambda}\right)^2 h^2 \left(c_{\Phi}^2 - |c_{\Phi 0}|^2\right)} \left[\int_{-\infty}^{\infty} \frac{S_0(\omega)^2}{1 + hS_0(\omega)} d\omega \right]^{-1}. \qquad (10.8.3)$$

From this it is clear that the best scanning is symmetric scanning of the phase center, where $c_{\phi 0} = 0$. Then (10.8.3) takes form

$$S_{\text{опт}} = \frac{\pi}{\left(\frac{\pi d}{\lambda}\right)^2 h^2 c_\phi^2} \left[\int_{-\infty}^{\infty} \frac{S_0(\omega)^2}{1 + hS_0(\omega)} d\omega\right]^{-1}, \qquad (10.8.4)$$

coinciding with (10.4.8) with replacement of μ_a by $\pi d/\lambda$, and of c^2 by c_ϕ.

Regarding magnitudes of coefficients c_ϕ and s_ϕ^2 it is possible to conduct reasoning analogous to the reasoning regarding coefficients c^2 and s^2 for the method of pattern scanning (§ 10.4). Here it will become clear that if it is required to measure only one angle, the best scanning will be jumps of the phase center to two extreme positions in the plane of this angle, where $c_\phi^2 = 1$ and $s_\phi^2 = 0$. If, however, it is required to measure angles in two planes with identical accuracy, the law of scanning of the phase center one should select in such a manner that $c_\phi^2 = s_\phi^2 = 1/2$. Subsequently, we assume namely the latter.

At low frequencies of scanning, obviously, $S_{\text{опт}} \to \infty$, i.e., measurement of angular coordinates is impossible.

As $h \to \infty$ (decrease of noises) we obtain

$$S_{\text{опт}} = \frac{\pi}{\left(\frac{\pi d}{\lambda}\right)^2} \left\{\sum_{k=-\infty}^{\infty} |c_{k\phi}|^2 \int_{-\infty}^{\infty} \frac{S_0(\omega + k\Omega) - S_1(\omega)}{S_0(\omega)} d\omega\right\}^{-1}. \qquad (10.8.5)$$

Thus, for the method of phase center scanning due to fluctuations of the phase of the signal for finite Ω there exists error of measurement differing from zero even with complete elimination of noises.

For investigation of the general case we produce by formula (10.8.1) a calculation using approximation (10.3.26) for $S_0(\omega)$. Let us consider the case of uniform circular scanning of the phase center, i.e., with $c_{\phi\pm1} = 1/2$, and with $i \neq \mp 1$, $c_{\phi i} = 0$. Here we obtain

$$S_{\text{опт}} = \frac{1}{\left(\frac{\pi d}{\lambda}\right)^2 \Delta f_c} \frac{\sqrt{1+h}(1+\sqrt{1+h})}{h^2}\left[1 + \frac{(1+\sqrt{1+h})^2}{\zeta^2}\right], \qquad (10.8.6)$$

where $\zeta = \Omega/2\Delta f_c$.

Curves of the dependence of $S_{\text{опт}}$ on h, calculated by this formula (for different values of ζ), are shown in Fig. 10.22. From the curves it is clear that with growth of h spectral density $S_{\text{опт}}$ monotonically drops to a certain finite quantity, seeking 0 as $\zeta \to \infty$. With decrease of ζ it monotonically increases, increasing to infinity.

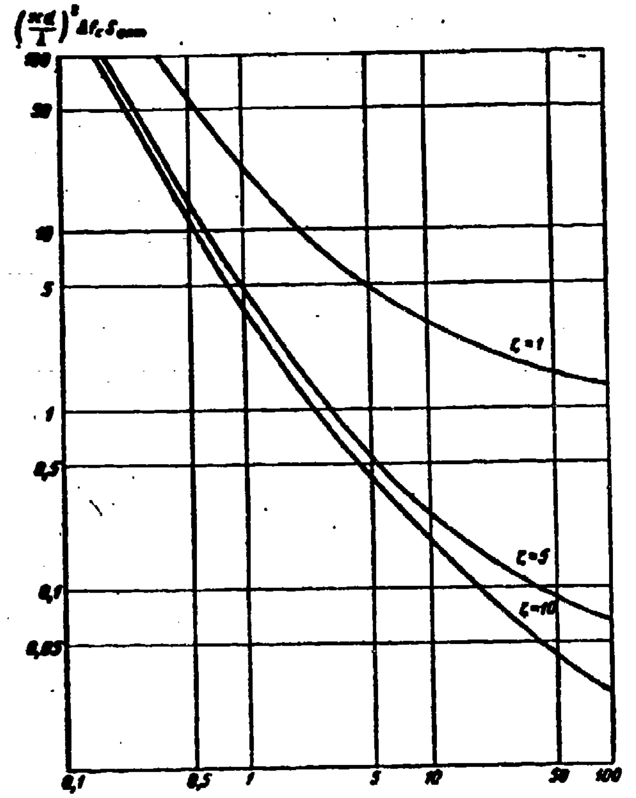

Fig. 10.22. The dependence of $S_{опт}$ on h for the method of phase center scanning.

10.8.3 Investigation of the Synthesized Circuit for the Method of Phase Center Scanning

Let us consider now the question of the real accuracy of the circuit of Fig. 10.21. We assume that filters in this circuit have pulse response h(t) cos $[\omega_{пр}t + \theta(t)]$. We introduce also designations analogous to (10.5.2). The output signal of the circuit of Fig. 10.21 here will have the form

$$z(t) = \mathrm{Re}\int_{-\infty}^{t} \dot{h}^*(t-\tau)e^{i\omega_0\tau}y(\tau)u^*(\tau)\,d\tau \times \int_{-\infty}^{t} \dot{h}(t-\tau)e^{i\omega_0\tau}y(\tau)\,iu(\tau)\,\delta(\tau)\,d\tau, \qquad (10.8.7)$$

where $y(\tau)$ — received signal.

Subsequently we shall limit ourselves to the case of uniform circular scanning of the phase center in the aperture plane, where $\sigma(t) = d\cos\Omega t$.

Calculation by the usual rules using formula (10.5.5) and (10.5.11) gives

$$K_д = \frac{P_0\left(\frac{\pi d}{\lambda}\right)}{2\pi\Delta f_e}\int_{-\infty}^{\infty} \mathrm{Re}\, H_1(i\omega)\, H^*_2(i\omega)\,[S_0(\omega) - S_0(\omega+\Omega)]\,d\omega, \qquad (10.8.8)$$

$$S_{\text{ЭКВ}} = \frac{2\pi}{\left(\frac{\pi d}{\lambda}\right)^2 h^2} \int_{-\infty}^{\infty} [|H_1(i\omega)|^2 |H_2(i\omega)|^2 + |H_1(i\omega)|^2 \times$$

$$\times |H_2(i\omega + i\Omega)|^2] [1 + hS_0(\omega)] [1 + hS_0(\omega + \Omega)] d\omega \times$$

$$\times \left\{ \int_{-\infty}^{\infty} \operatorname{Re} H_1(\omega) H^*_2(i\omega) [S_0(\omega) - S_0(\omega + \Omega)] d\omega \right\}^{-2}. \qquad (10.8.9)$$

It is also easy to calculate that systematic error $\Delta = 0$. From (10.8.9) it is clear that nonidentity of gain-frequency responses little affects accuracy of the circuit; nonidentity of phase-frequency responses leads to decrease of $K_{Д}$ and, consequently, to increase of $S_{ЭКВ}$. Here considerable divergence of phase responses can even lead to change of the sign of $K_{Д}$.

Calculation by formula (10.8.9) we perform on the assumption of identity of the filters, approximating their frequency responses by formula (10.5.12). Here we obtain for $S_{ЭКВ}$ an expression coinciding with (10.5.17), only factor $\zeta^2(1 + x)^2/\mu_a \Delta f_c \; [\zeta^2 + 2(1 + x)^2]^2$ before the braces must be replaced by the simpler $\zeta^2 + (1 + x)^2/(\pi d/\lambda)^2 \Delta f_c \; \zeta^4$. This will introduce definite changes in the law of the dependence of $S_{ЭКВ}$ on h, x and ζ. In particular, for small x we will have the following expression for $S_{ЭКВ}$:

$$S_{\text{ЭКВ}} = \frac{1}{2\left(\frac{\pi d}{\lambda}\right)^2 \Delta f_c x} \frac{\zeta^2 + 1}{\zeta^4} \left(1 + \frac{\zeta^2 + 2}{h} + \frac{\zeta^2 + 1}{h^2}\right), \qquad (10.8.10)$$

from which we see infinite growth of $S_{ЭКВ}$ as $\zeta \to 0$. For large x we obtain

$$S_{\text{ЭКВ}} = \frac{2x^4}{\left(\frac{\pi d}{\lambda}\right)^2 \Delta f_c h^2 \zeta^4}. \qquad (10.8.11)$$

Here we see the exceptional criticality of the circuit to expansion of the passband of the filters. Equivalent spectral density increases proportionally to x^4. However, the dependence on the frequency of scanning is preserved. Increase of the frequency of scanning leads to considerable decrease of the equivalent spectral density.

With respect to limiting cases for ζ we note that for large ζ the expression for $S_{ЭКВ}$ will coincide with (10.5.20) (with replacement of μ_a by $\pi d/\lambda$), i.e., in this case accuracy of the considered circuit coincides with the accuracy of the circuit of Fig. 10.8 for the method of pattern scanning. For small ζ we have

$$S_{\text{ЭКВ}} = \frac{(1 + x)^4}{\left(\frac{\pi d}{\lambda}\right)^2 \Delta f_c \zeta^4} \left(2 + 1/x + 2\,\frac{2 + 1/x}{h} + \frac{x + 2 + 1/x}{h^2}\right), \qquad (10.8.12)$$

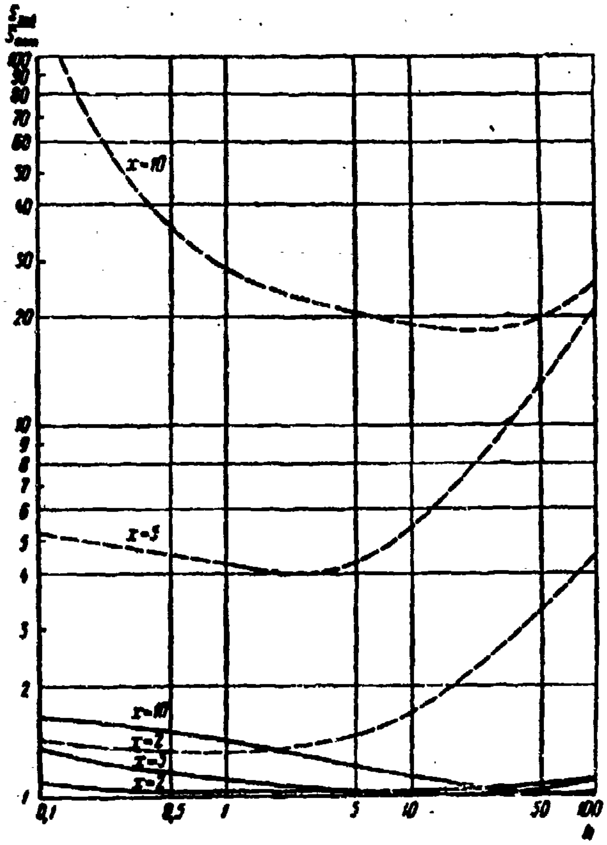

Fig. 10.23. The dependence of $S_{\text{экв}}/S_{\text{опт}}$ on h for a quasi-optimum circuit of the radio channel of a goniometer with phase center scanning: —— ζ = 100; - - - ζ = 10.

i.e., the drop of accuracy is proportional to ζ^4.

Curves of the dependence of $S_{\text{экв}}/S_{\text{опт}}$ on h, calculated with these same approximations, are shown in Fig. 10.23.

As can be seen from these curves, with high frequencies of scanning ($\zeta \approx 100$) the considered circuit sufficiently well realizes potential accuracy of the method in a wide range of variation of x(x = 2-10). For lower frequencies of scanning (ζ = 10) accuracy of the circuit already will sharply depend on x (for h = 5 change of x from 2 to 10 leads to increase of $S_{\text{экв}}$ by approximately 20 times).

Thus, results of investigation of the method of phase center scanning can be summarized in the following way: this method is applicable only with sufficiently high frequencies of scanning ($\zeta \approx 100$), where it ensures accuracy equal to the accuracy of high-frequency scanning of the directional pattern (or the method of scanning with compensation or IAC), only $\mu_a = \pi d/\lambda$ and the received powers are identical; the synthesized circuit of the method of phase scanning is not very critical to the degree of matching of filters with the spectrum of fluctuations of signal only at very high frequencies of scanning; already at $\zeta \approx 10$ it becomes so critical that it is possible to place in doubt the expediency of its practical realization.

From the obtained results it is clear that in many cases the method of phase center scanning is less desirable in comparison with the above-considered amplitude methods of direction finding.

§ 10.9. Method of Instantaneous Phase Comparison of Signals

10.9.1. Optimum Circuits and Potential Accuracy for IPC

The method of instantaneous phase comparison of signals (IPC) is the phase analog of the IAC method. In IPC circuits the presence of several outputs of the antenna array with separated phase centers means that phase fluctuations of the received signals differ only by a constant component, and, consequently, there exist the prerequisites for compensation of phase fluctuations. This indicates that from the IPC method we are justified in expecting higher accuracy of measurements during operation on a fluctuating signal, than from the method of phase center scanning.

The circuit of the optimum radio channel for IPC is easy to obtain from the general circuit of Fig. 10.7 if we there set n = 2 (two receiving antennas), $U_{ai}(t, \alpha) = 1$ (amplitudes of the received signals are identical), and we introduce $\Phi_i(t, \alpha)$ according to expressions (10.2.8). The circuit of the radio channel of IPC is obtained here in the form depicted in Fig. 10.24, which can easily be absolutely identically transformed to the form depicted in Fig. 10.25.

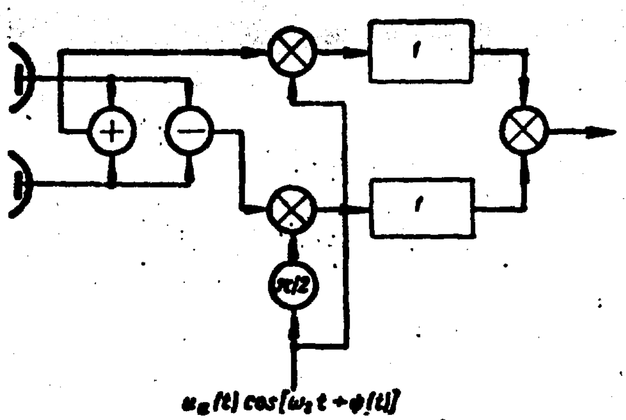

Fig. 10.24. Variant of the optimum circuit with formation of the sum and difference signals of the radio channel of a goniometer for IPC. 1) optimum filters with frequency response $H_{1\,\text{опт}}(i\omega)$.

In both variants of optimum IPC circuits signals from the output of the antenna system after transformations of the type of addition or subtraction of these signals enter two channels. There the signals are heterodyned, where heterodyne signals proceeding into these channels are shifted in phase $\pi/2$. Then the signals are filtered by an optimum filter of type (10.3.25) and are multiplied. Both circuits are very simple and are similar to the circuit of the radio channel with phase center scanning (if in the latter we exclude multiplication by the function varying according to the law of scanning).

Potential accuracy of IPC can be calculated by formula (10.3.31) if there we set n = 2, $U_{ai}(t, \alpha) = 1$, and introduce $\Phi_i(t, \alpha)$ by formula (10.2.8). Then

$$S_{\text{опт}} = \frac{2\pi}{\left(\frac{\pi d}{\lambda}\right)^2 k_\Sigma^2} \left[\int_{-\infty}^{\infty} \frac{S_0(\omega)^2}{1 + k_\Sigma S_0(\omega)} d\omega \right]^{-1}. \tag{10.9.1}$$

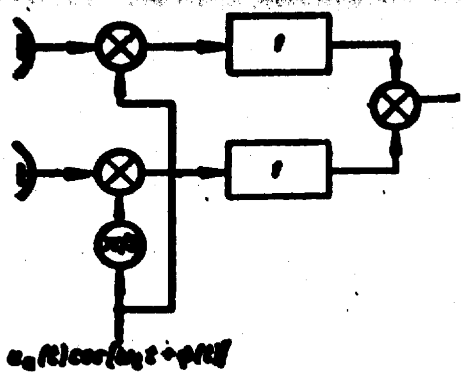

Fig. 10.25. Variant of the optimum circuit (without formation of the sum and difference signals) of the radio channel of a goniometer for IPC. 1) optimum filters with frequency response $H_{1\ \text{опт}}(i\omega)$.

This formula will completely coincide with (10.6.5), characterizing the potential accuracy of the method of scanning with compensation and IAC, if there we replace μ_a on by $\pi d/\lambda$ (as we agreed, $c^2 = 1/2$). It follows from this that potential accuracy of IPC is identical with potential accuracy of IAC (or the method of scanning with compensation), if the total received powers in both cases are identical and if $\mu_a = \pi d/\lambda$. Accuracy of the method of phase center scanning nears the accuracy of IPC only at high frequencies of scanning (if the received total powers in both cases are identical).

The dependence of $S_{\text{опт}}$ on different factors was studied in sufficient detail in § 10.6.

10.9.2. Investigation of Synthesized Circuits for IPC

Let us analyze real accuracy of synethesized circuits of IPC. All assumptions about circuits and designations we keep the same as in the preceding paragraph. We consider first the circuit of Fig. 10.24. The output signal of this circuit has the form

$$z(t) = \operatorname{Re} \int_{-\infty}^{t} \dot{h}^*_1(t-\tau)\,[y_1(\tau)+y_2(\tau)]\,u^*(\tau)\,e^{-i\omega_0\tau}\,d\tau \times \\ \times \int_{-\infty}^{t} \dot{h}_2(t-\tau)\,[y_1(\tau)-y_2(\tau)]\,iu(\tau)\,e^{i\omega_0\tau}\,d\tau, \tag{10.9.2}$$

where $y_1(t)$ — are the received signals.

Substituting this expression in (10.5.5), (10.5.6), (10.5.11) and (10.5.22), we can calculate $K_Д$, Δ, $S_{\text{экв}}$ and $S_{\text{пар}}$. It is interesting to note that the results here are absolutely the same as for the circuit of Fig. 10.15 or for the circuit of Fig. 10.17, only μ_a is replaced by $\pi d/\lambda$, i.e., $K_Д$ is expressed by formula (10.7.2), $S_{\text{экв}}$ by formula (10.7.3), $\Delta = 0$, and $S_{\text{пар}}$ (with identical filters) is expressed by formula (10. 7. 6). Investigation of accuracy of these circuits was conducted in detail in § 10.7. We recall the basic results of this investigation: with identical channels the circuit well realizes potential accuracy of measurement; the degree of matching of filters with the spectrum of signal fluctuations weakly influences

accuracy of the circuit; the greatest danger to identity of channels is nonidentity of phase-frequency responses of filters, leading to decrease of $K_Д$ and increase of $S_{ЭКВ}$.

The circuit of Fig. 10.25 in its properties, as it turns out, considerably differs from the circuit of Fig. 10.24. The output signal of the circuit of Fig. 10.25 has, obviously, the form (10.9.2), only under the sign of the first integral instead of the combination of received signals there should be $y_1(\tau)$, and under the second integral – $y_2(\tau)$. Substitution of this expression in (10.5.5) after simple calculations gives for $K_Д$ again the expression (10.7.2) (with replacement of μ_a by $\pi d/\lambda$). However, the other characteristics of the circuit turn out to be different. In particular, $S_{ЭКВ}$ for this circuit equals

$$S_{\text{экв}}=\frac{\pi}{\left(\frac{\pi d}{\lambda}\right)^2 h_\Sigma^2}\left\{\int_{-\infty}^{\infty}[|H_1(i\omega)|^2|H_2(i\omega)|^2(1+h_\Sigma S_0(\omega))^2-\right.$$
$$-\frac{1}{4}h_\Sigma^2 S_0(\omega)^2\operatorname{Re}H_1(i\omega)^2H^*{}_2(i\omega)^2]\,d\omega+$$
$$\left.+\frac{1}{2}h_\Sigma^2\int_{-\infty}^{\infty}\operatorname{Im}H_1(i\omega)H^*{}_2(i\omega)S_0(\omega)\,d\omega\right\}\times\left[\int_{-\infty}^{\infty}\operatorname{Re}H_1(i\omega)H^*{}_2(i\omega)S_0(\omega)\,d\omega\right]^{-2}. \qquad (10.9.3)$$

Furthermore, systematic error Δ for this circuit is different from zero and is equal to

$$\Delta=-\frac{1}{2\frac{\pi d}{\lambda}}\frac{\int_{-\infty}^{\infty}\operatorname{Im}H_1(i\omega)H^*{}_2(i\omega)S_0(\omega)\,d\omega}{\int_{-\infty}^{\infty}\operatorname{Re}H_1(i\omega)H^*{}_2(i\omega)S_0(\omega)\,d\omega}. \qquad (10.9.4)$$

With identical filters, obviously, accuracy of both circuits of Fig. 10.24 and Fig. 10.25 are identical. However, nonidentity of filters in the circuit of Fig. 10.25 leads to considerably more serious consequences than in the circuit of Fig. 10.24. Analysis of expression (10.9.3) shows that with nonidentical filters in the circuit of Fig. 10.25 there occurs decrease of $K_Д$ and, consequently, increase of $S_{ЭКВ}$ to the same measure as for the circuit of Fig. 10.24. However, nonidentity of filters here leads to the appearance of a component of equivalent spectral density caused by nonlinear signal transformation. This component of equivalent spectral density causes, in particular, error of measurement in the considered circuit even with total absence of noises. Finally, a very negative property of the considered circuit is the presence of systematic error with nonidentical filters. Let us note that the basic influence on accuracy of the circuit is rendered only by nonidentity

of phase-frequency responses of the filters. To the above we must add that the circuit of Fig. 10.25 is more complicated technically than the circuit of Fig. 10.24, since it, like the circuit of Fig. 10.18, requires for measurement of the angles in two planes four UPCh's, four mixers with inversion of phase modulation, etc., while the circuit of Fig. 10.23 in this respect is similar to the circuit of Fig. 10.17 and requires only three UPCh's, etc. On the basis of this we can draw conclusions concerning the obvious advantages of the circuit of the IPC method of Fig. 10.24 over the circuit of Fig. 10.25. Let us note that investigation of real properties of circuits leads to this conclusion. In ideal conditions both circuits are identical.

Thus, the IPC method has the same potential accuracy as IAC or the method of scanning with compensation, if μ_a and $\pi d/\lambda$ are identical. Accuracy close to the potential for the IPC method can be realized with success in practice by the circuit of Fig. 10.24.

§ 10.10. The Method of Flat Scanning (Optimum Circuits)

Let us study the method of flat scanning or the method of tracking by pulse packs. With this method, as was shown in § 10.2, there is realized tracking of pulse packs of the received signal by an electronic circuit. Pulse packs are received at the moments of passage of the directional pattern through the target during its scanning in a certain angular sector, in which this target is located.

Thus, the method of tracking by pulse packs no longer belongs to methods with a tracking antenna, and the preceding results, in particular, general results of § 10.3, here are not directly applicable. Synthesis of the optimum circuit here should be done anew.

10.10.1. Synthesis of an Optimum Circuit for the Method of Flat Scanning

Let us assume that the directional pattern (for power) of the antenna array of a goniometer using the method of tracking by pulse packs has in plane of the measured angle α symmetric form $Gg(\varphi)$. For $g(\varphi)$ we select the following normalization:

$$\frac{1}{\Omega T_{\pi}} \int_{-\infty}^{\infty} g(\varphi)^2 \, d\varphi = 1, \qquad (10.10.1)$$

where Ω – angular velocity of motion of the pattern over the sector;

T_{π} – period of repetition of pulse packs.

Antenna gain is accounted for by coefficient G. Here the received signal can be recorded in the form (10.3.3), where $\Phi(t, \alpha) = 0$ (the antenna array during reception does not introduce in the received signal any phase modulation), and $U_a(\tau, \alpha)$ is determined by formula (10.2.6). The amplitude and phase modulations of the sounding signal are arbitrary.

We introduce one assumption with respect to the character of fluctuations of the received signal. The assumption we make here reduces to the fact that reflecting properties of the target do not change during the time during which the directional pattern passes over the target. This will lead the signal inside the pack to be rigidly correlated or, so to speak, the pack will be "harmoniously fluctuating." The degree of correlatedness of different packs may, in general, be arbitrary, since the period of repetition of the packs almost always is comparible with the time of correlation of fluctuations of the signal. The assumption of "harmoniously fluctuating" packs is very signfificant later.

Furthermore, we shall subsequently consider that the period of repetition T_r is considerably less than the duration of a pulse pack. This assumption, in general, is more limiting than the preceding one; however, in many cases it to greater or lesser extent is realized. Furthermore, it also will considerably facilitate our finding of optimum operations for the method of tracking by pulse packs.

To find the operation of the optimum radio channel for the method of tracking by packs we need, as usual, to first construct the likelihood functional $L(\alpha)$ of parameter α. For this we must solve integral equation (10.3.9) or, ultimately, equation (10.3.14). Solution of the last equation for the method of tracking by pulse packs involves considerable difficulties, since here that method which we used in examining goniometers with a tracking antenna is unapplicable. However here very useful is our assumption of "harmoniously fluctuating packs." Really, function $U_a(t_2, \alpha)^2$, under the sign of the integral in (10.3.14) differs from zero only in the environments of moments $iT_\Pi + \alpha/\Omega$, determined by the width of the directional pattern and the rate of scanning (i – integers). We assume for definitiveness that

$$t-\Delta t \leq (n-m)T_\Pi + \alpha/\Omega < (n-m+1)T_\Pi + \\ + \alpha/\Omega < \ldots < nT_\Pi + \alpha/\Omega \leq t. \quad (10.10.2)$$

The assumption of harmonious fluctuations of the signal inside the packs means that during a time of the order of duration of one pack the correlation function of fluctuations of signal $\rho(t)$ practically does not vary. We also consider the assumption of smallness of the period of repetition of the signal T_r as compared to the

duration of the pack, thanks to which it is possible under the sign of the integral in (10.3.15) to replace $|u(t_2)|^2$ by its time-averaged value, which is equal to 1. Here from equation (10.3.15) we can obtain the following expression (see also Chapter VI):

$$\Delta f_c h \sum_{k=n-m}^{n} \rho_{ik} v_{kl} + \rho_{il} + v_{il} = 0, \tag{10.10.3}$$

where h, as before, is the ratio of mean signal power to the power of noise in the band of fluctuations of the signal,

$$\rho_{ik} = \rho[(i-k)T_\text{п}], \quad v_{ik} = v(iT_\text{п} + \alpha/\Omega,\ kT_\text{п} + \alpha/\Omega). \tag{10.10.4}$$

This equation is solved using discrete Fourier transforms. Considering

$$S_{0T_\text{п}}(\omega) = T_\text{п} \Delta f_c \sum_{k=-\infty}^{\infty} \rho(kT_\text{п}) e^{-i\omega kT_\text{п}} \tag{10.10.5}$$

it is simple to obtain

$$v_{jl} = -\frac{1}{2\pi \Delta f_c} \int_{-\pi/T_\text{п}}^{\pi/T_\text{п}} \frac{S_{0T_\text{п}}(\omega) e^{i\omega(l-j)T_\text{п}}}{1 + hS_{0T_\text{п}}(\omega)}\, d\omega. \tag{10.10.6}$$

We note that, as can be seen from (10.10.6), v_{jl} does not depend on α. Thus, we could find, not the whole function $v(t_1, t_2)$, but only a set of its values in discrete moments of time (10.10.2). However, this very set of values is necessary to us for construction of the operation of the optimum radio channel.

The operation of the optimum radio channel $z(t)$ for the considered case should be found from relationship

$$\int_{t-\Delta t}^{t} z(\tau)\, d\tau = \frac{\partial \ln L(\alpha)}{\partial \alpha}\bigg|_{\alpha=\hat{\alpha}}, \tag{10.10.7}$$

where $\hat{\alpha}$ – estimated value of angle α, introduced in the radio channel from the output of the tracking circuit [this is the distinction between (10.10.7) and (10.3.12)].

Hence

$$\int_{t-\Delta t}^{t} z(\tau) d\tau = -\frac{\partial}{\partial \alpha} \frac{\sqrt{P_c}}{N_0} \operatorname{Re} \int_{t-\Delta t}^{t} \int_{t-\Delta t}^{t} U_a(t_1, \alpha) U_a(t_2, \alpha) \times$$
$$\times u(t_1) u^*(t_2) v(t_1, t_2) y(t_1) y(t_2) e^{i\omega_0(t_1 - t_2)} dt_1 dt_2 \Big|_{\alpha=\hat{\alpha}} \tag{10.10.8}$$

Due to the presence of function $U_a(t_1, \alpha)U_a(t_2, \alpha)$ in (10.10.8) under the sign of the integral the integrand will differ from zero only in the environment of moments of time (10.10.2) where function $v(t_1, t_2)$ takes value v_1. In connection with this (10.10.8) can be rewritten in the form

$$\int_{t-\Delta t}^{t} z(\tau)\,d\tau = -\frac{\partial}{\partial\alpha}\frac{\sqrt{P_c}}{N_0}\sum_{i,k=n-m}^{n} v_{ik}\,\mathrm{Re}\, z_i z^*_k, \tag{10.10.9}$$

where

$$z_i = \int_{lT_r+\alpha/\Omega-\delta t}^{lT_r+\alpha/\Omega+\delta t} U_a(t,\alpha)\,u(t)\,y(t)\,e^{i\omega_0 t}\,dt; \tag{10.10.10}$$

δt — intervals of time of the order of the duration of a pulse pack.

Consider now quantity h_{0k}, related to v_{ik} by relationship

$$T_\pi \sum_{k=-\infty}^{\infty} h_{0(i-k)} h_{0(k-l)} = -v_{il}\,h\Delta f_c. \tag{10.10.11}$$

If we introduce the discrete Fourier transform of sequence h_{0k}

$$H_{0T_\pi}(i\omega) = T_\pi \sum_{k=-\infty}^{\infty} h_{0k}\,e^{i\omega k T_\pi}, \tag{10.10.12}$$

from (10.10.11) it is easy to find that

$$|H_{0T_\pi}(i\omega)|^2 = \frac{hS_{0T_\pi}(\omega)}{1+hS_{0T_\pi}(\omega)}, \tag{10.10.13}$$

where $S_{0T_\pi}(\omega)$ is expressed by formula (10.10.15). With the help of h_{0k} expression (10.10.9) can be rewritten (with accuracy to a proportionality factor) in the form

$$\int_{t-\Delta t}^{t} z(\tau)\,d\tau \sim \frac{\partial}{\partial\alpha}\sum_{k=n-m}^{n} T_\pi \left|\sum_{l=n-m}^{n} h_{k-l} z_l\right|^2, \tag{10.10.14}$$

Then, considering (10.10.10), we can write

$$\sum_{l=n-m}^{n} h_{k-l} z_l = \sum_{l=n-m}^{n} h_{k-l} \int_{lT_r+\alpha/\Omega-\delta t}^{lT_r+\alpha/\Omega+\delta t} U_a(t,\alpha)\,u(t)\,y(t)\,e^{i\omega_0 t}\,dt =$$
$$= \int_{t-\Delta t}^{t} h_0(kT_r+\alpha/\Omega-t)\,U_a(t,\alpha)\,u(t)\,y(t)\,e^{i\omega_0 t}\,dt, \tag{10.10.15}$$

where function $h_0(t)$ is introduced in such a way that $h_0(kT_r) = h_{0k}$; values of this function in the remaining moments of time are not essential and can be anything, inasmuch as due to filtering properties of function $U_a(t, \alpha)$ they simply disappear. Substituting (10.10.15) in (10.10.14), we obtain

$$\int_{t-\Delta t}^{t} z(\tau)d\tau \sim \frac{\partial}{\partial\alpha}\sum_{k=n-m}^{n} T_\pi \left|\int_{t-\Delta t}^{t} h_0(kT_\pi+\alpha/\Omega-\tau)\,U_a(\tau,\alpha)\,\times\right.$$
$$\left.\times\, u(\tau)\,y(\tau)\,e^{i\omega_0\tau}d\tau\right|^2 \approx$$
$$\approx \frac{\partial}{\partial\alpha}\int_{t-\Delta t}^{t} ds \left|\int_{t-\Delta t}^{t} h_0(s-\tau)\,U_a(\tau,\alpha)\,u(\tau)\,y(\tau)\,e^{i\omega_0\tau}\,d\tau\right|^2$$

or

$$z_0(s) \approx \frac{\partial}{\partial\alpha}\left|\int_{t-\Delta t}^{t} h_0(s-\tau)U_a(\tau, \alpha)\,u(\tau)\,y(\tau)e^{i\omega_0\tau}\,d\tau\right|^2. \tag{10.10.16}$$

Thus, in the considered case the optimum operation already has a familiar form, derived earlier (see Chapter VI). However, the pulse response of the optimum filter $h_0(t)$ is assigned by the set of its values h_{0k} in moments of time kT_{Π} by relationships (10.10.12) and (10.10.13). In the remaining moments of time values of pulse response are arbitrary.

Let us consider this optimum filter in greater detail. We introduce for the correlation function of fluctuations of the signal the approximation $\rho(t) = e^{-2\Delta f_c |t|}$, i.e., we approximate the spectrum of fluctuations by expression (10.3.26). Here, from (10.10.5) we obtain

$$S_{0T_\Pi}(\omega) = \frac{\Delta f_c T_\Pi (1-\rho_c^2)}{\rho_c^2 - 2\rho_c \cos\omega T_\Pi + 1}, \tag{10.10.17}$$

where $\rho_c = e^{-2\Delta f_c T_\Pi}$.

By virtue of (10.10.13) we have

$$|H_{0T_\Pi}(i\omega)|^2 = \frac{hT_\Pi \Delta f_c (1-\rho_c^2)}{hT_\Pi \Delta f_c (1-\rho_c^2) + \rho_c^2 - 2\rho_c \cos\omega T_\Pi + 1}. \tag{10.10.18}$$

It is of interest to find the pulse response $h_0(t)$ of the optimum filter in the form

$$h_0(t) = ce^{-2\Delta f_{0\Pi T} t}, \tag{10.10.19}$$

i.e., the gain-frequency response of this filter has the form

$$|H_0(i\omega)|^2 = \frac{c}{1+\left(\frac{\omega}{2\Delta f_{0\Pi T}}\right)^2}.$$

Here $|H_{0T_\Pi}(i\omega)|^2$, defined by relationship (10.10.18), should be the Fourier transform of sequence $ce^{-2\Delta f_{0\Pi T} kT_\Pi}$. We bring (10.10.18) to form

$$|H_{0T_\Pi}(i\omega)|^2 = \frac{hT_\Pi \Delta f_c (1-\rho_c^2)}{A^2} \Big/ \left|1 - \frac{B}{A} e^{-i\omega T_\Pi}\right|^2, \tag{10.10.20}$$

where

$$A = \frac{1}{2}\sqrt{hT_\Pi \Delta f_c (1-\rho_c^2) + (1+\rho_c)^2} + \sqrt{hT_\Pi \Delta f_c (1-\rho_c^2) + (1-\rho_c)^2};$$

$$B = \frac{1}{2}\sqrt{hT_\Pi \Delta f_c (1-\rho_c^2) + (1+\rho_c)^2} - \sqrt{hT_\Pi \Delta f_c (1-\rho_c^2) + (1-\rho_c)^2}. \tag{10.10.21}$$

From this it is easy to find that the pulse response can be taken namely in the form (10.10.19),

where

$$\Delta f_{\text{опт}} = \frac{\ln A/B}{2T_{\text{п}}}, \tag{10.10.22}$$

A and B are determined by formulas (10.10.21), and

$$c = \frac{\sqrt{hT_{\text{п}}\Delta f_{\text{c}}(1-\rho_{\text{c}})}}{A}.$$

Thus, the optimum filter, as also in the preceding cases, has a frequency response similar to the spectrum of fluctuations of the signal; however the width of the filter passband has a more complicated dependence on h. The passband also essentially depends on the degree of correlatedness of neighboring pulse packs, i.e., on quantity $T_{\text{п}}\ \Delta f_{\text{c}}$. In particular, when $T_{\text{п}}\ \Delta f_{\text{c}} \gg 1$ (neighboring pulse packs fluctuate independently) and $\tau_{\text{п}} \ll T_{\text{п}}$ the passband of the optimum filter should satisfy condition

$$\frac{1}{T_{\text{п}}} < \Delta f_{\text{опт}} < \frac{1}{\tau_{\text{п}}}, \tag{10.10.23}$$

where $\tau_{\text{п}}$ — duration of the pack.

Consequently, the optimum filter in this case should integrate every pack, where response of the filter to a given pack should attenuate before the arrival of the following pack. In Fig. 10.26 there is depicted the dependence of $\Delta f_{\text{опт}}$ on h for different values of $T_{\text{п}}\ \Delta f_{\text{c}}$. As can be seen from this figure, with growth of h, $\Delta f_{\text{опт}}$ increases; however the rate of this growth drops with increase of $T_{\text{п}}\ \Delta f_{\text{c}}$ (i.e., with constant h with growth of the number of correlated packs the passband of the filter narrows). Physically these laws are understandable.

Operation (10.10.6) can easily be reduced to a real form analogously to how we obtained expression (10.3.29). Here there become clear the block diagrams of the optimum radio channel realizing operation (10.10.16). We can offer two block diagrams of the optimum radio channel, depicted in Figs. 10.27 and 10.28. The circuit of Fig. 10.27 is obtained when in expression (10.10.6) we carry out exact differentiation with respect to parameter α. In this circuit the output signal from the antenna is first heterodyned, as in other optimum circuits, and then enters two channels. In these channels the signal is gated. Gate pulses in one channel in form should coincide with the form of the pack, and in the other channel — with the derivative of the pack. The time position of the gate pulses is established in

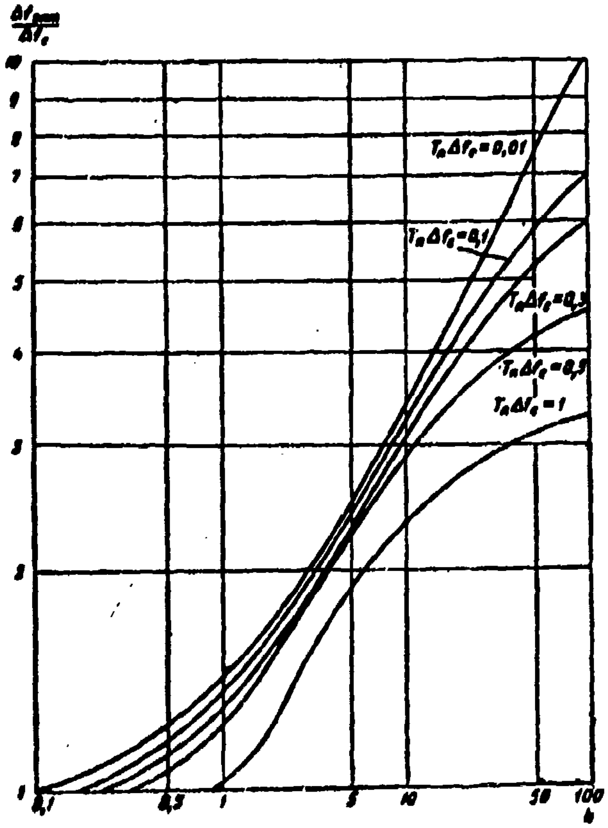

Fig. 10.26. Dependence of the bandwidth of the optimum filter $\Delta f_{OПТ}$ on h for goniometers using the method of flat scanning.

accordance with the measured delay of the packs (i.e., in accordance with the measured angular position of the target). After gating, the signals are filtered by optimum filters with response (10.10.18) and are multiplied.

The block diagram of Fig. 10.28 is obtained if we replace calculation of the derivative in (10.10.6) approximately by calculation of the difference. The signal in this circuit after heterodyning as in the preceding case, enters two channels where it is gated. Gate pulses in both channels coincide in form with the form of the pulse pack; however, they are detuned relative to each other in time by a certain time interval. The time position of these gate pulses is established in accordance with the measured delay of the packs; however their relative detuning is preserved. Then the signals are filtered, detected by a square-law detector, and one signal is subtracted from the other.

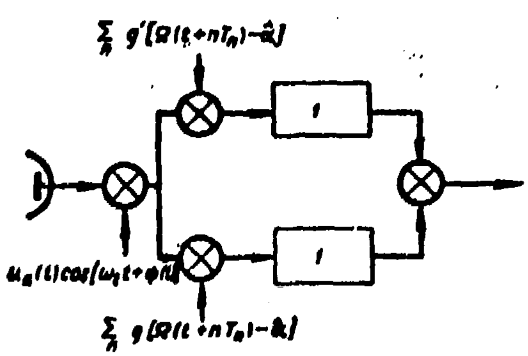

Fig. 10.27. Optimum circuit with differentiation of gates of the radio channel of a goniometer with flat scanning of the directional pattern: 1) optimum filters with discrete frequency response $H_{OT\,П}(i\omega)$.

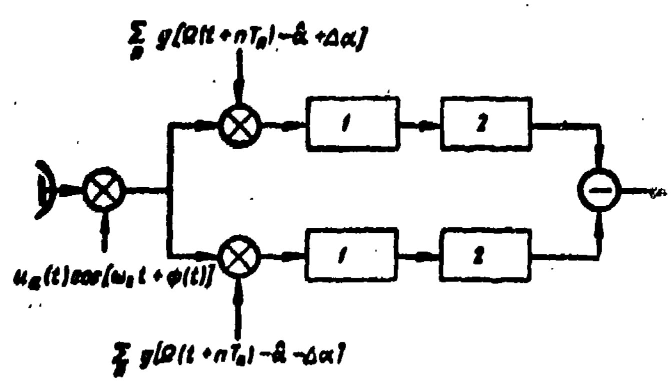

Fig. 10.28. Optimum circuit with detuned gates of the radio channel of a goniometer with flat scanning of the directional pattern: 1) optimum filters with discrete frequency response $H_{OT\,П}(i\omega)$; 2) square-law detectors.

The obtained circuits are very similar to the circuits of the optimum radio channel of a range finder, if we ignore the form of the optimum filters and of the gate pulses. This is understandable, since in both cases the matter ultimately leads to measurement of delay of the signal relative to the beginning of the period. Filtration was different due to the different fluctuating properties of the received signal: if in range finders pulses of the signal whose delay was measured were rigidly correlated, the correlation coefficient of the equivalent packs of pulses in the method of angular tracking by packs no longer is equal to unity. The form of gate pulses is determined in the given case by the form of the pulse packs.

10.10.2. Equivalent Spectral Density of Optimum Circuits for the Method of Flat Scanning

Let us consider now the potential accuracy of measurements using the method of flat scanning. Equivalent spectral density in this case can be presented in the form

$$S_{\text{опт}} = \left\{ \frac{2\Delta f_{c}}{\Delta t} \int_{t-\Delta t}^{t} [1 + \rho(\tau, \tau)] \, |u(\tau)|^{2} \times \right.$$
$$\left. \times \left[\sum_{i} u'_{\text{н}} \left(\tau - iT_{\text{н}} - \frac{\hat{\varepsilon}}{\Omega} \right) \right]^{2} d\tau \right\}^{-1} = \frac{2\pi}{c^{2}h^{2}} \left\{ \int_{-\pi/T_{\text{н}}}^{\pi/T_{\text{н}}} \frac{S^{2}_{0\Gamma_{\text{н}}}(\omega)}{1 + hS_{0\Gamma_{\text{н}}}(\omega)} d\omega \right\}^{-1}, \tag{10.10.24}$$

where

$$c^{2} = \frac{1}{\Omega T_{\text{н}}} \int_{-\infty}^{\infty} g'(\varphi)^{2} d\varphi. \tag{10.10.25}$$

The form of the directional pattern affects accuracy through quantity c^2. Let us calculate this magnitude for some approximation of the directional pattern. If w take a Gaussian directional pattern

$$g(\varphi) = 1{,}15 \sqrt{\frac{\Omega T_{\text{н}}}{\Delta \varphi}} \, e^{-2,8 \left(\frac{\varphi}{\Delta \varphi} \right)^{2}} \tag{10.10.26}$$

[taking into account normalization (10.10.1)], where $\Delta\varphi$ — width of the directional pattern of a level of half power, then it is easy to find that

$$c^{2} = \frac{2{,}8}{\Delta \varphi^{2}}. \tag{10.10.27}$$

From (10.10.25) it is clear that c^2 increases infinitely with increase of the steepness of the leading edge of the pack.

Let us consider formula (10.10.24) in greater detail. Obviously, as $h \to \infty$ quantity $S_{\text{ОПТ}} \to 0$, i.e., in the absence of noises there does not exist fluctuating error of target tracking, in spite of fluctuations of the received signal. As $h \to 0$

$$S_{\text{опт}}=\frac{1}{c^2h^2T_\text{п}\Delta f_c^2}\Big[\sum_{k=-\infty}^{\infty}\rho\,(kT_\text{п})^2\Big]^{-1}, \tag{10.10.28}$$

i.e., accuracy decreases inversely proportionally to the square of the signal-to-noise ratio. We also give the formula occurring for $T_\text{п} \ \Delta f_c \gg 1$, i.e., with independently fluctuating pulse packs:

$$S_{\text{опт}}=\frac{1}{c^2h\Delta f_c}\left(1+\frac{1}{hT_\text{п}\Delta f_c}\right). \tag{10.10.28'}$$

To more fully judge the laws governing variation of $S_{\text{опт}}$, we calculate it by formula (10.10.24) with approximation of the spectrum of signal fluctuations by expression (10.3.26). Here, as was shown, $S_{\text{от}_\text{п}}(\omega)$ is expressed by formula (10.10.5). Substituting (10.10.5) in (10.10.24), we obtain

$$S_{\text{опт}}=\frac{1}{\Delta f_c c^2h^2}\sqrt{(hT_\text{п}\Delta f_c)^2+2hT_\text{п}\Delta f_c\frac{1+\rho_c^2}{1-\rho_c^2}+1}\times\frac{1+\sqrt{(hT_\text{п}f_c)^2+2hT_\text{п}\Delta f_c\frac{1+\rho_c^2}{1-\rho_c^2}+1}}{h(T_\text{п}\Delta f_c)^2+2T_\text{п}\Delta f_c\frac{1+\rho_c^2}{1-\rho_c^2}}. \tag{10.10.29}$$

Curves of the dependence of $S_{\text{опт}}$ on h for different values of $T_\text{п} \ \Delta f_c$ are shown in Fig. 10.29. As can be seen from this figure, $S_{\text{опт}}$ drops rather rapidly with increase of h. Very curious is the dependence of $S_{\text{опт}}$ on $T_\text{п} \ \Delta f_c$. With growth of $T_\text{п} \ \Delta f_c$ for an assigned magnitude of h quantity $S_{\text{опт}}$ drops, i.e., accuracy of angle measurement increases. The physical explanation of this phenomenon is the following. Increase of $T_\text{п} \ \Delta f_c$ for a fixed h means that there occurs decrease of the frequency of repetition of packs with increase of the energy in each pack. As a result there is increased, if it is possible to express it thus, the energy of signal sections which retain their coherence during reflection from a fluctuating target. Increase of accuracy due to this effect depends on h. For large h it is insignificant; for h = 1, $S_{\text{опт}}$ will decrease approximately by a factor of 4 (with change of $T_\text{п} \ \Delta f_c$ from 0.3 to 100).

In conclusion we note that for the method of tracking by packs it is sometimes profitable to take as the power characteristic the ratio of maximum signal power (which would exist if the directional pattern of the antenna always "looked" at the target) to the power of noise in the band of signal fluctuations. Designating this characteristic by h_m, it is easy to find its relationship to h [with approximation

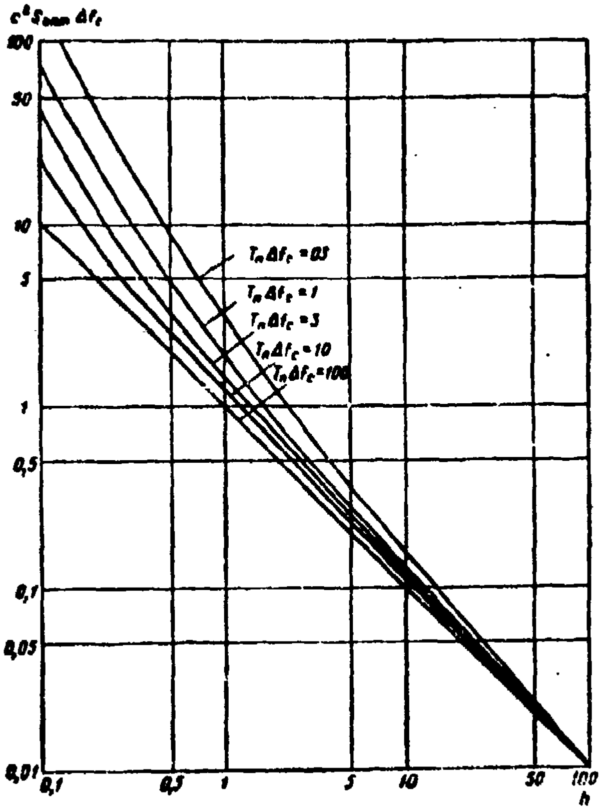

Fig. 10.29. Dependence of $S_{\text{опн}}$ on h for the method of flat scanning of the directional pattern.

(10.10.26) of the directional pattern]

$$h_m = 1{,}32\frac{\Omega T_{\Pi}}{\Delta\varphi}h. \qquad (10.10.30)$$

If we designate by Φ the magnitude of the examined angular sector and consider that $\Omega T_{\Pi} = \Phi$, from (10.10.30) we obtain

$$h_m = 1{,}32\frac{\Phi}{\Delta\varphi}h. \qquad (10.10.31)$$

We note that the order of magnitude of h_m for the method of tracking by pulse packs is comparable with the order of magnitude h for methods with a tracking antenna (since with these methods the directional pattern of the antenna system always "looks" at the target). Inasmuch as usually $\Delta\varphi \ll \Phi$, for the method of tracking by pulse packs $h \ll h_m$, i.e., the method of tracking by pulse packs in terms of power is considerably inferior to methods with a tracking antenna. However, here it is necessary to consider other tactical functions of the method of tracking by pulse packs (tracking of many targets), compelling us to put up with this loss in terms of power.

§ 10.11. Investigation of Synthesized Circuits for the Method of Flat Scanning

Let us pursue the study of real accuracy of the synthesized circuits of Figs. 10.27 and 10.28 for the method of flat scanning. We consider here certain basic factors of the type of various nonidentities and imperfectnesses affecting accuracy of measurement of angles. First of all, the gate pulses fed to different channels of the synthesized circuits are practically formed each separately, and therefore will not be matched in form both with each other, and also with the form of the pulse pack. Thus, the gates must be considered different and differing in form from the pulse pack.

Filters in the circuits should be matched with the spectrum of signal fluctuations. However, as already repeatedly noted, this matching cannot be realized

sufficiently accurately practically, and during the analysis of the frequency response of the filters it should be considered, in general, arbitrary. Regarding identity of filters in each of the circuits, it is possible to consider it sufficiently good.

10.11.1. Investigation of a Circuit with Differentiation of Gate Pulses

Let us consider first the circuit of Fig. 10.27. We introduce for characteristics of the filters designation (10.5.2). urthermore, trains of gate pulses entering different channels of this circuit we designate by $u_1(t)$ and $u_2(t)$. Then the output signal can easily be reduced to the form:

$$z(t)=\operatorname{Re}\int_{-\infty}^{t}\dot h(t-\tau)u_1(\tau-\hat\tau)u(\tau)y(\tau)e^{i\omega_0\tau}d\tau\times\int_{-\infty}^{t}\dot h^*(t-\tau)u_2(t-\hat\tau)u^*(\tau)y(\tau)e^{-i\omega_0\tau}d\tau, \quad (10.11.1)$$

where $\hat\tau$ — delay of gate pulses issued by the tracking system.

By formula (10.5.5) it is easy to find that the slope of the discrimination characteristic of the considered circuit is determined by

$$K_R=-\frac{P_c}{4\pi\Delta f_c}\int_{-\pi/T_\pi}^{+\pi/T_\pi}|H_{T_\pi}(i\omega)|^2 S_{0T_\pi}(\omega)d\omega\times[C'_{10}(0)C_{20}(0)+C_{10}(0)C'_{20}(0)], \quad (10.11.2)$$

where $H_{T_\pi}(i\omega)=T_\pi\sum_{k=-\infty}^{\infty}\dot h(kT_\pi)e^{ik\omega T_\pi}$ — discrete Fourier transform of the pulse response envelope of the filters;

$$C_{ij}(\tau)=\frac{1}{T_\pi}\int_0^{T_\pi}u_i(t)u_j(t-\tau)dt; \quad (10.11.3)$$

$$u_0(t)=u(t);$$

strokes designate derivatives (note that results obtained here are very similar to results of Chapter VII, dealing with range meters).

Furthermore, in this case there will exist systematic error equal to

$$\Delta=\frac{C_{10}(0)C_{20}(0)+\dfrac{C_{12}(0)}{h}\displaystyle\int_{-\pi/T_\pi}^{\pi/T_\pi}|H_{T_\pi}(i\omega)|^2d\omega\times\left[\int_{-\pi/T_\pi}^{\pi/T_\pi}|H_{T_\pi}(i\omega)|^2S_{0T_\pi}(\omega)d\omega\right]^{-1}}{C'_{10}(0)C_{20}(0)+C_{10}(0)C'_{20}(0)}. \quad (10.11.4)$$

As can be seen from (10.11.4), systematic error is absent if $C_{12}(0) = C_{10}(0)C_{20} \times (0) = 0$. This condition, obviously, means that one of the gate pulses should be symmetric and the other, asymmetric (with a symmetric pulse pack).

Equivalent spectral density in this case turns out to be equal to

$$S_{\text{экв}}=\frac{\pi}{h^4}\int\limits_{-\pi/T_{п}}^{\pi/T_{п}}|H_{T_{п}}(i\omega)|^4\{C_{11}(0)C_{22}(0)+C_{12}^2(0)+$$

$$+hS_{0T_{п}}(\omega)[C_{11}(0)C_{20}^2(0)+C_{22}(0)C_{10}^2(0)+$$

$$+2C_{12}(0)C_{20}(0)C_{10}(0)]+2h^2S_{0T_{п}}(\omega)C_{01}^2(0)C_{20}^2(0)\}\,d\omega\times$$

$$\times\left\{[C'_{10}(0)C_{02}(0)+C_{10}(0)C'_{02}(0)]\times\int\limits_{\pi/T_{п}}^{-\pi/T_{п}}|H_{T_{п}}(i\omega)|^2S_{0T_{п}}(\omega)\,d\omega\right\}^{-2}. \qquad (10.11.5)$$

From this expression it is clear that when $C_{01}(0)\ C_{02}(0) \neq 0$ there exists a component of error caused by nonlinear transformation of the signal; it does not depend on h and will give error of measurement with complete elimination of noises. This component of error disappears together with systematic error.

It is difficult to study in more detail the dependence of $S_{\text{экв}}$ on different parameters in view of the complexity of expression (10.11.5). We will conduct this study in two stages. First, we assume gating to be ideal; equivalent spectral density during ideal gating we shall designate by $S_{\text{иин}}$. It is easy to find that

$$S_{\text{иин}}=\frac{\int\limits_{-\pi/T_{п}}^{\pi/T_{п}}|H_{T_{п}}(i\omega)|^4[1+hS_{0T_{п}}(\omega)]\,d\omega}{\frac{c^2h^2}{\pi}\left[\int\limits_{-\pi/T_{п}}^{\pi/T_{п}}|H_{T_{п}}(i\omega)|^2S_{0T_{п}}(\omega)\,d\omega\right]^2}. \qquad (10.11.6)$$

We produce by formula (10.11.6) calculation for the most typical approximations of characteristics of the filter and of the spectrum of fluctuations of the signal. We shall approximate the response envelope of the filters by expression $\dot{h}(t) = e^{-2\Delta f_{\Phi}t}$. Then by the formula (10.11.2) we have

$$H_{T_{п}}(i\omega)=\frac{T_{п}(1-\rho_{\Phi}^2)}{\rho_{\Phi}^2-2\rho_{\Phi}\cos\omega T_{п}+1}, \qquad (10.11.7)$$

where $\rho_{\Phi}=e^{-2\Delta f_{\Phi}T_{п}}=e^{-2\Delta f_{c}T_{п}x}$;

$x=\frac{\Delta f_{\Phi}}{\Delta f_{c}}$ — ratio of the effective passband of the filter to the width of the band of fluctuations of the signal.

For $S_{0T_{п}}(\omega)$ we introduce approximation (10.10.17). Substituting (10.10.17) in (10.11.6), we obtain

$$S_{\text{мин}}=\frac{1}{2c^2h^2T_{\text{п}}\Delta f_c^2}\left\{\frac{1+\rho_\phi^2}{1-\rho_\phi^2}+hT_{\text{п}}\Delta f_\phi\frac{1-\rho_\phi^2}{(1-\rho_c\rho_\phi)(\rho_c-\rho_\phi)}\times\right.$$
$$\left.\times\left[\rho_c\frac{1+\rho_c\rho_\phi}{1-\rho_c\rho_\phi}-\rho_\phi\frac{(1+\rho_\phi^2)(1-\rho_c^2)}{(1-\rho_\phi^2)^2}\right]\right\}\left(\frac{1+\rho_c\rho_\phi}{1-\rho_c\rho_\phi}\right)^{-2}. \qquad (10.11.8)$$

The curve of the dependence of $S_{\text{мин}}/S_{\text{опт}}$ on h, where $S_{\text{опт}}$ is calculated with these same approximations and is given by formula (10.10.29), is shown in Fig. 10.30. As can be seen from the figure, nonmatching of filters with the spectrum of fluctuations has greater influence, the larger quantity $T_{\text{п}}\Delta f_c$. The difference between $S_{\text{мин}}$ and $S_{\text{опт}}$ can attain a very considerable magnitude (a factor of 10 or more).

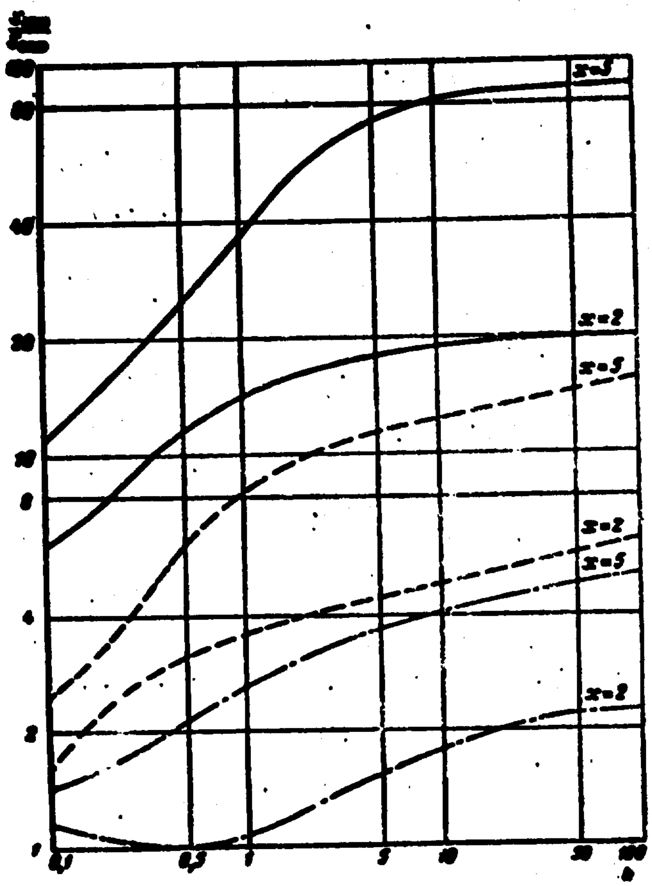

Fig. 10.30. The dependence of $S_{\text{мин}}/S_{\text{опт}}$ on h for circuits of the method of flat scanning:
——— $T_{\text{п}}\Delta f_c=10$; – – – $T_{\text{п}}\Delta f_c=3$; —·—·— $T_{\text{п}}\Delta f_c=1$.

Knowing the dependence of $S_{\text{мин}}/S_{\text{опт}}$ on basic parameters of the circuit, we consider now ratio $S_{\text{экв}}/S_{\text{мин}}$. This ratio essentially depends on the form of the gate pulses. For simplicity we shall consider that gate pulses possess the necessary symmetry. The influence of asymmetry of gate pulses was already estimated qualitatively. If we consider, for instance, gate pulses $u_1(t)$ symmetric, and gate pulses $u_2(t)$ asymmetric, in (10.11.5) we have the following simplification: $C_{12}(0) = 0$, $C_{20}(0) = 0$, $C'_{10}(0) = 0$. Taking into account this simplification, it is easy to see that for sufficiently large h ($h \geq 10$) ratio $S_{\text{экв}}/S_{\text{мин}}$ will not depend on characteristics of the filters, but will depend on characteristics of the gate pulses:

$$\frac{S_{\text{экв}}}{S_{\text{мин}}}=c^2\frac{C_{11}(0)+C_{22}(0)\,C_{10}^2(0)}{C_{10}^2(0)\,C'_{20}(0)^2}. \qquad (10.11.9)$$

We produce calculation by formula (10.11.9). For this we approximate the directional pattern of the antenna system by formula (10.10.26). Pulse packs here have form

$$u_0(t) = \sqrt{\frac{T_п}{\tau_п}}\, 1{,}15\, e^{-2{,}8\left(\frac{t}{\tau_п}\right)^2}, \tag{10.11.10}$$

where $\tau_п = \Omega\Delta\varphi$ — duration of the pack at level 0.5.

Gate pulses $u_1(t)$ we approximate also by a Gaussian function

$$u_1(t) = e^{-2{,}8\left(\frac{t}{\tau_{1c}}\right)^2}, \tag{10.11.11}$$

where τ_{1c} – duration of the gate pulse at level 0.5.

Gate pulses $u_2(t)$ we consider derivatives of Gaussian pulses, i.e.,

$$u_2(t) = t e^{-2{,}8\left(\frac{t}{\tau_{2c}}\right)^2}. \tag{10.11.12}$$

With such approximations all coefficients in (10.11.9) are easily calculated, as a result of which we obtain

$$\frac{S_{экв}}{S_{мин}} = \left(\frac{1+\xi^2\eta^2}{2\xi\eta}\right)^2, \tag{10.11.13}$$

where $\xi = \frac{\tau_п}{\tau_{1c}}$; $\eta = \frac{\tau_{1c}}{\tau_{2c}}$.

Thus, always $S_{экв}/S_{мин} \geq 1$. The sign of equality is attained when $\xi\eta = 1$. The graph of the dependence of $S_{экв}/S_{мин}$ on ξ for different values of η is shown in Fig. 10.31. As can be seen from this figure, imperfectness of gating can lead to very great worsening of accuracy of measurement.

Having dependence $S_{мин}/S_{опт}$ (Fig. 10.30) and $S_{экв}/S_{мин}$ (Fig. 10.31), we can, if we so desire, find the ratio for $S_{экв}/S_{опт}$. We recall once again that Fig. 10.31 is constructed for large h.

The spectral density of parametric fluctuations will be given for the case of ideal gating. It turns out to be equal to

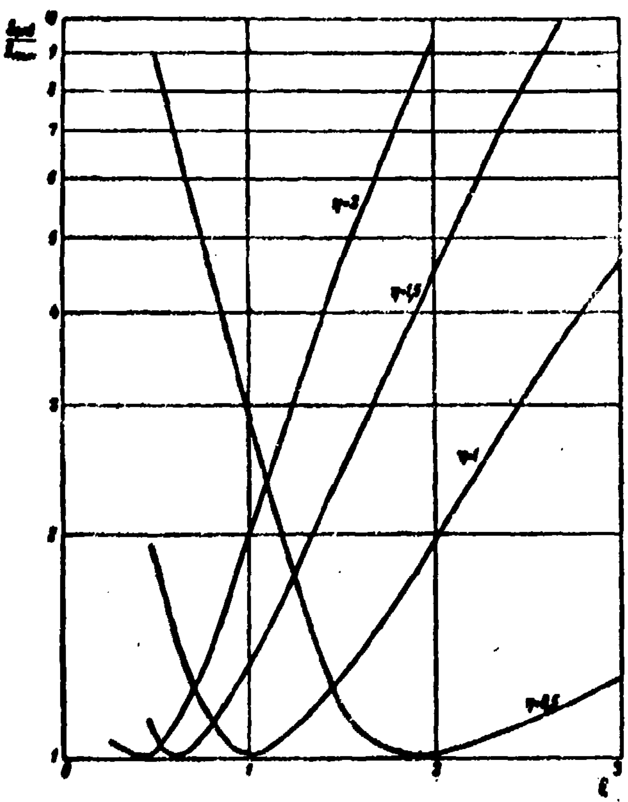

Fig. 10.31. The dependence of $S_{экв}/S_{мин}$ on the relationship of durations of the pack and gate pulses in a circuit with differentiation of gate pulses.

$$S_{nap}=2\pi\frac{\int\limits_{-\pi/T_n}^{\pi/T_n}|H_{T_n}(i\omega)|^4 S_{0T_n}^2(\omega)\,d\omega}{\left[\int\limits_{-\pi/T_n}^{\pi/T_n}|H_{T_n}(i\omega)|^2 S_{0T_n}(\omega)\,d\omega\right]^2}. \qquad (10.11.14)$$

As can be seen from this, it does not depend on h. If the filter has a sufficiently wide band, $S_{пap}$ turns out to be equal to $T_П \coth T_П \Delta f_c$. With decrease of Δf_c quantity $S_{пap}$ grows. When $T_П \Delta f_c \gg 1$ we have $S_{пap} \approx T_П$. If the filter has a sufficiently narrow band, then $S_{пap} \approx T_П \coth \Delta f_c T_П x$, where $x = \Delta f_ф / \Delta f_c$. Laws governing the change of $S_{пap}$ here are as before, only the approximation of $S_{пap}$ to $T_П$ with increase of Δf_c is slower.

10.11.2. Investigation of a Circuit with Detuned Gate Pulses

Let us turn to consideration of the circuit of Fig. 10.28. All designations we leave the same. The output signal of this circuit, obviously, has form

$$z(t)=\left|\int\limits_{-\infty}^{t}\dot h(t-\tau)u_1(\tau-\hat\tau-\delta)y(\tau)u(\tau)e^{i\omega_0\tau}d\tau\right|^2-\left|\int\limits_{-\infty}^{t}\dot h^*(t-\tau)u_2(\tau-\hat\tau+\delta)y(\tau)u^*(\tau)e^{-i\omega_0\tau}d\tau\right|^2. \qquad (10.11.15)$$

From formula (10.5.5) it is easy to find that the slope of the discrimination characteristic of the given radio channel is equal to

$$K_x=\frac{P_0[C'_{10}(\delta)C_{10}(\delta)-C'_{20}(-\delta)C_{20}(-\delta)]}{2\pi\Delta f_c}\times\int\limits_{-\pi/T_n}^{\pi/T_n}|H_{T_n}(i\omega)|^2 S_{0T_n}(\omega)\,d\omega. \qquad (10.11.16)$$

Systematic error with the general assumptions with respect to the form of gate pulses differs from zero and is equal to

$$\Delta=\frac{C_{10}(\delta)^2-C_{20}(-\delta)^2+C_{11}(0)-C_{22}(0)}{C'_{10}(\delta)C_{10}(\delta)-C'_{20}(-\delta)C_{20}(-\delta)}\times\frac{h}{2\pi\Delta f_c x}\int\limits_{-\pi/T_n}^{\pi/T_n}|H_{T_n}(i\omega)|^2 S_{0T_n}(\omega)\,d\omega. \qquad (10.11.17)$$

As in the preceding case, systematic error is caused by nonidentity of gating. Systematic error will be absent if $C_{11}(0) = C_{22}(0)$ and $C_{10}(\delta)^2 = C_{20}(-\delta)^2$. These conditions in general are rather rigid and practically signify identity of gate pulses in both channels. Obviously, to ensure identity of gate pulses in two channels is practically more difficult than simply to ensure symmetry of gate pulses in each channel, which was required for elimination of systematic error in the preceding case. Thus, from this point of view the circuit of Fig. 10.28 one should consider somewhat more critical to imperfectnesses of gating than the circuit of Fig. 10.27.

Equivalent spectral density for the considered circuit is equal to

$$S_{\text{экв}} = \frac{\pi}{2h} \int\limits_{-\pi/T_\text{п}}^{\pi/T_\text{п}} |H_{T_\text{п}}(i\omega)|^2 \{C_{11}(0)^2 + C_{22}(0)^2 - C_{12}(2\delta)^2 -$$
$$- C_{21}(2\delta)^2 + 2hS_{0T_\text{п}}(\omega)\,[C_{11}(0)\,C_{10}(\delta)^2 + C_{21}(0)\,C_{20}(-\delta)^2 -$$
$$- C_{12}(2\delta)\,C_{10}(\delta)\,C_{20}(\delta) - C_{21}(-2\delta)\,C_{20}(-\delta)\,C_{10}(\delta)] +$$
$$+ h^2 S^2_{0T_\text{п}}(\omega)\,[C_{10}(\delta)^2 - C_{20}(-\delta)^2]^2\}\, d\omega \times$$
$$\times \left\{ \int\limits_{-\pi/T_\text{п}}^{\pi/T_\text{п}} |H_{T_\text{п}}(i\omega)|^2 S_{0T_\text{п}}(\omega)\, d\omega \times \right. \qquad (10.11.18)$$
$$\left. \times [C'_{10}(\delta)\,C_{10}(\delta) - C_{20}(-\delta)\,C'_{20}(-\delta)] \right\}^{-2}.$$

From the obtained expression it is clear that for arbitrary imperfectness of gating in the expression for $S_{\text{экв}}$ there exists a component caused by nonlinear transformation of the signal. It will give fluctuating error even with complete absence of noises. This component of error disappears when $C_{10}(\delta)^2 = C_{20}(-\delta)^2$, i.e., when there is no systematic error.

In investigating accuracy of the considered circuit it is also important to establish the influence on basic characteristics of the circuit of detuning of gate pulses δ. Here it is necessary to note the following: theoretically, the considered circuit more exactly approximates the optimum, the less δ; however, in practice, for small δ we will obtain small values of the slope of the discrimination characteristic $K_Д$, which will lead to worsening of dynamic characteristics of the system. Consequently, detuning δ must be selected, in general, noticeably differing from zero. In order to investigate the dependence of $K_Д$ on δ, we introduce approximation (10.11.10) for $u_0(t)$, and gate pulses $u_1(t)$ and $u_2(t)$ we assume identical and approximate them by expression (10.11.11). Here, as it is easy to calculate,

$$C'_{10}(\delta)\,C_{10}(\delta) - C'_{20}(-\delta)\,C_{20}(-\delta) = \frac{22.4\xi^2 z}{(1+\xi^2)^2 \Delta\varphi}\, e^{-5.6\frac{\xi^2 z^2}{1+\xi^2}}, \qquad (10.11.19)$$

where $\xi = \frac{\tau_и}{\tau_0}$; $z = \frac{|\delta|}{\tau_и}$; $\Delta\varphi$ — width of the directional pattern.

The graph of the dependence of $C'_{10}(\delta)C_{10}(\delta) - C'_{20}(-\delta)C_{20}(-\delta)$ on z when $\xi = 1$ is in Fig. 7.17 (there this dependence is given also for other approximations of the pulse pack and gate pulses; however, for understanding qualitative laws one approximation suffices). Analysis of Fig. 7.17 of Chapter VII shows that $C'_{10}(\delta)C_{10}(\delta) - C'_{20}(\delta)C_{20}(-\delta)$, and consequently, also the slope of the discrimination characteristic $K_Д$ attain a maximum at $z \approx 0.4$, i.e., when time detuning of gate pulses is

approximately 40% of the duration of the pack. Thus, namely such detuning is desirable in practice.

Investigation of the dependence of $S_{экв}$ on different parameters, as in the preceding case, we produce in two stages. First we consider cases of ideally matched gate pulses and zero detuning. The value of $S_{экв}$ here we designate by $S_{мин}$. It is not difficult to see that $S_{мин}$ in this case is expressed again by formula (10.11.8).

Studying ratio $S_{мин}/S_{опт}$, we examine, as in the preceding case, ratio $S_{экв}/S_{мин}$. For simplicity we shall consider that gate pulses are identical. Influence of nonidentity of gate pulses we have already studied qualitatively. Here, as it is easy to see, for sufficiently large h ($h \geqq 10$) we have

$$\frac{S_{экв}}{S_{мин}} = c^2 [C_{11}(0) C_{10}(\delta)^2 + C_{11}(0) C_{10}(-\delta)^2 - C_{11}(2\delta) C_{10}(\delta) C_{01}(\delta) - C_{11}(-2\delta) C_{10}(-\delta) C_{10}(\delta)] \times [C'_{10}(\delta) C_{10}(\delta) - C_{10}(-\delta) C'_{10}(-\delta)]^{-2}. \quad (10.11.20)$$

Using the earlier Gaussian approximations for pulse packs and gate pulses, we obtain

$$\frac{S_{экв}}{S_{мин}} = \left(\frac{1+\xi^2}{2\xi}\right)^2 \frac{1-e^{-5,6\xi^2 z^2}}{5,6\xi^2 z^2} e^{-5,6\frac{\xi^2 z^2}{1+\xi^2}}. \quad (10.11.21)$$

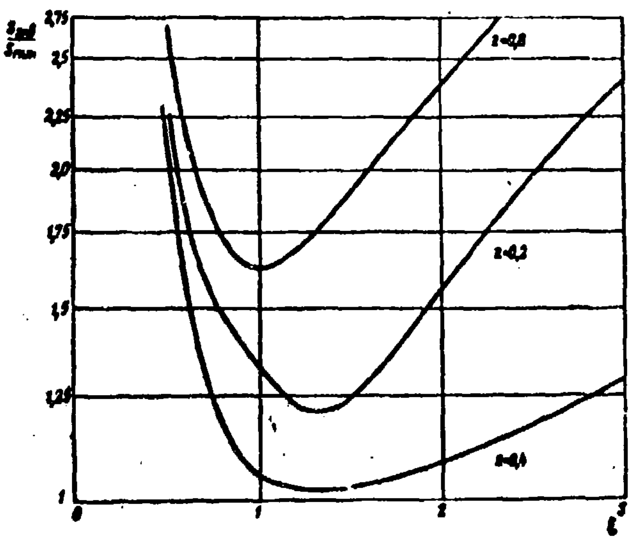

Fig. 10.32. The dependence of $S_{экв}/S_{мин}$ on the relationship of durations of the pack, gate pulses and detuning between gate pulses in a circuit with detuned gate pulses.

The graph of the dependence of $S_{экв}/S_{мин}$ on ξ for different values of z is shown in Fig. 10.32. As can be seen from this figure, the value $z = 0.4$ is characteristic in that approximately at this z there occurs the local minimum of $S_{экв}/S_{мин}$, very close to unity. Thus, mismatch of gate pulses, if only gate pulses in the various channels are identical, with detuning $\delta \approx 0.4\tau_{и}$ leads to considerable worsening of accuracy as compared with the case of ideal gating.

Finishing the present section, we shall say several words on comparison of the circuits of Figs. 10.27 and 10.28 for the method of tracking by pulse packs.

In accuracy the circuits are almost identical. If, however, we talk about those small differences which nevertheless take place, then it is possible to note somewhat lower criticality of the circuit of Fig. 10.27 to imperfectness of gating: to eliminate systematic error in the circuit of Fig. 10.27 it is necessary to ensure only symmetry of gate pulses, while in the circuit of Fig. 10.26 gate pulses must be identical.

§ 10.12. Comparison of Methods of Direction Finding

For the practice of radar goniometry it is extraordinarily interestingly to have comparative characteristics both of methods of angular direction finding and also of separate circuits of radar goniometer radio channels. Such comparison and basic conclusions from it were produced in the preceding sections; however it is of interest to systematically expound the questions pertinent here.

Potential accuracy corresponding to different methods of direction finding in most cases is expressed by formulas of the same time. Namely for methods of pattern scanning with high frequencies of scanning, scanning with compensation, IAC, phase center scanning with high frequencies of scanning, IPC, and, finally, flat scanning (when h > 1) minimum equivalent spectral densities are expressed by the single formula

$$S_{\text{out}}=\left[c^2h^2\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{S_0^2(\omega)}{1+hS_0(\omega)}\,d\omega\right]^{-1}. \qquad (10.12.1)$$

In (10.12.1) by h it is necessary to understand the ratio of the total mean signal power received by the antenna system of the goniometer to the power of noise acting in one channel, and coefficient c^2 is expressed through parameters of the goniometer antenna system and differs for different methods of direction finding;

for methods of pattern scanning, scanning with compensation and IAC

$$c^2=\mu_a^2, \qquad (10.12.2)$$

where μ_a – gain factor of the directional pattern in the direction of the axis of axis of scanning or the equisignal axis;

for the method of phase center scanning and IPC

$$c^2=\left(\frac{\pi d}{\lambda}\right)^2, \qquad (10.12.3)$$

where d – amplitude of oscillations of the phase center or the distance between phase centers, λ – wavelength;

for the method of flat scanning

$$c^2 = \frac{1}{\Omega T_{\text{п}}} \int_{-\infty}^{\infty} g'(\varphi)^2 \, d\varphi, \qquad (10.12.4)$$

where Ω — angular velocity of pattern scanning;

$T_{\text{п}}$ — period of repetition of packs;

$g(\varphi)$ — power directional pattern (appropriately normalized);

in particular, for a Gaussian directional pattern

$$c^2 = \frac{2{,}8}{\Delta\varphi^2}, \qquad (10.12.5)$$

where $\Delta\varphi$ — width of the directional pattern for the level of half power.

In all the enumerated cases, when equivalent spectral densities are expressed by formula (10.12.1), it is rather simple to compare methods of direction finding. For this it is necessary to compare the signal-to-noise ratios h and coefficients c^2 for the different methods of direction finding. During comparison of signal-to-noise ratios it is possible to indicate the following: if the aperture area of the antenna system of the goniometer is fixed, then in the method of scanning with compensation, IAC and IPC there is received all the power reaching the aperture of the antenna system; in the methods of pattern and phase scanning only half the power is received, i.e., the signal-to-noise ratio for these methods is approximately half that for the above-mentioned methods.

In the method of flat scanning, as already noted, the mean received power is usually a minute share (of the order of $\Delta\varphi/\Phi$, where $\Delta\varphi$ — width of the directional pattern; Φ — angular dimensions of the scan sector) of the power which would be received by an antenna with the same aperture using the IAC method. However, in comparison of the method of flat scanning with methods IAC, IPC and others, it is necessary to consider other tactical functions of the method of flat scanning, justifying the indicated power losses. Thus, if we ignore the method of flat scanning, of the remaining methods of direction finding from the point of view of the signal-to-noise ratio best are the method of scanning with compensation, and the methods of IAC and IPC, where all the power reaching the aperture of the antenna system is used.

Now we produce comparison of gain factors c. Obviously, c depends on the geometric characteristics of the aperture of the antenna system. Here, for the same form and area of the aperture gain factors c for the methods of pattern scanning, scanning with compensation and IAC completely coincide. If, for instance, we have a square aperture with dimension d in the plane of the angle interesting us, for

these methods

$$c^2 = \mu_a^2 \approx \left(\frac{\pi d}{2\lambda}\right)^2, \tag{10.12.6}$$

where λ — wavelength.

Thus, comparison of the methods of pattern scanning, scanning with compensation and IAC absolutely definitely testifies to the advantages of IAC and the method of scanning with compensation over the method of pattern scanning. Here the method of scanning with compensation and IAC are equivalent.

We produce now comparison of the phase method with IAC. In a number of cases the advantage of the phase method is obvious. This occurs when the antenna system aperture consists of two parts, separated a considerable distance d from each other. Here quantity $(\pi d/\lambda)^2$ considerably exceeds the possible gain factors of the directional patterns, which can be provided due to the dimensions of the aperture of the antenna system.

Less clear is the question of comparison of methods of IAC and IPC if the aperture is of simple form, for instance square. If the dimension of the aperture in the plane of the angle interesting is d, the gain factor of the directional pattern which we will have for the IAC method is expressed by formula (10.12.6). But it is absolutely obvious that if we use the IPC method, dividing the aperture into two parts, we obtain a distance between phase centers equal to d/2. Putting d/2 instead of d in (10.12.3) and comparing the obtained expression with (10.12.6), we see that IAC and IPC are equivalent. Thus, with an aperture of the assigned simple form it is possible to use IPC or IAC.

Thus, consideration of slopes c did not change our conclusion concerning the equivalence of IAC, IPC and scanning with compensation and their advantage with respect to methods of pattern scanning and phase center scanning, which in the same conditions are equivalent to one another.

For comparison of the method of flat scanning, for instance with IAC, it is necessary to compare the square of the gain factor of the directional pattern in IAC μ_a^2 with c^2 (10.12.4). With a Gaussian approximation of the directional pattern instead of (10.12.4) we have expression (10.12.5) for c^2. With such an approximation of the directional for IAC we have $\mu_a^2 \approx 2/\Delta\varphi^2$. Thus, for the method of flat scanning on the assumption of equality of the mean received powers there is a somewhat better result. However, as already noted, comparison of the method of flat scanning with IAC, IPC and other methods is not quite just in view of the difference of the

tactical function of these methods.

We have till now left from consideration methods of pattern scanning and phase scanning in cases when the frequencies of scanning are not too high. However, this gap is easily filled if we recall results of comparison of these methods in the shown cases with cases of high-frequency scanning. Such comparison was conducted earlier and showed that accuracy of methods of pattern scanning and phase scanning rather sharply drops with decrease of the frequency of scanning. Numerically, the drop of accuracy with decrease of frequency of scanning can be found from graphs of Figs. 10.11 and 10.22.

Thus, we have compared the considered methods of direction finding. The best one should recognize is the methods of scanning with compensation, IAC and IPC. The worst results come from methods of pattern scanning and phase center scanning, especially at low frequencies of scanning.

§ 10.13. Optimum Measurement of Angular Coordinates by Antennas of Phased Array Type

Till now we were occupied with synthesis and study of optimum radar meters of angular coordinates with assigned methods of direction finding. By this we understood that there is assigned the structure of the antenna array of the goniometer, i.e., there are determined operations produced on the electromagnetic field in the aperture of the antenna array. Such a formulation of the problem is limited. It does not give a complete answer to this question: if we are assigned the geometric form and dimensions of the aperture of the antenna array, what method of direction finding will be best?

Certainly, it is possible by comparing all methods of direction finding to establish the best of them. However there is no guarantee that there will not be found a certain method of direction finding, not considered by us, which gives greater accuracy of angle measurement. In order to obtain an answer to this question, it is necessary to synthesize an optimum goniometer, originating, not from an assigned method of direction finding, but from an assigned aperture of the antenna system.

In one case, very important and interesting, the technique developed in the preceding sections permits us to carry out complete synthesis of the goniometer circuit, including operations produced in the antenna system. We are talking about the case when there are used antennas of the phased array type.

Such antennas, as it is known, consist of a set of discrete radiators (usually isotropic), each of which is fed current with its own amplitude and phase. Selection

of amplitudes and phases of feed of the radiators permits us to obtain from the system of radiators various forms of directional patterns. Clearly, the sum of the effective areas of the radiators is approximately equal to the area occupied by all the radiators so that power losses due to the discrete structure [illegible]nna arrays are insignificant. Interest in antennas of the phased a[illegible]esent has sharply increase.

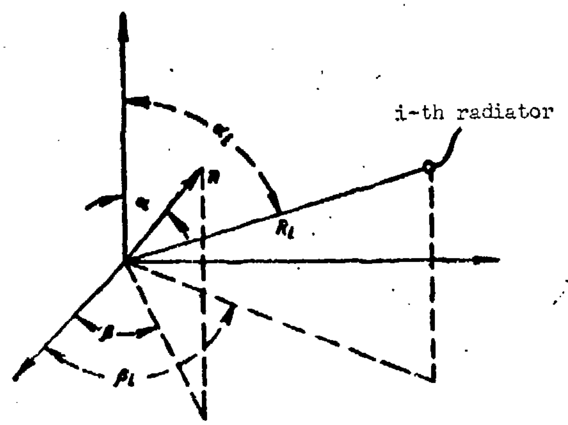

Fig. 10.33. Finding the form of signals received by elements of an antenna array.

During use of antennas of the phased array type the received signals from every element of the antenna usually are first processed separately up to and including transformation of frequencies, and only then are they cross-processed. This cross-processing determines the structure of the device, which should be selected optimally.

Let us assume that the antenna array consists of n elements, having (in a certain system of coordinates) polar coordinates R_i, α_i, β_i ($i = 1, 2, \ldots, n$). Vector **n** of the normal to the incident wave (which we consider planar in the vicinity of the antenna) we assume has coordinates α and β (Fig. 10.33). If we designate by φ_0 the phase of the incident wave at the origin of coordinates, the phase of the signal received by the i-th element of the antenna will be equal to

$$\Phi_i(t,\ \alpha,\ \beta)=\varphi_0+\frac{2\pi}{\lambda}(R_i\sin\alpha_i\sin\alpha\cos(\beta_i-\beta)+R_i\cos\alpha_i\cos\alpha).$$

Then the received signals are recorded in the form

$$y_i(t)=\sqrt{P_{0i}}\,\mathrm{Re}\,E(t)\,u(t)\times$$
$$\times e^{i\left[\omega_0 t+\frac{2\pi}{\lambda}(R_i\sin\alpha_i\sin\alpha\cos(\beta_i-\beta)+R_i\cos\alpha_i\cos\alpha)\right]}+\sqrt{N_{0i}}\,n_i(t),$$

where P_{0i} – power of the signal received by the i-th element of the antenna,

N_{0i} – spectral density of noises acting in the channel of this element.

Subsequently to facilitate calculation we shall limit our consideration to the most important practical case of a flat square uniform array with identical elements (Fig. 10.34). Here for definitiveness we shall consider the variant with a tracking antenna, i.e., in the process of target tracking the actual array turns to match the normal to the array plane with the direction to the target. Considering here mismatch with respect to one of the coordinates equal to zero, as we did in all cases,

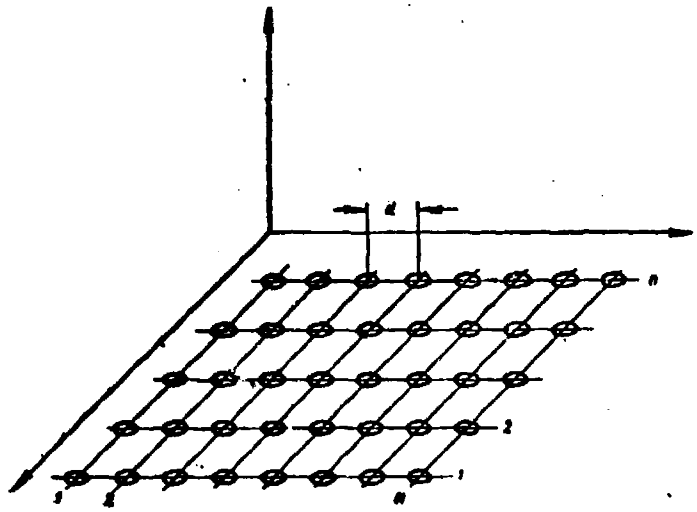

Fig. 10.34. Square uniform antenna array.

we can consider the array oriented so that one of its sides (we call it transverse) is parallel to the leading edge of the incident wave. Then signals received by elements of transverse rows will be identical. This is equivalent to observing n signals of power $mP_{c\,1}$, where $P_{c\,1}$ – power received by one element; m – number of elements in the transverse row; n – number of elements in the longitudinal row. Denoting by d the distance between elements in the longitudinal row of the array, we see that the phase in the i-th row of radiators ($i = 1, 2, \ldots, n$, Fig. 10.34) is equal to

$$\Phi_i(t,\ \alpha)=\frac{2\pi}{\lambda}d\sin\alpha. \tag{10.13.1}$$

Thus, the received signals turn out to be equal to

$$y_i(t)= =\sqrt{mP_{c1}}\,\mathrm{Re}\,E(t)\,u(t)\,e^{i\left(\omega_0 t+\frac{2\pi i}{\lambda}d\sin\alpha\right)}+\sqrt{N_0}\,n_i(t). \tag{10.13.2}$$

We now use ~~a~~ general expression for the operation of the optimum radio channel (10.3.29). Substituting (10.3.1) in this formula, we obtain

$$z(\tau)=\frac{16\Delta f_c h^2}{mP_{c1}\lambda}\pi d\int_{t-\Delta t}^{t}h_{1\text{опт}}(\tau-s)\cos\omega_{\text{пр}}(\tau-s)\times\sum_{i=1}^{n}iy_i(s)\,u_a(s)\sin[\omega_r s+\psi(s)]\,ds\times$$
$$\times\int_{t-\Delta t}^{t}h_{1\text{опт}}(\tau-s)\cos\omega_{\text{пр}}(\tau-s)\times\sum_{i=1}^{n}y_i(s)\,u_a(s)\cos\left(\omega_r s+\psi(s)\right)ds. \tag{10.13.3}$$

The block diagram realizing this operation is depicted in Fig. 10.35. As can be seen from this diagram, the signal from each element of the antenna is separated into two channels, in which it is heterodyned by voltages shifted relative to each other in phase by $\pi/2$. Signals heterodyned with one phase are added, are filtered by an optimum filter with gain-frequency response curve (10.3.29) and enter the

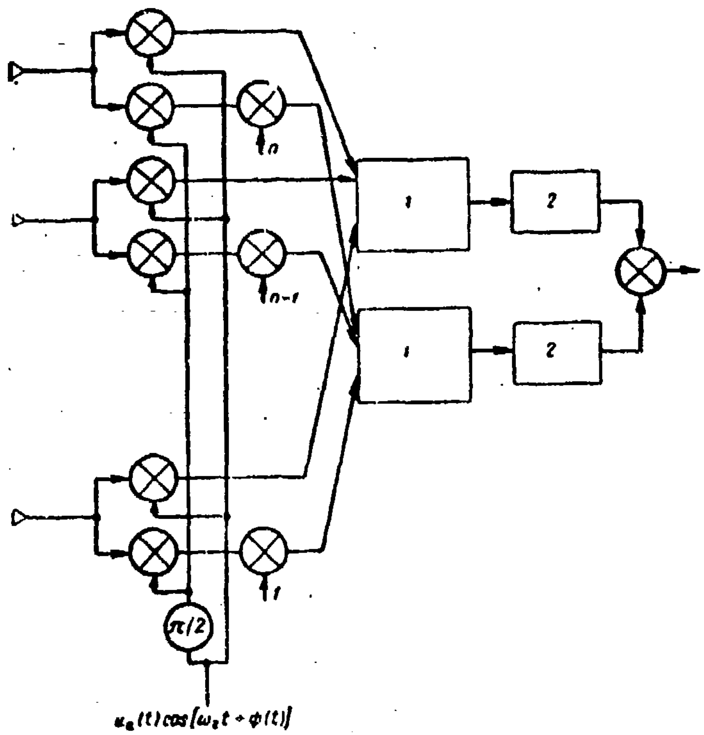

Fig. 10.35. Block diagram of an optimum receiver during measurement of angular coordinates with the help of an antenna of phased array type: 1) adder; 2) optimum filters with frequency response $H_{1\,\text{опт}}(i\omega)$.

phase detector as reference voltage. Signals heterodyned by voltage with the other phase, are added with scales 1, 2, ..., n, are filtered by the same filter and enter the other input of the phase detector. Summation with scales has a simple physical meaning: the further a given radiator is removed from the 1-th, the greater the information about angular coordinates in the signal from this radiator and the greater the weight with which this signal should be considered. Note that the weights with which we consider signals from different radiators are determined ambiguously: as scales there can be used quantities c, c + 1, ..., c + n - 1, where c is any number, including a negative number. Operation of the circuit in no way is changed by this.

We now calculate potential accuracy of measurement of angular coordinates by a phased array. For this we use general formula (10.3.32). Substituting there expression (10.13.1) for the equivalent spectral density of measurement of one angle (taking into account the usual assumption that we have to measure both angles with identical accuracy), we obtain the following formula:

$$S_{\text{опт}}=$$

$$=\frac{1}{h_\Sigma^2\left(\frac{2\pi d}{\lambda}\right)^2\frac{1}{n}\left[\sum_{l=1}^{n} l^2-\frac{1}{n}\left(\sum_{l=1}^{n} l\right)^2\right]\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{S_0(\omega)\,d\omega}{1+h_\Sigma S_0(\omega)}}, \qquad (10.13.3)$$

where h_Σ — ratio of total signal power received by all elements of the array to the spectral density of noise in one channel;

d — distance between neighboring radiators;

λ — wavelength.

We calculate the sum in (10.13.3); finally we find

$$S_{\text{опт}}=\frac{1}{\dfrac{h_{\Sigma}^{2}}{2\pi}\displaystyle\int_{-\infty}^{\infty}\frac{S_{0}(\omega)\,d\omega}{1+h_{\Sigma}S_{0}(\omega)}\,\frac{n^{2}-1}{3}\left(\frac{\pi d}{\lambda}\right)^{2}}. \qquad (10.13.4)$$

It is interesting for comparison to consider analogs of the methods of IAC and IPC during use of phased arrays as antennas and to compare accuracy with the potential accuracy found by us.

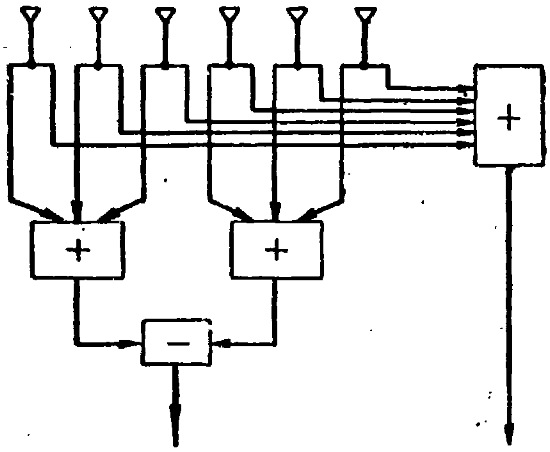

Fig. 10.36. Coupling of radiators of the array with the IAC method.

In the IAC method all signals received by radiators are added, and there will be formed a difference signal, representing the difference between the sum of the signals of one half of the radiators and the sum of the signals of the other half of the radiators (Fig. 10.36). Further processing will only be on these two signals. Here, it is easy, producing the shown operations on the signals (10.13.2), to find that the sum signal is proportional to

$$g_{+}(\alpha)=\cos\frac{n}{2}\frac{\pi d}{\lambda}\sin\alpha,$$

(sum directional pattern), and the difference is proportional to

$$g_{-}(\alpha)=\sin\frac{n}{2}\frac{\pi d}{\lambda}\sin\alpha$$

(difference directional pattern). Comparing these expressions with (10.4.8) and considering normalization (10.3.1), it is easy to calculate the gain factor of the directional pattern entering all formulas for accuracy of the IAC method:

$$\mu_{a}=\frac{n}{2}\frac{\pi d}{\lambda}, \qquad (10.13.5)$$

i.e., the equivalent spectral density of the IAC method is equal to

$$S'_{\text{опт}}=\frac{1}{\dfrac{h_{\Sigma}^{2}}{2\pi}\displaystyle\int_{-\infty}^{\infty}\frac{S_{0}(\omega)\,d\omega}{1+h_{\Sigma}S_{0}(\omega)}\,\frac{n^{2}}{4}\left(\frac{\pi d}{\lambda}\right)^{2}}. \qquad (10.13.6)$$

Comparing this expression with (10.13.4), we see that the optimum method of measurement of angles ensures somewhat higher accuracy than IAC; however this gain is insignificant. When $n > 6$

$$\frac{S'_{\text{out}}}{S_{\text{out}}} = \frac{4}{3}\,\frac{n^2-1}{n^2} \approx \frac{4}{3}.$$

We shall now consider the IPC method. With this method radiators are coupled as shown in Fig. 10.37, i.e., the signals of two groups of radiators are added separately, and subsequently only the two sum signals thus obtained are processed. These two signals are easily found by corresponding summation of signals (10.13.2). It is easy to find that one of the sum signals differs from the second only by complex factor $e^{in\,\pi d/\lambda\,\sin\alpha}$. From this it is clear that the equivalent base of the considered IPC method is equal to nd/2. Consequently, potential accuracy of this method, too, is expressed by formula (10.13.6). Thus, accuracies of IAC and IPC are identical and somewhat lower than the accuracy of the optimum method of direction finding.

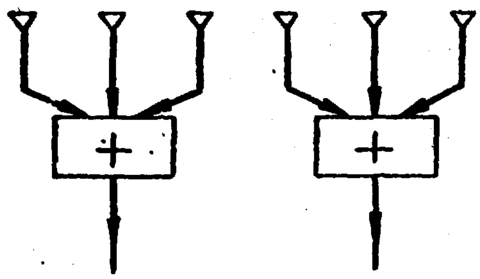

Fig. 10.37. Coupling of radiators of the array in the IPC method.

In the obtained results on synthesis of the optimum method of direction finding using antennas of phased array type it would have been possible to pass to the limit $n \to \infty$, $d \sim 1/n$. Here, a discrete array is turned into an antenna with a continuous aperture [51].

Thus, from consideration of antennas of phased array type it is clear that there exists an optimum method of processing the field in the aperture of an antenna system (method of direction finding) which does not reduce to the widely applied IAC or IPC methods. However, accuracy of measurement provided by the optimum method insignificantly exceeds accuracy of IAC or IPC (by 18% for equivalent spectral density), which has no special practical value.

§ 10.14. Analysis of Accuracy of Tracking Radar Goniometers

In the preceding sections we studied in detail radio channels of tracking radar goniometers. However accuracy of measurement of angular coordinates of a target by a tracking goniometer is determined to a considerable extent by the structure of the smoothing circuits. In Chapter VI it was shown that mean square fluctuating error

is determined by the characteristic of the radio channel $S_{\text{экв}}$ and characteristics of the smoothing circuits. Thus, for instance, during application of linear smoothing circuits with constant parameters error is determined by effective bandwidth of the system $\Delta f_{\text{эф}}$, which depends basically from the form of the smoothing circuits. Furthermore, by smoothing circuits we determine dynamic error, constituting a considerable share of total error of tracking a target by angles.

The present section considers different types of smoothing circuits and analyzes accuracy of measurement of angular coordinates by tracking radar goniometers.

10.14.1. Influence of a System of Automatic Gain Control on Accuracy of Tracking Goniometers

In analysis of accuracy of tracking radar goniometers it is necessary first of all to consider the influence of the system of automatic gain control. From results of Chapter II it follows that the AGC system can be considered in first approximation a linear inertial amplifier of the signal amplitude envelope, the gain factor of which depends on the level of amplitude of the signal. Action of AGC leads basically to normalization of the output signal of the amplifier covered by the AGC loop, so that the mean value of signal amplitude remains constant. In this approximation the AGC system, as it is easy to see, does not affect the magnitude of equivalent spectral density at the input of the discriminator.

However for analysis of accuracy of a tracking goniometer as a whole the AGC system must be taken into account. Actually, due to automatic gain control there is established a definite gain factor of the radio channel, depending on the signal-to-noise ratio. This leads to dependence of the effective bandwidth and other characteristics of the closed tracking system on the signal-to-noise ratio, which in a certain way affects fluctuating and dynamic error of a tracking goniometer.

In goniometers using one antenna the gain factor of the radio channel with automatic gain control is calculated just as in § 7.10. The AGC loop in these goniometers covers the UPCh, whose passband is sufficiently wide as compared to the band of subsequent filters. The input signal of the radio channel has the form (10.3.3). Gain of the UPCh K_y is established in such a way that

$$K_y^2 \lim_{T\to\infty} \frac{1}{T} \int_0^T \overline{y_1(t)^2}\, dt = \text{const},$$

i.e.,

$$K_y 2 (2N_0 \Delta f_y + P_0) = \text{const},$$

where N_0 — spectral density of noise;

Δf_y — bandwidth of the amplifier covered by the AGC loop.

Hence, as it is easy to see, the slope of the discrimination characteristic is equal to

$$K_Д = \frac{K_{0Д}}{1 + \frac{1}{h}\frac{\Delta f_y}{\Delta f_c}} = \frac{K_{0Д}}{1 + \frac{y}{h}}, \qquad (10.14.1)$$

where $y = \Delta f_y / \Delta f_c$ — ratio of the band of amplification of the UPCh to the band of the signal;

$K_{ОД}$ — slope of the discrimination characteristic in the absence of noises;

h — ratio of mean signal power to the power of noise in the band of signal fluctuations.

In circuits of goniometers using several antennas there exists a variety of methods of coupling in automatic gain control. Most frequently automatic gain control is coupled in one of the channels, simultaneously controlling amplification of the remaining channels. In this case the slope of the discrimination characteristic of the goniometer has the form (10.14.1); however, by h it is necessary to understand the signal-to-noise ratio of only the one channel in which the AGC system is coupled. In a number of cases to the AGC system there is simultaneously inserted voltage from several channels, taken with certain scales. The slope of the discrimination characteristic will again be expressed by formula (10.12.1), where by h it is necessary to understand the weighted sum of the signal-to-noise ratios in the channels to which the AGC system is coupled. Quantity $K_Д$ is calculated analogously in more complicated cases, too.

Besides the influence on the transmission factor of the radio channel automatic gain control also leads to decrease of the spectral density of parametric fluctuations. Decrease of the spectral density of parametric fluctuations is caused by the demodulating action of the AGC system. This decrease can be easily calculated with the help of results of Chapter II. According to (2.7.54) the spectral density of parametric fluctuations $S^{(1)}_{пар}$ with the presence of an AGC system is connected with the magnitude of $S_{пар}$ calculated in the preceding sections by relationship

$$S^{(1)}_{пар} = S_{пар}\frac{1}{n^2}, \qquad (10.14.2)$$

where n — a coefficient which depends on parameters of the AGC system and the signal-to-noise ratio, where

$$n^2 = n_0^2 \frac{h}{h_0}.$$

Here h_0 — the signal-to-noise ratio corresponding to a signal equal to the delay level of the AGC system; n_0 — the value of parameter n corresponding to the delay level (see also Paragraph 7.10.1).

10.14.2. Errors of Goniometers with Smoothing Circuits with Constant Parameters

Usually in smoothing circuits of radar goniometers there are used filters with constant parameters. Here it is essential that the structure of the smoothing circuits of tracking goniometers in distinction from range finders and speed meters turns out in a number of cases to be rather complicated, inasmuch as smoothing circuits include such elements as actuator motors, power amplifiers, etc., necessary for rotation of the antenna.

Changes of smoothing circuits here can be carried out only by introduction of additional correcting links.

Smoothing circuit transfer functions used in practice have one of the following forms:

$$H_{I}(p)=\frac{D}{p}\frac{1+pT_2}{(1+pT_1)(1+pT_3)},\quad H_{II}(p)=\frac{D}{p}\frac{(1+pT_2)^2}{(1+pT_1)^2(1+pT_3)},$$

$$H_{III}(p)=\frac{D}{p}\frac{1+pT_2}{(1+pT_1)(1+pT_3)^2},\quad H_{IV}(p)=\frac{D}{p}\frac{(1+pT_2)^2}{(1+pT_1)^2(1+pT_3)^2}, \qquad (10.14.3)$$

where $T_1 > T_2 > T_3$.

Smoothing circuits of a very simple type in the form of an integrator (RC-filter) or a double integrator (double RC-filter) with correction, studied in detail in the preceding chapters, in the practice of radar goniometers are almost never met. Therefore, we will pursue our study of meters with smoothing circuits of type (10.14.3), using methods of approximation for calculation, developed in radar automation, since obtaining of exact results here is very difficult in view of the complexity of these expressions.

Smoothing circuits are characterized by Q-factor D and time constants T_1, T_2, T_3.

However, in practice, smoothing circuits are better characterized by so-called generalized parameters, having considerably greater physical and technical meaning: Q-factor D, cutoff frequency ω_c and reserve phase stability φ_c.

Cutoff frequency ω_c is determined by relationship

$$|H(i\omega_c)|^2=1, \qquad (10.14.4)$$

the reserve of phase stability

$$\varphi_c = \pi + \arg H(i\omega_c). \tag{10.14.5}$$

These parameters usually are assigned.

The relationship of time constants T_1, T_2 and T_3 with parameters D, ω_c and φ_c, in general, is rather complicated; however with accuracy sufficient for practical applications it can be fixed as follows.

For an example we shall consider only function $H_I(p)$ from (10.14.3). Results for the remaining transfer functions are established absolutely analogously. From (10.14.4) we have

$$|H_1(i\omega_c)| = \frac{D}{\omega_c} \frac{\sqrt{1+\omega_c^2 T_2^2}}{\sqrt{1+\omega_c^2 T_1^2}\sqrt{1+\omega_c^2 T_3^2}}.$$

It is easy to see that $1/T_2 < \omega_c < 1/T_3$; considering $\sqrt{1+x^2} \approx x$ for $x > 1$ and $\sqrt{1+x^2} \approx 1$ for $x < 1$, for $H_I(p)$ we obtain

$$\omega_c = \frac{T_2}{T_1} D. \tag{10.14.6}$$

The same result is obtained for transfer function $H_{III}(p)$. For transfer functions of type $H_{II}(p)$ and $H_{IV}(p)$ in an analogous way we find

$$\omega_c = D\left(\frac{T_2}{T_1}\right)^2. \tag{10.14.7}$$

We consider now the phase-frequency responses. We have

$$\arg H_1(i\omega_c) = -\frac{\pi}{2} - \operatorname{arc\,tg} \omega_c T_1 + \operatorname{arc\,tg} \omega_c T_2 - \operatorname{arc\,tg} \omega_c T_3.$$

Considering that arc tg $x \approx \pi/2 - 1/x$ when $x > 1$ and arc tg $x \approx x$ when $x < 1$, it is easy to find that for $H_I(p)$

$$\beta_c = \frac{1}{\omega_c T_2}\left(1 - \frac{\omega_c}{D}\right) + \omega_c T_3, \tag{10.14.8}$$

where

$$\beta_c = \frac{\pi}{2} - \varphi_c. \tag{10.14.9}$$

For transfer functions $H_{II}(p)$, $H_{III}(p)$ and $H_{IV}(p)$ analogous expressions will have the form

$$\beta_c = \frac{2}{\omega_c T_2}\left(1 - \sqrt{\frac{\omega_c}{D}}\right) + T_3\omega_c,\ \beta_c = \frac{1}{\omega_c T_2}\left(1 - \frac{\omega_c}{D}\right) + 2T_3\omega_c,$$
$$\beta_c = \frac{2}{\omega_c T_2}\left(1 - \sqrt{\frac{\omega_c}{D}}\right) + 2T_3\omega_c. \tag{10.14.10}$$

Analysis of errors of the used approximations shows that errors here are fully permissible [52].

As can be seen from expressions (10.14.6) and (10.14.10), time constants T_1, T_2 and T_3 are determined by parameters D, ω_c and φ_c ambiguously, i.e., there exists a broad class of systems with assigned D, ω_c and φ_c, but with different time constants T_1, T_2, and T_3. Inasmuch as basic parameters D, ω_c and φ_c are fixed, these systems possess approximately identical dynamic properties; the difference of time constants T_1, T_2 and T_3 plays a less essential role. However, from the point of view of operation most profitable is the system for which ratio T_2/T_3 is minimum [52]. Here, in particular, the system has a phase-frequency response changing little near the cutoff frequency, which ensures constancy of φ_c with change of amplification. Therefore in practice there are used basically systems possessing for assigned D, ω_c and φ_c a minimum value of ratio T_2/T_3. We subsequently will limit our consideration to such systems.

Using expressions (10.14.8) and (10.14.10), in Table 10.1 we give minimum values of ratio T_2/T_3 for fixed D, ω_c and β_c.

From this table it is easy to obtain expressions for time constants T_1, T_2 and T_3 in terms of basic parameters of the smoothing circuits, D, ω_c and φ_c, given in Table 10.2.

Table 10.1

$H(p)$	$\left(\frac{T_2}{T_3}\right)_{min}$
$H_I(p)$	$\frac{4}{\beta_c^2}\left(1-\frac{\omega_c}{D}\right)$
$H_{II}(p)$	$\frac{8}{\beta_c^2}\left(1-\sqrt{\frac{\omega_c}{D}}\right)$
$H_{III}(p)$	$\frac{8}{\beta_c^2}\left(1-\frac{\omega_c}{D}\right)$
$H_{IV_a}(p)$	$\frac{16}{\beta_c^2}\left(1-\sqrt{\frac{\omega_c}{D}}\right)$

What has been presented can be summarized in the following way. Smoothing circuits of goniometer systems are characterized by basic parameters D, ω_c and φ_c. Transfer functions of smoothing circuits have form (10.14.3), where T_1, T_2 and T_3 are expressed through D, ω_c and φ_c with the help of Table 10.2.

Numerical values of basic parameters for real tracking systems have different values. The reserve phase stability usually is selected in such a manner that $20^\circ \leq \varphi_c \leq 60^\circ$. Usually $\varphi_c \approx 3$[illegible], and ω_c and D_c can have different values depending upon the assignment of the radar goniometer: $D_c \approx 150$–500 1/sec, $\omega_c \approx 8$–400 1/sec. Hence with Table 10.2 it is easy to establish the order of time constants T_1, T_2, and T_3.

Table 10.2

$H(p)$	T_1	T_2	T_3
$H_I(p)$	$\frac{2}{\omega_c\beta_c}\left(\frac{D}{\omega_c}-1\right)$	$\frac{2}{\omega_c\beta_c}\left(1-\frac{\omega_c}{D}\right)$	$\frac{\beta_c}{2\omega_c}$
$H_{II}(p)$	$\frac{4}{\omega_c\beta_c}\left(\sqrt{\frac{D}{\omega_c}}-1\right)$	$\frac{4}{\omega_c\beta_c}\left(1-\sqrt{\frac{\omega_c}{D}}\right)$	$\frac{\beta_c}{2\omega_c}$
$H_{III}(p)$	$\frac{2}{\omega_c\beta_c}\left(\frac{D}{\omega_c}-1\right)$	$\frac{2}{\omega_c\beta_c}\left(1-\frac{\omega_c}{D}\right)$	$\frac{\beta_c}{4\omega_c}$
$H_{IV}(p)$	$\frac{4}{\omega_c\beta_c}\left(\sqrt{\frac{D}{\omega_c}}-1\right)$	$\frac{4}{\omega_c\beta_c}\left(1-\sqrt{\frac{\omega_c}{D}}\right)$	$\frac{\beta_c}{4\omega_c}$

Let us turn to study of fluctuating errors. These errors, as it is known, are expressed through the effective bandwidth of the tracking system.

Let us calculate effective bandwidths of closed systems for all types of smoothing circuits (10.14.3). They are determined by expression

$$\Delta f_{эф}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{K_д H(i\omega)}{1+K_д H(i\omega)}\right|^2 d\omega, \tag{10.14.11}$$

where $K_Д$ – the slope of the discrimination characteristic of the radio channel.

Substituting for $H(i\omega)$ consecutively the expressions from (10.14.3) and calculating this integral in every case, we obtain (calculated by M. M. Kreymerman):

for $H_I(p)$

$$\Delta f_{эф}=\frac{1}{4}\,\frac{K_д D^2 T_2^2+K_д D(T_1+T_2)}{(T_1+T_2)(1+K_д DT_2)-T_1T_2K_д D}; \tag{10.14.12}$$

for $H_{II}(p)$

$$\begin{aligned}\Delta f_{эф}=\frac{1}{4}\{&[(1+2K_д DT_2)K_д^2D^2T_2^4+2T_1T_2K_д^2D^2(T_1+2T_2)+\\&+K_д DT_1(T_1+2T_2)(T_2+2T_1+K_д DT_2^2)-\\&-K_д DT_1^2(1+2DT_2)][-T_1^2T_2(1+2K_д DT_2)^2-\\&-K_д DT_1^2(T_1+2T_2)^2+T_1(T_1+2T_2)(T_2+2T_1+\\&+K_д DT_2^2)(1+2K_д DT_2)]^{-1};\end{aligned} \tag{10.14.13}$$

for $H_{III}(p)$

$$
\begin{aligned}
\Delta f_{\text{эф}} = \frac{1}{4} [& D^2 K_{\text{д}}^2 T_2^2 (2T_1 + T_3) - K_{\text{д}} D T_1 T_3 (1 + K_{\text{д}} D T_2) + \\
& + K_{\text{д}} D (T_1 + 2T_3)(2T_1 + T_3] \cdot [-T_1 T_3 (1 + K_{\text{д}} D T_2)^2 - \\
& - T_3 K_{\text{д}} D (2T_1 + T_3)^2 + (T_3 + 2T_1)(T_1 + 2T_3) \times \\
& \times (1 + K_{\text{д}} D T_2)]^{-1};
\end{aligned}
\tag{10.14.14}
$$

for $H_{IV}(p)$

$$
\begin{aligned}
\Delta f_{\text{эф}} = \frac{1}{4} \{ & K_{\text{д}}^2 D^2 T_2^4 [T_1 T_3 K_{\text{д}} D - 2 (T_1 + T_3)(1 + 2DT_2)] + \\
& + 2K_{\text{д}}^2 D^2 T_2^2 T_1 T_3 (KDT_2^2 + 2T_1 + 2T_3) - \\
& - 4K_{\text{д}}^2 D^2 T_2^2 (T_1 + T_3)(T_1^2 + T_3^2 + 4T_1 T_3) - \\
& - 2K_{\text{д}}^2 D^2 T_1^2 T_3^2 (T_1 + T_3) + K_{\text{д}} D T_1 T_3 (K_{\text{д}} D T_2^2 + 2T_1 + \\
& + 2T_3)^2 + 4K_{\text{д}} D T_1 T_3 (T_1 + T_3)^2 (1 + 2DT_2) - \\
& - 2K_{\text{д}} D (T_1 + T_3)(T_1^2 + T_3^2 + 4T_1 T_3)(K_{\text{д}} D T_2^2 + 2T_1 + \\
& + 2T_3)\} \{-T_1^3 T_3^3 K_{\text{д}}^2 D^2 + 4K_{\text{д}} D T_1^2 T_3^2 \quad (T_1 + T_3)(1 + \\
& + 2T_2 K_{\text{д}} D) + K_{\text{д}} D T_1 T_3 (T_1^2 + T_3^2 + 4T_1 T_3)(K_{\text{д}} D T_2^2 + \\
& + 2T_1 + 2T_3) - 2K_{\text{д}} D (T_1 + T_3)(T_1^2 + T_3^2 + 4T_1 T_3)^2 - \\
& - T_1^3 (1 + 2K_{\text{д}} D T_2)(K_{\text{д}} D T_2^2 + 2T_1 + 2T_3^2) - \\
& - 4T_1 T_3 (T_1 + T_3)^2 (1 + 2K_{\text{д}} D T_2)^2 + 2(T_1 + T_3)(T_1^2 + T_3^2 + \\
& + 4T_1 T_3)(1 + 2K_{\text{д}} D T_2)(K_{\text{д}} D T_2^2 + 2T_1 + 2T_3)\}^{-1}.
\end{aligned}
\tag{10.14.15}
$$

Expressions (10.14.12)-(10.14.15) give exact values of the effective passbands of the closed system. It is possible to express them in parameters D, φ_c and ω_c according to the formulas (10.14.6)-(10.14.10). For large values of $K_{\text{Д}} D/\omega_c$ ($K_{\text{Д}} D/\omega_c \geq 10$), which practically always takes place, in Table 10.3 there are given simple approximate expressions for effective bands of a closed tracking system.

Inasmuch as $K_{\text{Д}}$ depends on the signal-to-noise ratio in the radio channel, $\Delta f_{\text{эф}}$ also will depend on this ratio. We note that for large reserves of phase stability ($\varphi_c \geq 50^{\circ}$) $\Delta f_{\text{эф}}$ for all types of transfer function has the same asymptotic form

$$\Delta f_{\text{эф}} = \frac{K_{\text{д}} \omega_c}{4}.$$

Knowing the effective band of the closed tracking system, we can write an expression for fluctuating error of angle measurement, which is equal to $\sigma^2_{\text{фл}} = 2S_{\text{эв}} \Delta f_{\text{эф}}$. To decrease fluctuating error it is necessary, obviously, to decrease $\Delta f_{\text{эф}}$. As can be seen from Table 10.3, to decrease $\Delta f_{\text{эф}}$ it is necessary to decrease ω_c and $K_{\text{Д}}$; however it is necessary to consider that dynamic errors here will increase. Optimum values of parameters of smoothing circuits must be selected from

considerations of a compromise between dynamic and fluctuating errors.

Table 10.3

$H(p)$	$\Delta f_{эф}$
$H_{I}(p)$	$\omega_c \frac{K_Д + \beta_c/2}{4 - \beta_c^2}$
$H_{II}(p)$	$K_Д\omega_c \frac{K_Д + \frac{3}{8}\beta_c}{K_Д(4 - \beta_c^2) - 2\beta_c}$
$H_{III}(p)$	$\omega_c \frac{K_Д(1 - \beta_c^2/16) + \frac{1}{2}\beta_c}{4 - \beta_c^2 - \frac{1}{2}K_Д\beta_c}$
$H_{IV}(p)$	$K_Д\omega_c \frac{K_Д(1 - \beta_c^2/16) + \frac{3}{8}\beta_c}{K_Д(4 - \beta_c^2) - \beta_c}$

Several words about the condition $DK_Д/\omega_c \gg 10$ used by us, allowing to obtain a very simple expressions for effective bands. This condition is realized practically always if $h \gg 1$. With decrease of h, as can be seen from expression (10.14.1), $K_Д$ starts to drop, and with sufficiently small values of h approximate expressions for effective bands of Table 10.3 will already be incorrect. This is necessary to consider during practical calculations.

As an example we shall calculate fluctuating error of measurement of an angle by a system using IAC with the circuit of Fig. 10.17 (or the method of scanning with compensation with the circuit of Fig. 10.15). We assume that smoothing circuits have a transfer function of type $H_I(p)$. We consider that automatic gain control is closed through the sum channel.

For calculation of fluctuating error we use formula (10.7.5) for $S_{экв}$ and (10.14.1) for $K_Д$ (where h is replaced by h_Σ — the signal-to-noise ratio in the sum channel, and $K_{ОД} = 1$). The expression for the effective band we take from Table 10.3. We have

$$\sigma_{\phi x}^2 = 2S_{экв}\Delta f_{эф} = \frac{1}{\mu_x^2 \Delta f_0}\left(\frac{2 + \frac{1}{x}}{h_\Sigma} + \frac{x + 2 + \frac{1}{x}}{h_\Sigma^2}\right) \times$$

$$\times \frac{\omega_c}{2}\frac{2K_Д + \beta_c}{4 - \beta_c^2}. \tag{10.14.16}$$

We take in (10.14.16) $x = 5$, $\varphi_c = 30°$. Ratio $\omega_c/4\Delta f_c$ we will change. Curves of the dependence of $\sigma^2_{\text{фл}}$ on h_Σ for different $\omega_c/4\Delta f_c$ and y shown in Fig. 10.38.

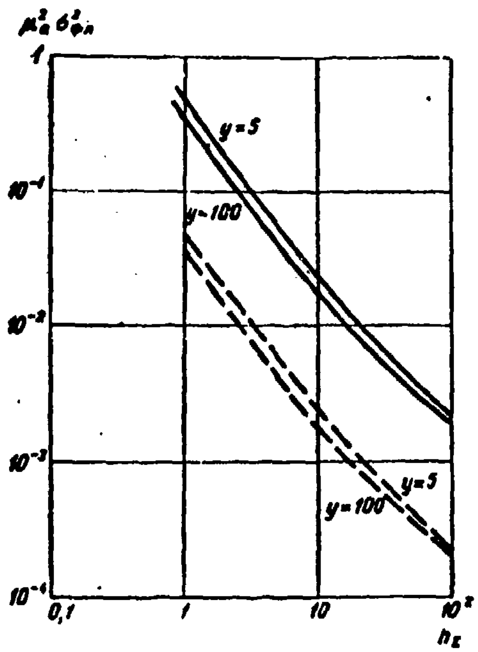

Fig. 10.38. Example of the dependence of fluctuating error of a tracking goniometer on the signal-to-noise ratio. —— $\omega_c/4\Delta f_c = 0.1$; - - - $\omega_c/4\Delta f_c = 0.01$.

When using these graphs it is necessary to consider that expression (10.14.16) is valid, as we already noted, when $DK_Д/\omega_c \geq 10$, i.e., if $D = 400$ 1/sec and $\omega_c = 10$ 1/sec, expression (10.14.16) is valid only when $K_Д \geq 1/4$. This means that the curves of Fig. 10.33 are valid only up to values $1/(1 + y/h_\Sigma) \geq 1/4$. For instance, at $y = 5$ we should have $h_\Sigma \geq 1.6$; at $y = 100$ we have considerably greater limitations on the permissible range of values of h_Σ ($h_\Sigma \geq 33$). In order to have a dependence of $\sigma^2_{\text{фл}}$ on h_Σ, valid for all values of h_Σ, it is necessary in (10.14.16) to substitute the exact value of the effective band of the closed system (10.14.12)-(10.14.15). This would lead to very bulky expressions, and therefore we will limit ourselves to the given approximate calculations, covering, incidentally, the most interesting cases for practice.

We shall now study dynamic errors of tracking goniometer systems. First of all we shall consider an input of the form

$$\vartheta(t) = \vartheta_0(t) + \sum_{i=1}^{m} \mu_i \vartheta_i(t), \qquad (10.14.17)$$

where $\vartheta_i(t)$ – known functions;

μ_i – random normally distributed coefficients with characteristics

$$\overline{\mu_i} = 0, \quad \overline{\mu_i \mu_k} = M_{ik}.$$

The presentation of an input in the form (10.14.17) is rather general and covers a large number of practically important cases. This we already discussed in detail in the preceding chapters.

In Chapter VI we showed that mean square dynamic error in the considered case will be expressed by formula

$$\sigma^2_{\text{дин}} = \sum_{i,k=1}^{m} M_{ik} \varepsilon_i(t) \varepsilon_k(t), \tag{10.14.18}$$

where functions $\varepsilon_k(t)$ have Laplace transforms

$$E_k(p) = \frac{\Theta_k(p)}{1 + K_д H(p)};$$

$\Theta_k(p)$ — Laplace transform of functions $\vartheta_k(t)$;

$H(p)$ — transfer function of the smoothing circuits.

For smoothing circuits of any of the types considered here, having astaticism of the 1st order, for large t we have

$$\varepsilon_k(t) = \frac{\dot{\vartheta}_k(t)}{K_д D}. \tag{10.14.19}$$

Formula (10.14.19) is most often used in practice for estimating dynamic errors of tracking systems. Relationships (10.14.18) and (10.14.19) show that the steady-state value of dynamic error only exists when functions $\vartheta_i(t)$ have a bounded 1st derivative. Otherwise dynamic error grows without limit. For instance, in the case of linear functions

$$\vartheta_k(t) = a_{0k} + a_{1k} t \tag{10.14.20}$$

we have

$$\varepsilon_k(t) = \frac{a_{1k}}{K_д D}. \tag{10.14.21}$$

Mean square dynamic error here is equal to

$$\sigma^2_{\text{дин}} = \frac{\sigma_0^2}{K_д D^2}, \quad \sigma_0^2 = \sum_{i,k=1}^{m} M_{ik} a_{1i} a_{1k} \tag{10.14.22}$$

Thus, dynamic error is inversely proportional to the slope of the discrimination characteristic $K_д$ and Q-factor D. With decrease of $K_д$, taking place with decrease of the signal-to-noise ratio, dynamic error increases. For example, in Fig. 10.39 there is given the dependence of $D\sigma_{\text{дин}}/\sigma_0$ on h. Analogously we can consider error with more complicated inputs of form (10.14.17).

In the considered simple case dynamic and fluctuating components of total tracking error are minimized independently. To lower the first component it is necessary to increase Q-factor D; to lower the second it is necessary to increase the cutoff frequency ω_c. However, consideration of more complicated inputs leads to more complicated conditions of a minimum of overall error, ensuing from the compromise between dynamic and fluctuating errors.

Let us consider dynamic errors with purely random inputs. As it is known, during depletion [? processing (text misprint)] of a stationary random input with spectral density $S_\mu(\omega)$ there exists dynamic error, the mean square of which is equal to

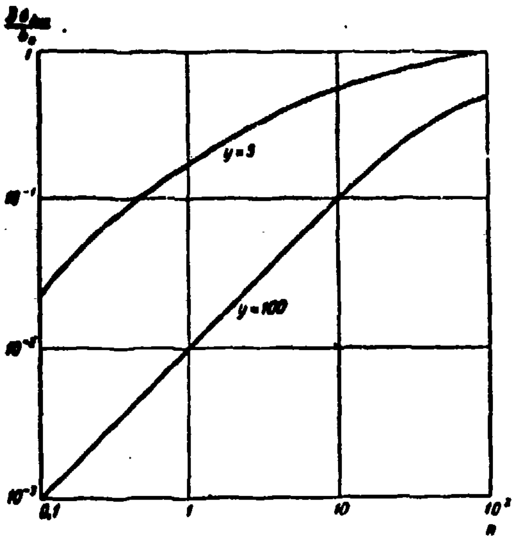

Fig. 10.39. Example of the dependence of dynamic error of a tracking goniometer on the signal-to-noise ratio with an input in the form of known functions with random coefficients.

$$\sigma^2_{дин} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{S_\mu(\omega)}{|1 + K_д H(i\omega)|^2} d\omega. \quad (10.14.23)$$

For calculation by this formula we approximate $S_\mu(\omega)$ by expression

$$S_\mu(\omega) = \begin{cases} S_\mu, & |\omega| < \omega_\mu, \\ 0, & |\omega| > \omega_\mu. \end{cases} \quad (10.14.24)$$

Let us consider particular cases occurring with different relationships of ω_μ, T_1, T_2, T_3.

We turn to transfer function $H_I(p)$. We shall use approximations

$$|H_1(i\omega)| \approx \frac{D}{\omega},\ \omega < \frac{1}{T_1},\quad |H_1(i\omega)| \approx \frac{D}{\omega^2 T_1},\ \frac{1}{T_1} < \omega < \frac{1}{T_2}, \quad (10.14.25)$$

(which already were used in deriving formula (10.14.6)). We note that when $\omega < 1/T_2$ quantity $|H_I(i\omega)| \gg 1$.

Here

$$\sigma^2_{дин} = \frac{S_\mu}{K_д^2\, 2\pi} \int_{-\omega_\mu}^{\omega_\mu} \frac{d\omega}{|H(i\omega)|^2} = \begin{cases} \dfrac{1}{3\pi} \dfrac{S_\mu \omega_\mu^3}{K_д^2 D^2}, & \omega_\mu < \dfrac{1}{T_1}, \\ \dfrac{S_\mu \omega_\mu^3}{3\pi K_д^2 D^2} \left(\dfrac{3}{5} \omega_\mu^2 T_1^2 - \dfrac{3}{5} \omega_\mu^3 T_1^3 + 1 \right), & \\ \dfrac{1}{T_1} < \omega_\mu < \dfrac{1}{T_2}. & \end{cases} \quad (10.14.26)$$

Analogously for the remaining types of transfer functions we obtain the following results: in range $T_1\omega_\mu < 1$ quantity $\sigma^2_{дин}$ in all cases is identical and is expressed by formula (10.14.26); in range $1/T_1 < \omega_\mu < 1/T_2$ for transfer function $H_{III}(p)$ the square of error $\sigma^2_{дин}$ has the same form as for $H_I(p)$, i.e., is expressed by formula (10.14.26), and for functions $H_{II}(p)$ and $H_{IV}(p)$ by

$$\sigma^2_{\text{дин}} = \frac{1}{3\pi}\frac{S_\mu \omega_\mu^3}{K_\text{д}^2 D^2}\left(\frac{3}{7}\omega_\mu^4 T_1^4 - \frac{13}{7T_1^3\omega_\mu^3} + 1\right), \quad \frac{1}{T_1} < \omega_\mu < \frac{1}{T_2}. \tag{10.14.27}$$

Note that the obtained formulas occur, of course, only for the case of $K_\text{Д}$ which is not too small [so that it is valid to disregard the one in the denominator of the integrand in (10.14.23)]. Time constant T_1 in formulas (10.14.26) and (10.14.27) can be replaced, if desired, bv the expression containing parameters ω_c , D and φ_c from Table 10.3.

In Fig. 10.40 we constructed from formula (10.14.26) the dependence of $K_\text{Д}\sigma_\text{дин}/\sigma_0$($\sigma_0 = 2S_\mu\omega_\mu$ – mean square value of input) on the width of the spectrum of input ω_μ for different ω_c. During calculation of this dependence we assumed D = 400 1/sec, $\varphi_c = 30°$, $T_2 \approx 0.1$ sec. Note that a section of this dependence to ω_μ = 1 cps is valid when $DK_\text{Д}T_\Gamma$ [?] >> 1, i.e., $K_d \gg 1.25\cdot 10^{-3}$ [for y = 100 in formula (10.14.1) the permissible signal-to-noise ratio h >> 10^{-1}]. The section ω_μ > 1 cps is valid practically only at $K_\text{Д} \approx 1$, i.e., for rather low noise levels.

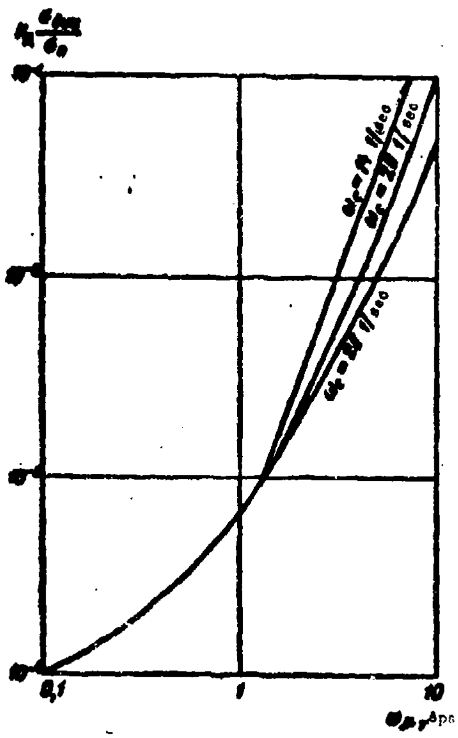

Fig. 10.40. Example of the dependence of dynamic error of a tracking goniometer on the width of the spectrum of a random input.

Let us turn to the question of optimum selection of parameters of smoothing circuits in the presence of a random input. This question we shall consider purely illustratively, inasmuch as complete analysis is very difficult to perform here. Let us assume that the random input has a square spectrum (10.14.24) with $1/T_1 < \omega_\mu < 1/T_2$. If the smoothing circuit has transfer function $H_I(p)$, total error of tracking is equal (when $\omega_\mu T_1 \gg 1$)

$$\sigma^2 = \sigma^2_{\text{фл}} + \sigma^2_{\text{дин}} = 2\omega_c \frac{K_\text{д} + \frac{1}{2}\beta_c}{4 - \beta_c^2} S_{\text{экв}} + \frac{S_\mu \omega_\mu^5 \cdot 4}{15\pi\omega_c^4 K_\text{д}^2 \beta_c^2}. \tag{10.14.28}$$

Varying this expression with respect to ω_c and β_c , it is easy to find that the optimum cutoff frequency is equal to

$$\omega_{\text{сопт}} = \frac{0{,}7}{\omega_\mu}\left(\frac{4-\beta_c^2}{K_д + \frac{1}{2}\beta_c}\right)^{\frac{1}{5}}\left(\frac{S_\mu}{S_{экв}}\right)^{\frac{1}{5}}\frac{1}{\beta_c K_д^{\frac{2}{5}}}, \tag{10.14.29}$$

and the optimum reserve of phase stability for $K_Д = 1$ $\beta_{C\ \text{опт}} = 2/5$, i.e., $\varphi_C = 67°$.

Substituting this value of φ_C in (10.14.32), we obtain (for $K_Д = 1$)

$$\omega_{\text{сопт}} = 0{,}88\omega_\mu\left(\frac{S_\mu}{S_{экв}}\right)^{\frac{1}{5}}. \tag{10.14.30}$$

Thus, the optimum cutoff frequency is proportional to the bandwidth of the spectrum of the random input; with growth of the spectral density of this input ω_C increases, and with growth of $S_{ЭКВ}$ it drops; however, the dependency of ω_C on these quantities is rather weak.

The given example shows that there exist optimum values of parameters of smoothing circuits, depending on the type of input. Optimum values depend also on the signal-to-noise ratio (through $K_Д$ and equivalent spectral density $S_{ЭКВ}$) and should be adjusted with change of this ratio. In an analogous way we can investigate cases of other inputs.

Let us consider now in brief the question of the influence of parametric fluctuations. As shown in Chapter VI, the presence of parametric fluctuations leads to increase of errors of measurement. With smoothing circuits with constant parameters this increase is expressed by relationship (6.2.39). As we have seen earlier, the spectral density of parametric fluctuations in almost all cases is expressed by formula (10.5.23) with (10.5.24). In Paragraph 10.14.1 it is shown that with AGC the spectral density of parametric fluctuations decreases according to relationship (10.14.2). From this we find that for the considered circuit error taking into account parametric fluctuations is expressed by formula

$$\sigma^2 = \sigma_0^2\left(1 + 2S_{пар}\Delta f_{эф}\frac{1}{n^2}\right) = \sigma_0^2\left(1 + \frac{\Delta f_{эф}}{\Delta f_c}\frac{1+3x+x^2}{x(1+x)}\frac{1}{n^2}\right),$$

where σ_0^2 — error ignoring parametric fluctuations, and the remaining designations are the same as before [see designations for formulas (10.5.24) and (10.14.2)].

10.14.3. Optimum Smoothing Circuits and Smoothing Circuits with Variable Parameters

Smoothing circuits considered in the preceding paragraph, as already noted, are practically the most commonly used in radar goniometers.

However, it is of interest to discuss other possible types of smoothing circuits.

Although the problem of their synthesis in principle is solved in the rather general assumptions about the character of the input (see § 6.8), an effective solution cannot always be obtained. The simplest solutions are obtained when the input is a stationary random process or a process with stationary increments. The smoothing filter here has constant parameters, depending on the statistical properties of the input. Here the smoothing circuits synthesized for the goniometer will not differ at all from smoothing circuits of a range finder or speed meter if properties of the input in all these cases are identical. Therefore all results obtained in Chapters VII and IX regarding optimum smoothing circuits almost without change can be tranferred to the case of measurement of angles. For instance, if the angular coordinate of the target is presented in the form of the double integral of white noise with spectral density B_2, the optimum smoothing filter is a double integrator with correction, with transfer function

$$H(p)=\frac{K_{И}(1+pT_k)}{p^2}, \tag{10.14.31}$$

where $K_{И}$ and T_k are expressed by characteristics of the radio channel $K_{Д}$ and $S_{ЭКВ}$ in the following way:

$$K_{И}=\frac{1}{K_{Д}}\sqrt{\frac{B_2}{S_{экв}}},\quad T_k=\sqrt[4]{4S_{экв}/B_2} \tag{10.14.32}$$

(coefficient B_2 has dimensionality [$\deg^2$ cps^3] and is equal to the mean square of the angular velocity developed by the target in 1 sec). Variance of total error of measurement in this case is equal to

$$\sigma^2=\sqrt{2}(B_2S^3_{экв})^{1/4}. \tag{10.14.33}$$

Let us consider, for example, a tracking goniometer in which there is used the IAC method. The circuit of the goniometer with smoothing circuits (10.14.31) is depicted in Fig. 10.15. From formulas (10.7.5) and (10.14.33) we find that total error is equal to

$$\sigma^2=\sqrt{2}\left[B_2\frac{1}{\mu_a^6\Delta f_c^3}\left(\frac{2+1/x}{h_\Sigma}+\frac{x+2+1/x}{h_\Sigma^2}\right)^3\right]^{1/4}. \tag{10.14.34}$$

Considering $\mu_a \approx 1.4/\Delta\varphi$, where $\Delta\varphi$ — width of the directional pattern, we reduce formula (10.14.34) to a form convenient for calculation

$$\left(\frac{\sigma}{\Delta\varphi}\right)^2=0{,}87\left[\frac{B_2}{\Delta\varphi^2\Delta f_c^3}\left(\frac{2+1/x}{h_\Sigma}+\frac{x+2+1/x}{h_\Sigma^2}\right)\right]^{1/4}.$$

Curves of the dependence of $\sigma/\Delta\varphi$ on h_Σ for various x are shown in Fig. 10.41. During calculation we assumed $B_2 = 10^{-7}$ deg²cps³, $\Delta\varphi = 1°$, $\Delta f_c = 30$ cps.

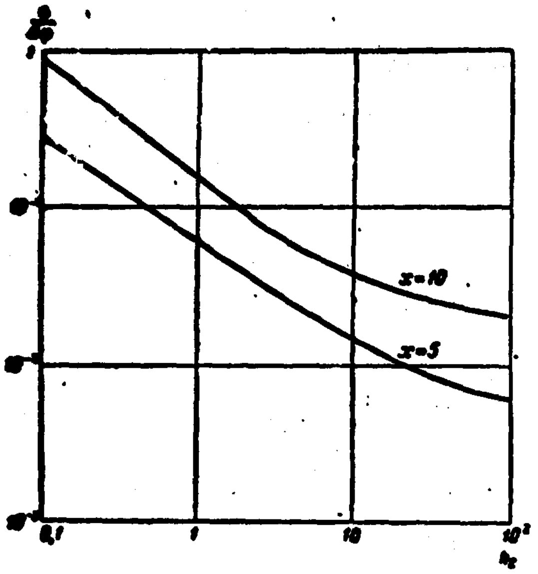

Fig. 10.41. The dependence of total error of an tracking goniometer on the signal-to-noise ratio with an input in the form of the double integral of white noise.

Note that parameters of the optimum smoothing filter (10.14.32) depend on the signal-to-noise ratio. With practical realization of such a filter they will be selected, of course, constant, and error can be expressed by a fromula differing from (10.14.33). This and similar cases were considered in detail in Chapters VII and IX. The results obtained there are easily generalized to the case of goniometers.

In many cases a minimum of errors in radar meters is ensured with the use of smoothing circuits with variable parameters. Practically such circuits in radar goniometers are almost never used; however, considering the prospects of development of technology, it is of interest to analyze error of radar goniometers using smoothing filters with variable parameters. Here again it is necessary to note that from the point of view of smoothing the goniometer has no individual specific character as compared to meters of other types. Therefore, results obtained for range finders and speed meters are almost completely transferable to the case of goniometers.

Let us briefly discuss basic propositions concerning smoothing circuits with variable parameters. If the angular target coordinate varies according to the law

$$\vartheta(t) = \sum_{i=1}^{m} \mu_i \vartheta_i(t) + \vartheta_0(t),$$

where μ_i — normally distributed random variables; $\overline{\mu_i} = 0$; $\overline{\mu_i \mu_k} = M_{ik}$; $\vartheta_i(t)$ — known functions, the optimum smoothing filter will have variable parameters. Its pulse response (with an optimum discriminator) and variance of error are expressed by formulas (6.8.54).

If, for instance, the angular target coordinate varies according to the law

$$\vartheta(t) = (\vartheta_1 + \vartheta_2 t) + (\mu_1 + \mu_2 t),$$

where $\overline{\mu_1^2} = \sigma_1^2$, $\overline{\mu_2^2} = \sigma_2^2$, according to (6.8.54) pulse response of the smoothing filter is equal to

$$G(t,\tau)=\frac{\sigma_1^2\left(1-\frac{\sigma_2^2}{S_{onr}}\frac{\tau^2}{6}\right)+\sigma_2^2 t\tau\left(1+\frac{\sigma_1^2}{S_{onr}}\frac{\tau}{2}\right)}{1+\frac{1}{S_{onr}}\left(\sigma_1^2\tau+\sigma_2^2\frac{\tau^3}{3}\right)+\frac{\sigma_1^2\sigma_2^2\tau^4}{12S_{onr}^2}}.$$

Variance of error of measurement is found in the form

$$\sigma^2=\frac{\sigma_1^2+\sigma_2^2t^2+\frac{1}{3S_{onr}}\sigma_1^2\sigma_2^2t^3}{1+\frac{1}{S_{onr}}\sigma_1^2t+\frac{1}{3S_{onr}}\sigma_2^2t^3+\frac{1}{12S_{onr}^2}\sigma_1^2\sigma_2^2t^4}.$$

With growth of t error seeks 0. For large t, obviously, we have

$$\sigma^2=\frac{4S_{onr}}{t}.$$

With arbitrary smoothing circuits with variable parameters error of tracking is determined by formula (6.2.15). No difficulties appear during use of this formula.

§ 10.15. Nonlinear Phenomena in Tracking Goniometers

The whole preceding analysis of tracking radar goniometers was based on their linearized presentation. Such a presentation is valid if tracking of angles is carried out with sufficiently high accuracy. In this case it turns out that the mean value of the output signal of the radio channel is proportional to the current angular mismatch, and the spectral density of noises does not depend on mismatch. However, these idealizations take place only with a comparatively high signal-to-noise ratio. In real conditions, especially in the case of the action of interferences, essentially decreasing the signal-to-noise, mismatch during tracking may be considerable.

Here the discrimination and fluctuation characteristcs of the radio channel of the goniometer become nonlinear functions of mismatch. Study of the process of tracking and of errors of goniometers in this regime represents an important and interesting task.

The task of investigating tracking systems in a nonlinear regime was considered in § 6.3. This problem is very complicated, and results obtained upon its solution are approximate and particular. However by them we can draw certain conclusions about nonlinear phenomena in tracking goniometers and find the order of numerical characteristics of these phenomena. We then use these results to study the phenomena of breakoff of tracking occurring with a sufficiently high level of noises, when increasing mismatch becomes comparable to the width of the directional pattern.

In order to best approximate conditions in which we produced analysis in § 6.3, we shall limit ourselves henceforth to the simplified situation when the fluctuation characteristic in required limits can be considered constant and equal to the characteristic $S_{\text{экв}}$ used everywhere earlier. This assumption is well justified, since due to demodulating properties of automatic gain control the fluctuation characteristic is essentially leveled. Furthermore, most interesting is the case of allowing for nonlinearity of the discrimination characteristic. In the preceding chapters, conforming to requirements of linear theory, we calculated only the slope of the discrimination characteristic at zero mismatch. Now we need, in general, knowledge of all the behavior of the discrimination characteristics.

The form of the discrimination characteristics of radar goniometers essentially depends on the methods of direction finding utilized and, in general, turns out to be very complicated. We will limit ourselves to consideration, as an example, of a goniometer using the IAC method. For a goniometer with IAC, in which reception of signals is carried out by two directional patterns having form $g_1(\varphi)$ and $g_2(\varphi)$, the received signals, as it is known, can be recorded in the form of formula (10.3.3), in which n = 2, and $U_i(t, \alpha)$ are determined by relationships (10.2.7). Determining for any of the circuits of the IAC method the error signal and averaging, we obtain (assuming identity of filters in the circuits) for the discrimination characteristic the following expression:

$$a(\alpha)=\frac{P_{c\Sigma}}{8\pi\Delta f_0}\int\limits_{-\infty}^{\infty}\operatorname{Re}H_1(i\omega)H^*_2(i\omega)S_0(\omega)\,d\omega\times$$

$$\times[g_1^2(\gamma_1+\alpha)-g_2^2(\gamma_2-\alpha)]=K[g_1^2(\gamma_1+\alpha)-g_2^2(\gamma_2-\alpha)]. \qquad (10.15.1)$$

Thus, the form of the discrimination characteristic essentially depends on the form of the directional pattern. We use the very wide-spread cosine approximation of the directional pattern, considering the diagrams identical:

$$g_1(\varphi)=g_2(\varphi)=\cos\frac{\pi}{2}\frac{\varphi}{\Delta\varphi},$$

where $\Delta\varphi$ — width of the directional pattern at the level of half power.

We also assume $\gamma_1 = \gamma_2 = \Delta\varphi/2$. Here,

$$a(\alpha)=K\sin\frac{\pi\alpha}{\Delta\varphi}. \qquad (10.15.2)$$

We obtained a discrimination characteristic in the form in which it was approximated in § 6.3. Consequently, it is possible to use directly the results of § 6.3, considering the half-width of the selected region $\Delta = \Delta\varphi$.

From results of § 6.3 it follows that the critical value of the ratio of mean square error of the linearized system $\sigma_{л}$ to the half-width of the selected region $\Delta = \Delta\varphi$ at which breakoff of tracking occurs is $(\sigma_{л}/\Delta\varphi)_{кр} \approx 0.2$ (by the criterion of sharp increase of the average time to the first breakoff) and $(\sigma_{л}/\Delta\varphi) \approx 0.1$ (by the criterion of sharp growth of mean square error).

Considering $\sigma_{л} = \sqrt{2\Delta f_{эф} S_{экв}}$ (dynamic errors are absent) and using expression $S_{экв}$ for the investigated circuit (10.7.5), it is easy to find the critical signal-to-noise ratio, at which failure occurs

$$\left(\frac{\sigma_{л}}{\Delta\varphi}\right)_{кр} = \frac{\sqrt{\dfrac{2+1/x}{h_{кр}} + \dfrac{x+2+1/x}{h_{кр}^2}}}{0{,}7\sqrt{\dfrac{\Delta f_c}{\Delta f_{эф}}}} = 0{,}2 \text{ or } 0{,}1.$$

Hence when $x = 1$ and $\Delta f_c/\Delta f_{эф} = 100$ we have $h_{кр} = 2.3$ (by the criterion of sharp growth of mean square error). The obtained magnitudes of $h_{кр}$ coincide in order with those that are known from practice.

Consideration of more complicated examples, accounting for the proximity of the obtained results, does not have meaning.

§ 10.16. Nontracking Radar Goniometers

We shall investigate radar goniometers not containing feedbacks and giving an estimate of the angular coordinate of a target at the output of an open circuit. There exist many tactical tasks in which application of just such goniometers is most expedient.

As a rule, nontracking goniometers work by the following scheme. Directional patterns of the antenna system of such goniometers cover the sector of possible values of the angular coordinates of the target. The goniometer consists of two basic units: a unit of primary processing of the signal and a unit of secondary processing. The unit of primary processing continuously issues an estimate of the angular coordinate of a target from the observed realization of a signal of duration Δt. Here time Δt, naturally, is sufficiently small so that the angular coordinate of the target will not vary. The unit of secondary processing smoothes the data of primary processing taking into account a priori laws of change of angular coordinates.

Synthesis of an optimum nontracking meter was performed in § 6.6. In the case of a rapidly fluctuating signal and for sufficiently large signal-to-noise ratios the optimum operation is given by expression (6.6.55). The structure of the optimum nontracking meter turns out to be very close to that which was described above.

The meter contains a unit of primary processing, called in Chapter VI an "estimator unit," which issues the maximum likelihood estimate of the measured parameter from the segment of realization of the signal during a time considerably exceeding the time of correlation of the signal, but considerably smaller than the time of correlation of the measured parameter. Then there follows a smoothing filter which has a pulse response given by relationship (6.6.53). Potential accuracy of nontracking meters here coincides with the accuracy of tracking meters.

Then we shall study synthesis of optimum nontracking goniometers using the IAC and IPC methods. For these methods it is possible to completely find the optimum circuit of a nontracking goniometer (for IAC, to be sure, with a certain approximation of the directional pattern). Other methods of direction finding in nontracking goniometers are practically not used.

10.10.1. Optimum Circuit of a Nontracking Goniometer with IAC

We start our consideration with the IAC method. With this method there is received a pair of signals of form (10.3.2), where $U_{ai}(t, \alpha)$ are determined by formula (10.2.5). Correlation functions of these signals is given by expression (10.3.7). If the interval of observation of signals is $(0, \Delta t)$, then the functional of distribution of probabilities of these signals has the form (10.3.8) [taking into account (10.3.10) and (10.3.11)]. So that this functional is completely determined, we need to find the matrix of inverse-correlation functions, solving equation (10.3.12) for this. The solution may again be sought in form (10.3.14), and for $v(t_1, t_2)$ we obtain equation (10.3.15), which (with identical noises) can be rewritten in the form

$$\frac{P_{c1}g_1(\gamma_1-\alpha)^2+P_{c2}g_2(\gamma_2+\alpha)^2}{2N_0}\int_0^{\Delta t}\rho(t_1-t_3)\,v(t_3, t_2)\,dt_3+\rho(t_1-t_2)+v(t_1, t_2)=0. \tag{10.10.1}$$

Henceforth we shall consider the patterns identical, and we introduce the approximation already used in the preceding section:

$$g_1(\varphi)=g_2(\varphi)=\cos^{n}\frac{\pi}{4}\frac{\varphi}{\Delta\varphi}, \tag{10.10.2}$$

where $\Delta\varphi$ – width of the directional pattern for the half-power level.

We also assume that $\gamma_1 - \gamma_2 = \Delta\varphi/2$, i.e., the patterns cross at the half-power level. Here (10.10.1) takes the following form:

$$\frac{P_{c2}}{2N_0}\int_0^{\Delta t}\rho(t_1-t_3)\,v(t_3, t_2)\,dt_3+\rho(t_1-t_2)+v(t_1, t_2)=0, \tag{10.10.3}$$

where $P_{c\,\Sigma}$ — total received power.

As can be seen from (10.16.3), function $v(t_1, t_2)$ will not depend on α.

We compose now the maximum likelihood equation for measurement of parameter α. Calculating by formula (10.3.11) it is easy to prove that $\partial \ln K/\partial\alpha = 0$. Therefore the maximum likelihood equation takes the form

$$\int_0^{\Delta t}\int_0^{\Delta t} \mathbf{Y}^+(t_1)\frac{\partial}{\partial\alpha}\mathbf{W}(t_1, t_2)\mathbf{Y}(t_2)\,dt_1dt_2 = -\sum_{i,j=1}^{2}\int_0^{\Delta t}\int_0^{\Delta t}\frac{\partial}{\partial\alpha}W_{ij}(t_1, t_2)\,y_i(t_1)\,y_j(t_2)\,dt_1dt_2 = 0. \qquad (10.16.4)$$

Substituting expression (10.3.14) in (10.16.4), it is possible to convert equation (10.16.4) to a rather simple form and to solve it explicitly.

Omitting simple calculations we give the final result:

$$\hat{\alpha} = \frac{\Delta\varphi}{\pi}\operatorname{arcctg}\Big\{\Big[\mathrm{Re}\int_0^{\Delta t}\int_0^{\Delta t} v(t_1, t_2)\,u(t_1)\,u^*(t_2)\,e^{i\omega_0(t_1-t_2)}\times y_1(t_1)\,y_1(t_2)\,dt_1dt_2 - \mathrm{Re}\int_0^{\Delta t}\int_0^{\Delta t} v(t_1, t_2)\,u(t_1)\,u^*(t_2)\times e^{i\omega_0(t_1-t_2)}\,y_2(t_1)\,y_2(t_2)\,dt_1dt_2\Big]\Big[2\mathrm{Re}\int_0^{\Delta t}\int_0^{\Delta t} v(t_1, t_2)\times u(t_1)\,u^*(t_2)\,e^{i\omega_0(t_1-t_2)}\,y_1(t_1)\,y_2(t_2)\,dt_1dt_2\Big]^{-1}\Big\}. \qquad (10.16.5)$$

In this expression we have to substitute function $v(t_1, t_2)$, found from equation (10.16.3).

We shall first consider the case of rapid fluctuations of the signal, when the time of correlation of fluctuations is considerably less than Δt. As it is easy to see, equation (10.16.3) in this case is identical to equation (10.3.17), and for $v(t_1, t_2)$ we obtain solution (10.3.18). Substituting (10.3.18) in expression (10.16.5), we can present the optimum operation of measurement in the following form:

$$\hat{\alpha} = \frac{\Delta\varphi}{\pi}\operatorname{arcctg}\Big\{\int_0^{t} d\tau\Big[\Big|\int_0^{\Delta t} h_{1\text{опт}}(\tau - s)\,y_1(s)\times u(s)\,e^{i\omega_0 s}ds\Big|^2 - \Big|\int_0^{\Delta t} h_{1\text{опт}}(\tau - s)\,y_2(s)\,u(s)\,e^{i\omega_0 s}ds\Big|^2\Big]\times\Big[\int_0^{\Delta t} d\tau\,\mathrm{Re}\int_0^{\Delta t} h_{1\text{опт}}(\tau - s)\,y_1(s)\,u(s)\,e^{i\omega_0 s}ds\int_0^{\Delta t} h_{1\text{опт}}(\tau - s)\times y_2(s)\,u^*(s)\,e^{-i\omega_0 s}ds\Big]^{-1}\Big\}, \qquad (10.16.6)$$

where the pulse response of the optimum filter $h_{1\ \text{опт}}$ is determined by relationship (10.3.23) [the gain-frequency response of the filter is given by formula (10.3.25)]. Expression (10.16.6) can be easily converted to real form [analogous to the transition from (10.3.24) to (10.3.29)].

The complete block diagram of an optimum nontracking meter using IAC can be presented in the form depicted in Fig. 10.42. Let us describe briefly the work of this circuit. Signals from the output of the antennas first are heterodyned (it is understood that heterodyne voltage has the proper amplitude modulation), are filtered by optimum filters with frequency response (10.3.25), are detected by square-law detectors, and are subtracted. The difference is integrated and enters a divider (as the dividend). To the second input of the divider (the divisor) there proceeds the integrated result of multiplication of signals from the output of the optimum filters (phase detection). The result of division enters an inertialess unit with nonlinear characteristic y = = arc ctg x (y — output, and x — input quantities).

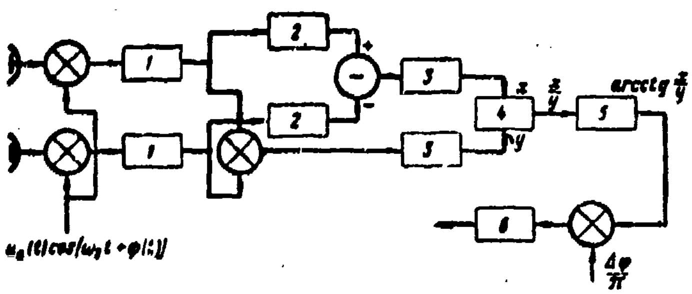

Fig. 10.42. Optimum circuit of a nontracking meter with IAC for rapid fluctuations of the signal: 1) optimum filters with frequency response $H_{1\ \text{опт}}(i\omega)$; 2) square-law detectors; 3) integrators; 4) dividers; 5) nonlinear unit; 6) smoothing circuits.

The output signal of the nonlinear unit, multiplied by $\Delta\varphi/\pi$ ($\Delta\varphi$ — width of the directional pattern), enters the smoothing circuits. Optimum smoothing circuits, as shown in § 6.6, are a linear filter with pulse response (6.6.53).

Let us consider the question of accuracy of measurement of angles in the synthesized circuit.

First of all we investigate accuracy of the estimator unit. In general in the presence in the circuit of an estimator unit estimates of deviations of its accuracy from optimality are very difficult to calculate. However, for an optimum circuit accuracy can easily be found. It is possible to show that the operation performed by the estimator unit gives an asymptotically efficient estimate of the angular coordinate for large values of $\Delta t \Delta f_c$, where Δf_c — band of fluctuations of the signal; Δt — time of observation. Variance of the efficient estimate, as it is known, is expressed by formula

$$\sigma^2_{э\phi} = \frac{1}{\left(\frac{\partial \ln L(\alpha)}{\partial \alpha}\right)^2}, \tag{10.16.7}$$

where $L(\alpha)$ — likelihood function.

Substituting expression (10.3.8) in (10.16.7), we obtain

$$\sigma^2_{э\phi} = \left\{-\frac{1}{2}\int_0^M\int_0^M \sum_{i,j=1}^{n} W'_{ij}(t_1, t_2) R'_{ij}(t_1, t_2)\, dt_1 dt_2\right\}^{-1} \tag{10.16.8}$$

(similar transformations we already made in § 10.3 during calculation of the minimum possible equivalent spectral density, which is proportional to the variance of the efficient estimate). We substitute expressions (10.3.7), (10.3.14) and (10.3.18) in (10.16.8); then

$$\sigma^2_{э\phi} = \left[\frac{\Delta t h^2_\Sigma \cdot 2{,}5}{\Delta\varphi^2 \cdot 2\pi}\int_{-\infty}^{\infty}\frac{S_0^2(\omega)}{1 + h_\Sigma S_0(\omega)}d\omega\right]^{-1}, \tag{10.16.9}$$

where $h_\Sigma = P_{c\ \Sigma}/2N_0\Delta f_c$ — ratio of signal power to the power of noise in the band of signal fluctuations;

$S_0(\omega)$ — spectrum of signal fluctuations.

Thus, variance of the efficient estimate decreases with growth of Δt (inversely proportional dependence). It is essential to note that the found variance does not depend on angle α, i.e., accuracy of measurement of the angular coordinate does not depend on the position of the target in the considered sector.

We shall not investigate expression (10.16.9), since with an accuracy of proportionality factors it coincides with expression (10.6.5), studied in detail in § 10.6. We indicate only the order of error; for $h_\Sigma = 10$ and $\Delta t\Delta f_c = 10$ we obtain $\sigma_{э\phi} = 0.064\ \Delta\varphi$.

We shall prove now the asymptotic efficiency of the estimate issued by the estimator unit. As it is known, an estimate of a parameter is close to efficient if

$$\sqrt{\overline{\left(\frac{\partial^2 L(\alpha)}{\partial \alpha^2} - \frac{\overline{\partial^2 L(\alpha)}}{\partial \alpha^2}\right)^2}} \ll -\frac{\overline{\partial^2 L(\alpha)}}{\partial \alpha^2} = \frac{1}{\sigma^2_{э\phi}}, \tag{10.16.10}$$

i.e., if the mean square value of deviations of $\partial^2 L(\alpha)/\partial\alpha^2$ from its mean value is negligible (see [8, 13]). As it is easy to show, for sets of normal signals of type (10.3.7) we have

$$\left(\overline{\frac{\partial^2 L(\alpha)}{\partial \alpha^2}-\overline{\frac{\partial^2 L(\alpha)}{\partial \alpha^2}}}\right)^2=\frac{1}{2}\sum_{i,j,k,l=1}^{2}\int_0^{\Delta t}\int_0^{\Delta t}\int_0^{\Delta t}\int_0^{\Delta t} W''_{ij}(t_1, t_2)\times$$

$$\times W''_{kl}(t_3, t_4) R_{ik}(t_1, t_3, \alpha) R_{jl}(t_2, t_4, \alpha)\, dt_1 dt_2 dt_3 dt_4. \qquad (10.16.11)$$

Substituting in (10.16.11) expressions (10.3.7), (10.3.14) and (10.3.18) [instead of $v(t_1, t_2)$] and using approximation (10.16.2), we obtain

$$\left(\overline{\frac{\partial^2 L(\alpha)}{\partial \alpha^2}-\overline{\frac{\partial^2 L(\alpha)}{\partial \alpha^2}}}\right)^2=\frac{h_\Sigma^2 \sqrt{\Delta f_c}}{2}\sqrt{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{S_0(\omega)^4}{[1+h_\Sigma S_0(\omega)]^2}\,d\omega}. \qquad (10.16.12)$$

From which we have

$$\frac{\sqrt{\left(\overline{\frac{\partial^2 L(\alpha)}{\partial \alpha^2}-\overline{\frac{\partial^2 L(\alpha)}{\partial \alpha^2}}}\right)^2}}{-\overline{\frac{\partial^2 L(\alpha)}{\partial \alpha^2}}}=\frac{2}{\sqrt{\Delta f_c}}\times\frac{\sqrt{\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{S_0(\omega)^4}{[1+h_\Sigma S_0(\omega)]^2}\,d\omega}}{\int_{-\infty}^{+\infty}\frac{S_0(\omega)^2}{1+h_\Sigma S_0(\omega)}\,d\omega}. \qquad (10.16.13)$$

The ratio of integrals has the order $1/\sqrt{\Delta f_c}$ [with a square approximation of $S_0(\omega)$ this ratio is equal to $1/\sqrt{2\pi\Delta f_c}$]. Thus, ratio (10.16.13) has the order $1/\sqrt{\Delta f_c \Delta t}$ and seeks 0 with growth of Δt (as compared to the time of correlation of fluctuations). It follows from this that for large $\Delta t \Delta f_c$ the maximum likelihood estimate becomes efficient, and operation (10.16.6) ensures measurement with the minimum possible error, given by formula (10.16.9).

Error of the goniometer after smoothing can easily be found for any linear smoothing circuits. Actually, by virtue of the asymptiotic efficiency of the estimate issued by the estimator unit the output signal of this unit will be equal to the sum of the true value of the angular coordinate and white noise with spectral density $\sigma^2_{\vartheta\phi}\Delta t$, where Δt — duration of the section of the signal processed in the estimator unit. Quantity $\sigma^2_{\vartheta\tilde{\phi}}\Delta t$, as can be seen from (10.16.9), does not depend on Δt and coincides with $S_{0\Pi\Pi}$, met during the study of tracking goniometers. From this it follows that tracking and nontracking goniometers with ideal construction of radio channels and identical structure of smoothing circuits (in the case of a tracking meter we have in mind the structure of the closed-loop system) will give identical accuracy of measurement.

In [46, 47] considerable attention was also paid to the case of a slowly fluctuating signal. For this case synthesis of meters, in general, was not carried out. However, it is possible to limit ourselves to synthesis of the unit of primary

processing, which gives the stimate of an angular coordinate for a small time, when this coordinate cannot change.

For such synthesis it is necessary in expression (10.16.5), giving the maximum likelihood estimate, to substitute function $v(t_1, t_2)$, calculated with various assumptions about the character of signal fluctuation. We assume that during the time of observation Δt the amplitude and phase of the signal cannot change (due to fluctuations of the target). As it was shown in Chapter VI, equation (10.16.3) has in this case the solution

$$v(t_1, t_2) = \text{const} = -\frac{1}{1+\frac{P_{c1}\Delta t}{2N_0}}. \tag{10.16.14}$$

Here (10.16.5) takes form

$$\hat{\alpha} = \frac{\Delta\varphi}{\pi}\,\mathrm{arcctg}\left\{\left[\left|\int_0^{\Delta t} y_1(t)\,u(t)\,e^{i\omega_0 t}\,dt\right|^2 - \left|\int_0^{\Delta t} y_2(t)\,u(t)\,e^{i\omega_0 t}\,dt\right|^2\right]\cdot\left[2\mathrm{Re}\int_0^{\Delta t} y_1(t)\,u(t)\,e^{i\omega_0 t}\,dt \times \int_0^{\Delta t} y_2(t)\,u^*(t)\,e^{-i\omega_0 t}\,dt\right]^{-1}\right\}. \tag{10.16.15}$$

The block diagram of a unit realizing operation (10.16.15) is depicted in Fig. 10.43. It contains a filter with pulse response envelope

$$h(t) = \begin{cases} 1, & 0 < t < \Delta t, \\ 0, & t > \Delta t. \end{cases}$$

This circuit differs from the preceding circuit in that basic accumulation in the estimator unit occurs immediately after heterodyning. Accuracy of the circuit of Fig. 10.43 is very difficult to calculate. It is possible to calculate, of course, variance of the efficient estimate for the considered case. Substituting in (10.16.8) expressions (10.3.7), (10.3.14) and (10.16.14), and also using approximation (10.16.2) for directional patterns of antennas,

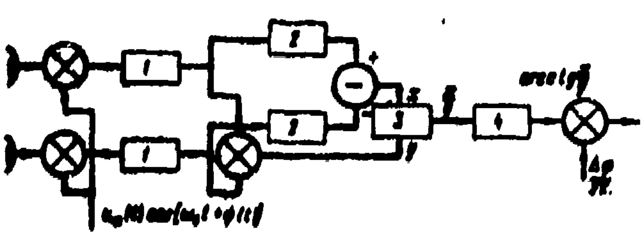

Fig. 10.43. Optimum circuit of the radio channel of a nontracking meter with IAC with slow fluctuations of the signal: 1) a filter whose low-frequency equivalent is an integrator with clearing; 2) square-law detectors; 3) dividers; 4) nonlinear unit.

we can obtain a very simple expression for variance of the efficient estimate of the angular coordinate α:

$$\sigma^2_{э\Phi} = 0{,}2\Delta\varphi^2\left(\frac{1}{q}+\frac{1}{q^2}\right), \tag{10.16.16}$$

where

$$q=\frac{P_{cz}\Delta t}{2N_0}$$

is the ratio of the total energy of the received signal (during time Δt) to the spectral density of noise.

However it is easy to prove that the maximum likelihood estimate in this case differs from the efficient one: ratio (10.16.3) in the considered case turns out to be constant and equal to 0.36. Consequently, it is only possible to state that variance of error of the circuit of Fig. 10.43 cannot be less than $\sigma^2_{э\Phi}$.

In order to find the accuracy of this circuit, it is necessary to directly to find the mathematical expectation and variance of quantity $\hat{\alpha}$, given by formula (10.16.15). We will not produce these calculations, but note only that as $q \to \infty$ estimate $\hat{\alpha} \to \alpha$, and with growth of q it decreases proportionally to 1/q.

10.16.2. Optimum Circuit of a Nontracking Goniometer with IPC

We now shall consider the phase method. With this method there is received a pair of signals of form (10.3.3), where $\Phi_1(t, \alpha)$ is determined by formula (10.2.8). Signal amplitudes are not modulated during reception, i.e., $U_{ai}(t, \alpha) = 1$. Correlation functions of these signals are given by expression (10.3.7), and their functional of the distribution of probabilities in interval $(0, \Delta t)$ has the form (10.3.8), and we need only find the matrix of inverse correlation functions. Solving equations (10.3.10) by methods already described, we can find that in the considered case the inverse correlation functions have form (10.3.14), where $v(t_1, t_2)$ satisfies equation (10.16.3). In contrast to the preceding case this is valid without any approximations.

The maximum likelihood equation, as before, is determined by expression (10.16.4). Substituting (10.3.14) in it, taking into account (10.3.2) and (10.2.8), we obtain likelihood equation

$$\frac{\partial}{\partial\alpha}\operatorname{Re} e^{-i\frac{2\pi d}{\lambda}\sin\alpha}\int_0^{\Delta t}\int_0^{\Delta t} v(t_1, t_2)\, y_1(t_1)\, y_2(t_2) \times u(t_1)\, u^*(t_2)\, e^{i\omega_0(t_1-t_2)}\, dt_1 dt_2 = 0.$$

From this it is already easy to obtain an explicit solution of the likelihood equation:

$$\sin\alpha=\frac{\lambda}{2\pi d}\operatorname{arcctg}\left\{\operatorname{Re}\int_0^{\Delta t}\int_0^{\Delta t}v(t_1,\ t_2)\,y_1(t_1)\,y_2(t_2)\times\right.$$
$$\times u(t_1)\,u^*(t_2)\,e^{i\omega_0(t_1-t_2)}\,dt_1dt_2\left[\operatorname{Im}\int_0^{\Delta t}\int_0^{\Delta t}v(t_1,\ t_2)\,y_1(t_1)\,y_2(t_2)\times\right.$$
$$\left.\left.\times u(t_1)\,u^*(t_2)\,e^{i\omega_0(t_1-t_2)}\,dt_1dt_2\right]^{-1}\right\}. \qquad (10.16.17)$$

The sector of uniqueness of measurement of an angle with the phase method, as shown in § 10.2, is $|\alpha| < \arcsin \lambda/2d$. When $d \gg \lambda$ the sector of uniqueness is small and in formula (10.16.17) $\sin \alpha$ can be replaced by α.

Let us consider in greater detail operation (10.16.17). Here it is necessary to substitute function $v(t_1,\ t_2)$, found from equation (10.16.3). Let us consider the case of rapid fluctuations. Here, using expression (10.3.18) for $v(t_1,\ t_2)$, we can reduce the optimum operation (10.16.17) to the form

$$\sin\alpha=\frac{\lambda}{2\pi d}\operatorname{arcctg}\left\{\int_0^{\Delta t}d\tau\operatorname{Re}\int_0^{\Delta t}h_{1\text{опт}}(\tau-s)\,y_1(s)\times\right.$$
$$\times u(s)\,e^{i\omega_0 s}\,ds\int_0^{\Delta t}h_{1\text{опт}}(\tau-s)\,y_2(s)\,u^*(s)\,e^{-i\omega_0 s}\,ds\times$$
$$\times\left[\int_0^{\Delta t}d\tau\operatorname{Im}\int_0^{\Delta t}h_{1\text{опт}}(\tau-s)\,y_1(s)\,u(s)\,e^{i\omega_0 s}\,ds\int_0^{\Delta t}h_{1\text{опт}}(\tau-s)\times\right.$$
$$\left.\left.\times y_2(s)\,u^*(s)\,e^{-i\omega_0 s}\,ds\right]^{-1}\right\}, \qquad (10.16.18)$$

where the pulse response of the optimum filter $h_{1\ \text{опт}}(t)$ is determined by relationship (10.3.23) [its frequency response has the form (10.3.25)]. The optimum circuit of a nontracking goniometer with IAC is represented in Fig. 10.44. Here signals from the output of the antennas first are heterodyned, where the output of one of the antennas is connected to two channels (we call them for briefness the 2nd and 3rd), to which there are fed heterodyne signals, shifted in phase $\pi/2$. All signals after heterodyning are filtered by optimum filters with gain-frequency response (10.3.25). After filtration the signals are multiplied, integrated, divided and fed to an inertialess nonlinear element with response $y = \operatorname{arc\,ctg} x$.

The output signal of the nonlinear element after multiplication by $\lambda/2\pi d$ enters

the smoothing circuits. The structure of the smoothing circuits remains as before: it is determined only by statistics of change of the angular coordinate.

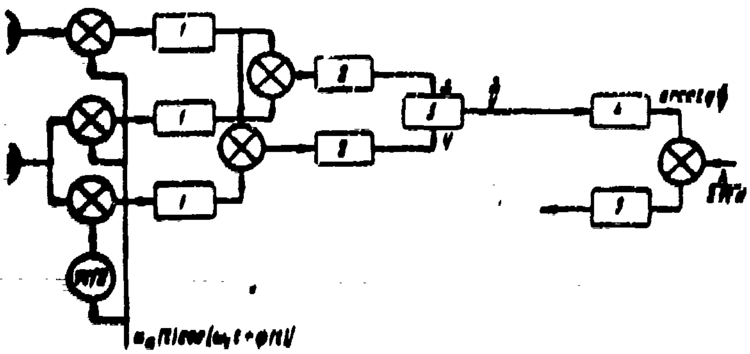

Fig. 10.44. Optimum circuit of a nontracking meter with IPC for fast signal fluctuations: 1) optimum filters with frequency response $H_{1\,\text{opt}}$ ($i\omega$); 2) integrators; 3) divider; 4) nonlinear element; 5) smoothing circuits.

Let us consider the accuracy of the phase method during fast fluctuations. Calculating by formula (10.16.8) the variance of the efficient estimate, it is easy to find that in the considered case it is equal to

$$\sigma^2_{\varphi\varphi} = \left[\frac{\Delta t h_2^2}{\left(\frac{\pi d}{\lambda}\right)^2} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{S_0(\omega)^2}{1 + h_2 S_0(\omega)}\, d\omega \right]^{-1}$$

(10.16.18)

and differs from (10.16.9) only by a constant factor.

Repeating the conclusions of the preceding paragraph, it is possible to prove that the maximum likelihood estimate found by us, expressed by formula (10.16.17), is asymptotically efficient for large Δt (in comparison with the time of correlation of signal fluctuations). Thus, operation (10.16.17) will ensure a minimum possible error with variance (10.16.18). Expression (10.16.18) for $\sigma^2_{\text{эф}}$ with accuracy of a proportionality factor coincides with expression (10.6.5), which was investigated in detail in § 10.6. Therefore, we shall not analyze $\sigma^2_{\text{эф}}$. We shall only show the order of error: for $h_2 = 10$, $\Delta t \Delta f_c = 1$ error $\sigma_{\text{эф}} \approx 0.01\ \lambda/\pi d$.

Accuracy of the synthesized goniometer in the smoothing circuits is the same as in the preceding case. The signal proceeding to the smoothing circuits is equal to the sum of the true value of the angular coordinate and white noise with spectral density $\sigma^2_{\text{эф}}\Delta t$. This quantity coincides with $S_{\text{экв}}$ met during the study of a goniometer with IPC. Thus, with ideal construction of radio chan[illegible] identical

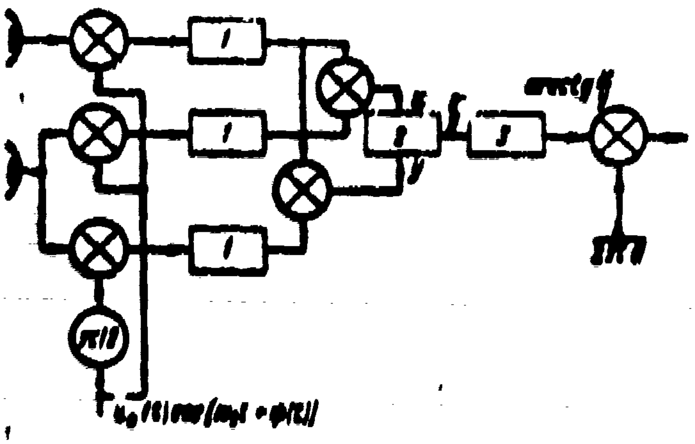

Fig. 10.45. The optimum circuit of the radio channel of a nontracking goniometer with IPC and slow fluctuations of the signal: 1) filters whose low-frequency equivalents are integrators with clearing 2) divider; 3) nonlinear element.

structure of the smoothing circuits accuracy of the nontracking and tracking IPC goniometers is identical.

Let us consider now the case of slow fluctuations, when $\Delta t \Delta f_0 \ll 1$. In this case, obviously, $v(t_1, t_2)$ will again be expressed by formula (10.16.14), and the operation of singling out the maximum likelihood angle with the phase method will take the form

$$\sin\alpha = \frac{\lambda}{2\pi d}\operatorname{arcctg}\left\{\operatorname{Re}\int_0^{\Delta t} y_1(t)\,u(t)\,e^{i\omega_0 t}\,dt \times \int_0^{\Delta t} y_2(t)\,u^*(t)\,e^{-i\omega_0 t}\,dt\left[\operatorname{Im}\int_0^{\Delta t} y_1(t)\,u(t)\,e^{i\omega_0 t}\,dt \times \int_0^{\Delta t} y_2(t)\,u^*(t)\,e^{-i\omega_0 t}\,dt\right]^{-1}\right\}. \quad (10.16.20)$$

The block diagram of a device realizing operation (10.16.20) can be presented in the form depicted in Fig. 10.[illegible]. The filter in this circuit has response envelope

$$h(t) = \begin{cases} 1, & 0 < t < \Delta t, \\ 0, & t > \Delta t. \end{cases}$$

With respect to accuracy of the found circuit it is possible to say the following. As also in the preceding case, it is easy to show that the estimate issued by this circuit is not efficient. Variance of the efficient estimate can be calculated and will determine an estimate from beneath of variance of error of the circuit. By simple calculation by formula (10.16.8), already known, it is easy to find that

$$\sigma^2_{\alpha э} = 0.49\left(\frac{\lambda}{\pi d}\right)\left(\frac{1}{q} + \frac{1}{q^2}\right). \quad (10.16.21)$$

where q is given by expression (10.16.11).

True accuracy of the considered circuit should be calculated directly. The magnitude of true accuracy we do not consider, since its determination is very

complicated and requires special investigation. We shall discuss this in greater detail in the concluding section of the present chapter.

§ 10.17. Influence of Interferences on Coherent Goniometers

In practice a goniometer is often subjected to the influence not only of a fluctuating signal and internal noises, but also to active and passive interferences, distorting the process of measurement or even completely disrupting it [70]. The number of interferences which it is possible to create for radar goniometers is, very great. However, to study the influence of each form of interference in the same detail as the influence of fluctuations of the signal and natural noises is hardly useful due to the large number of particular cases which can be encountered. Therefore, we shall study the influence of interferences basically in approximation, limiting ourselves to giving the simplest formulas.

First of all we shall classify possible interferences to goniometer devices, so that we see in what order we are to study them. We consider, first, active interferences. They can be sent from the target whose angular coordinates are being measured, and also from another target, located in an essentially different direction.

In the first case interference acts on the basic lobe of the directional pattern, and in the second — on side lobes. The influence of both types of interferences sharply differs. The effect of interference acting on the basic lobe, in turn, essentially depends on the form of interference, and also on the method of direction finding utilized in the considered goniometer. Interferences with amplitude modulation will strongly affect goniometers with pattern scanning and weakly affect other goniometers. Interference with great power; suppressing the useful signals through the AGC system, will have a different affect from interference with low power.

Thus, the number of possible cases here is very great. During the study of interferences we will widely use the fact that the influence of random interferences on a coherent radar, due to narrow-bandedness of its components, is equivalent to the influence of white noise (Chapter VII).

10.17.1. Influence of Broad-Band Interferences on Side Lobes

The influence of active interferences on side lobes of the directional pattern of radar goniometers is a very likely case. At the same time, the level of side lobes of the pattern in most cases is such that we cannot provide sufficient suppression of such interference.

The influence of random interference acting on side lobes can be estimated approximately rather simply. If the interference is broad-band, then it is equivalent to white noise, the magnitude of which does not depend on angular coordinates of the target, i.e., it is equivalent to the internal noise of the receiver. Certainly, in goniometers with pattern scanning such interference obtains amplitude modulation; however the side lobes have a more or less constant level, so that such modulation can, in most cases, be ignored. Formulas for accuracy of a goniometer under the influence of such interferences coincide with formulas for accuracy of goniometers in normal conditions (without interferences), in which the spectral density of noises N_0 must be replaced by $N_0 + N_П$, where $N_П$ — spectral density of the equivalent noise replacing interference. Spectral density $N_П$ should be calculated in the following way. If the influence is a broad-band noise interference with bandwidth $\Delta f_П$,

$$2N_П = \frac{P_П G_{ПП}}{(4\pi)^2} \frac{\lambda^2 \varkappa}{\Delta f_П d_П^2}, \qquad (10.17.1)$$

where $P_П$ — mean power of jamming transmitter;

$G_{ПП}$ — gain of the antenna of the jamming transmitter;

λ — wavelength of the considered radar;

$\varkappa$ — level of side lobes;

$d_П$ — distance to source of interferences.

If the influence is a pulse chaotic interference, then, using formula (7.14.9) derived in Chapter VII, we obtain

$$2N_П = \frac{P_{иП} G_{ПП} \lambda^2 \varkappa \nu \tau_П}{(4\pi)^2 d_П^2} \frac{1}{\sqrt{1 + \Delta f^2_{M\,эф} \tau_П^2}}, \qquad (10.17.2)$$

where $P_{иП}$ — pulse power of jamming transmitter;

ν — mean frequency of repetition of pulses of interference;

$\tau_П$ — duration of interference pulse;

$\Delta f^2_{M\,эф}$ — effective width of the spectrum of signal modulation.

Thus, the influence of broad-band interference on the side lobe leads simply to equivalent increase of natural noises, i.e., to equivalent decrease of the signal-to-noise ratio.

10.17.2. Influence of Broad-Band Interferences From the Target

Broad-band interference from the target (noise or pulse chaotic interference with a high frequency of repetition) is also a very common type of interferences. However, this kind of interference is intended primarily for suppression of such

functions of radars as measurement of range or speed, and for goniometers does not present serious danger. In fact, this interference itself carries information about angular coordinates, and the goniometer can measure angular coordinates of the target from the interference.

Let us consider in greater detail the influence of such interference on goniometers. First of all we note that, in view of the considerable excess of interference power over the signal, both the useful signal, and also natural noises will be suppressed, by the AGC system and measurement will be conducted exclusively of interference. Considering the broad-bandedness of the interference, it is possible to say that accuracy of measurement of angles will be determined by the same formulas with which we determined accuracy of goniometers in normal conditions, considering $h = \infty$ (internal noises are absent) and ratio $\Delta f_{ф} / \Delta f_{c} \ll 1$ (since the width of the interference band considerably exceeds the width of the passbands of filters in the goniometer). In both circuits with scanning (Figs. 10.9 and 10.10) there will exist fluctuating error with equivalent spectral density, equal, respectively, to

$$S_{\text{экв}} = \frac{1}{8\mu_a^2 \Delta f_{ф}}, \tag{10.17.3}$$

$$S_{\text{экв}} = \frac{1}{4\mu_a^2 \Delta f_{ф}}. \tag{10.17.4}$$

These formulas are obtained from formulas (10.5.17) and (10.5.22). From this it is clear that tracking of a target by interference radiated from it always occurs with limited accuracy. If, for instance, $\mu_a = 0.2$ 1/deg and $\Delta f_{ф} = 100$ cps, then for $S_{\text{экв}}$ we obtain the corresponding values

$$S_{\text{экв}} \approx 0{,}03 \ \text{deg}^2/\text{cps}, \quad S_{\text{экв}} \approx 0{,}06 \ \text{deg}^2/\text{cps}.$$

In many cases such magnitudes of $S_{\text{экв}}$ are already impermissible; therefore the considered interference for the method of pattern scanning of diagram will be rather dangerous, especially for the circuit of Fig. 10.10.

We note that for the circuit of Fig. 10.9, for which the frequency of scanning can be selected as high as one wishes, error of tracking by interference can be made minute, if the frequency of scanning considerably exceeds the width of the spectrum of interference. Fluctuating error here remains only due to the parametric component (if there are dynamic errors of tracking). The spectral density of parametric fluctuations, as it is easy to find from formulas (10.5.24) and (10.13.2), will be equal to

$$S_{нар}=\frac{1}{2\Delta f_\phi n^2}. \quad (10.17.5)$$

For circuits with multiplication of signals of the compensation method and, correspondingly, of IAC during tracking of a target by broad-band interference radiated from it we have $S_{экв} = 0$ [see (10.7.5) as $h \to \infty$]. This will take place even in the case of nonidentity of channels. However, as it is easy to see from (10.7.2), nonidentity leads to decrease of the slope of the discrimination characteristic $K_Д$, which will lead to worsening of dyamic properties of the tracking system during tracking of interference from the target.

In the circuit with subtraction of signals for scanning with compensation the considered interference will lead to the worst results. In the case of nonidentity of channels $S_{экв}$ will differ already from zero. If for simplicity we consider different only the gain of the channels, from expression (10.7.8) for $h \to \infty$ and $\Delta f_c \gg \Delta f_\phi$ it is easy to obtain

$$S_{экв}=\frac{1}{8\mu_a^2\Delta f_\phi}\left(\frac{1-k^2}{1+k^2}\right)^2, \quad (10.17.6)$$

where k – ratio of the squares of gain factors of channels of the radio channel.

For the IAC circuit with subtraction of signals for nonidentical channels $S_{экв}$ will also be expressed by formula (10.17.6).

Furthermore, here there will exist systematic error, absent in the circuit of the method of scanning with compensation

$$\Delta=\frac{1}{4\mu_a}\frac{1-k^2}{1+k^2}. \quad (10.17.7)$$

Thus, tracking of interference from a target will have error only in the case of nonidentity of channels. With identity of them results will be precisely the same as for circuits with multiplication of signals, i.e., $S_{экв} = 0$ and $\Delta = 0$. However, in both circuits here there will exist fluctuating error due to parametric fluctuations. The spectral density of parametric fluctuations will as before be expressed by formula (10.3.15). As already noted, this error is insignificant.

Absolutely analogously one can prove that during tracking of interference from a target using the method of phase center scanning there will exist very great fluctuating error with equivalent spectral density

$$S_{экв}=\frac{1}{2\left(\frac{\pi d}{\lambda}\right)^2\Delta f_\phi}\left(\frac{2\Delta f_c}{\Omega}\right)^4, \quad (10.17.8)$$

where $\Delta f_{п}$ — width of the spectrum of interference.

When using a goniometer with IPC (Fig. 10.23) we always will have $S_{экв} = 0$. Nonidentity of channels will lead only to decrease of $K_{Д}$ and worsening of dynamic properties of the tracking system.

When using IPC by the circuit of Fig. 10.24 interference from the target will have considerably stronger influence. For an example we shall consider the gain-frequency responses to differ by a constant $\Delta\varphi$. Then, during tracking of interference from a target there will exist fluctuating error with equivalent spectral density

$$S_{экв} = \frac{1}{4\left(\frac{\pi d}{\lambda}\right)^2} \operatorname{tg}^2 \Delta\varphi \left(\frac{1}{\Delta f_{\phi}} + \frac{1}{2\pi \Delta f_{\phi}^2} \int_{-\infty}^{\infty} |H(i\omega)|^4 d\omega\right) \approx$$

$$\approx \frac{3}{8\left(\frac{\pi d}{\lambda}\right)^2 \Delta f_{\phi}} \operatorname{tg}^2 \Delta\varphi. \qquad (10.17.9)$$

[obtained from formula (10.9.3) as $h \to \infty$].

Thus, fluctuating error here increases very fast with growth of nonidentity of phase-frequency responses. Systematic error existing in this circuit is equal to

$$\Delta = -\frac{1}{2\left(\frac{\pi d}{\lambda}\right)} \operatorname{tg} \Delta\varphi. \qquad (10.17.10)$$

With identical channels this circuit behaves just as the preceding one.

Consequently tracking of broad-band random interference from a target when using methods of scanning the directional pattern and phase center occurs fluctuating error differing from zero. This error, as rough calculation shows, is considerable. In methods of scanning with compensation, IAC and IPC with identical channels $S_{экв} = 0$, and fluctuating error (very insignificant) exists only due to parametric fluctuations. Fluctuating error appears in circuits without formation of sum and difference signals when there are nonidentical channels. In circuits with formation of sum and difference signals nonidentity of channel leads only to decrease of $K_{Д}$ and, consequently, to worsening of dynamic properties of tracking systems.

10.17.3. Influence of Active Interferences with L-F Modulation

As already noted, against goniometers using the method of pattern scanning there may be used interference (noise, pulse, return) with low-frequency amplitude modulation. With coincidence of frequencies of modulation of interference with the frequency of scanning this interference will, obviously, be very effective. For

goniometers using the method of phase center scanning, interference with low-frequency phase modulation will lead, obviously, to analogous results. We now pass to consideration of the influence of this kind of interference on goniometers.

Let us study the effect of interference from a target with special low-frequency amplitude modulation on goniometers using the method of pattern scanning, and also the compensation method. The fluctuating signal considered in the preceding paragraphs is not equivalent to such interference, since it contains definite phase modulation. We will produce approximate calculation of the influence of the considered form of interference, seeking, basically, qualitative conclusions. We assume that interference has the form

$$u_{\text{п}}[1+\xi(t)]\cos(\omega_0 t+\psi), \tag{10.17.11}$$

where $\xi(t)$ — modulating random process with a zero mean and small variance.

Let us consider first the method of pattern scanning in conditions of application of the circuit of Fig. 10.9. For simplicity we assume that the circuit is intended for work on an unmodulated signal (there is no inversion of phase modulation and corresponding transformation of amplitude modulation of the received signal), and also that scanning is uniform and conical. The received signal, after heterodyning, will obviously have form

$$u_{\text{п}}[1+\xi(t)](1+\mu_{\text{a}}a\cos\Omega t+\ldots)\cos(\omega_{\text{пр}}t+\psi). \tag{10.17.12}$$

Noises and the reflected signal we disregarded, considering interference sufficiently powerful. Considering the smallness of variance and the narrowness of the spectrum of process $\xi(t)$, for the output signal of the circuit we obtain

$$z(t)=\frac{u_{\text{п}}^2}{2}\left[\frac{\mu_{\text{a}}a}{2}+\int\limits_{-\infty}^{t}h(t-\tau)\xi(\tau)\cos\Omega\tau\,d\tau\right]. \tag{10.17.13}$$

Hence equivalent spectral density of fluctuating error is equal to

$$S_{\text{экв}}=\frac{2S_{\text{п}}(\Omega)}{\mu_{\text{a}}^2}\approx\frac{\sigma_{\text{п}}^2}{\mu_{\text{a}}^2\Delta f_{\text{п}}}, \tag{10.17.14}$$

where $S_{\text{п}}(\Omega)$ — spectral density of $\xi(t)$ at frequency Ω;

$\sigma_{\text{п}}^2$ — variance of $\xi(t)$;

$\Delta f_{\text{п}}$ — width of spectrum $\xi(t)$ (one-way).

Analogously it is easy to prove that for the circuit of Fig. 10.10 equivalent spectral density during tracking of the considered interference will be precisely the same. Analysis of expression (10.3.25) shows that the more exactly the modulation of interference is tuned to the frequency of scanning (i.e., $\Delta f_{\text{п}}$ decreases), the

greater the error of tracking.

Now we consider a circuit of scanning with compensation. In the circuit of Fig. 10.15, as it is easy to show, during tracking of the considered interference we will have $S_{экв} = 0$, i.e., this interference is well compensated. Nonidentity of channels will only lead to decrease of the slope of the discrimination characteristic.

In the circuit of Fig. 10.16 with identical channels we also will have $S_{экв} = 0$. However, with nonidentical channels there will appear fluctuating error with $S_{экв}$, equal to

$$S_{экв} = \frac{2S_{\pi}(\Omega)}{\mu_{\pi}^{2}}\left(\frac{1-k^{2}}{1+k^{2}}\right)^{2}, \qquad (10.17.15)$$

where k — the ratio of squares of the gain factors in the channels.

With identical channels k = 1 and $S_{экв} = 0$. However, even with identical channels in both the considered circuits fluctuating error due to parametric fluctuations will remain. Simple calculation shows that for both circuits, assuming identity of their channels

$$S_{пар} = \frac{S_{п}(0)}{n^{2}} = \frac{\sigma_{п}^{2}}{n^{2}\Delta f_{п}}, \qquad (10.17.16)$$

where all designations are the same as before.

If the spectrum of modulation of interference does not contain components at zero frequency, $S_{пар} = 0$, and guidance from such interference is accomplished, in general, without fluctuating errors.

Thus, interference from a target with amplitude modulation is very dangerous for the method of pattern scanning. For the method of scanning with compensation it is no longer dangerous. Fluctuating error will exist here only for the circuit without formation of sum and difference signals in the case of nonidentity of channels of the circuit. In the remaining cases $S_{экв} = 0$, and fluctuating error exists only due to parametric fluctuations, if only the spectrum of interference modulation contains components at zero frequency. This error has an insignificant relative magnitude, and for $S_{экв} \neq 0$ we can disregard it.

Absolutely analogously to the preceding we can consider the influence of interference with low-frequency phase modulation on goniometers using the method of phase center scanning. Recording interference in the form $u_{п} \cos[\omega_{0}t + \xi(t)]$, where $\xi(t)$ — random process with a zero mean and small variance, it is possible to easily show that the equivalent spectral density of fluctuating error of tracking of such

interference is equal to

$$S_{\text{экв}} \approx \frac{\sigma_{\text{п}}^2}{\left(\frac{\pi d}{\lambda}\right)^2 \cdot \Delta f_{\text{п}}},$$

where σ_{Π}^2 — variance of $\xi(t)$;

Δf_{Π} — width of spectrum $\xi(t)$.

10.17.4. Influence of Powerful Intermittent Interferences from Target

Let us consider the influence of certain types of interferences from the target, the influence of which reduces to artificial disturbance of the process of reception of the radar signal (for instance, due to excitation of transients in the receiver). We are talking about interference from the target, inasmuch as interference may cause disturbances of this type only with considerable excess of power over the signal, which can occur, basically, during influence of the interference of the main lobe.

This interference can be return, noise, pulse chaotic, etc. Specifics connected with interruptions do not vary from this. Duration of intervals of the presence of interference and the frequency of their repetition are selected such that there does not exist a steady-state regime of work of the receiver either with respect to interference or to the signal, and the receiver is always, as it were, in a prolonged transient regime. Such an effect appears during feed of intermittent interference of high power to input of a receiver with AGC. Let us consider this process in greater detail As already noted, in coherent circuits the main i-f amplifier, covered by the AGC system, has a sufficiently wide passband, so that inertia of the AGC system turns out to be usually considerably greater than the inertia of the amplifier covered by it. As was shown in Chapter 11, with certain additional limiting assumptions during feed to an amplifier input of a powerful interference it immediately becomes limited and emerges from the limitations after time

$$t_{\text{вых}} \approx T_{\Phi} \frac{K_0 u_{\text{п}} - u_{\text{огр}}}{u_{\text{п}} b K_1 (u_{\text{огр}} - E_3)}, \qquad (10.17.17)$$

where T_{Φ} — time constant of the filter of automatic gain control;

K_1 — gain in the AGC loop;

K_0 and b — parameters of linear approximation of the dependence of gain of the UPCh on control voltage of adjustment at a point, corresponding to steady-state regime for a given level of interference (see Chapter 11);

$u_{огр}$ and $E_з$ — level of limitation and voltage of delay of automatic gain control;

$u_п$ — amplitude of interference at the input of the UPCh.

Here there is est...ished such UPCh gain that the useful signal is completely suppressed. After turning off of interference the output signal of the UPCh is equal to zero and will rise to its normal magnitude approximately after time

$$t_{вкл} = T_ф \ln \frac{K_0 u_к - E_з}{K_{01} u_с - E_з} \frac{u_с b_1 K_1}{1 + u_к b K_1}, \qquad (10.17.18)$$

where K_{01} and b_1 — parameters of linear approximation of the dependence of UPCh gain on control voltage at a point corresponding to steady-state regime for a given input level of the useful signal.

Let us assume now that the duration of the time of continuous influence of interference is equal to $t_{вкл}$, and the duration of time when interference is turned off is equal to $t_{выкл}$. Here at the UPCh output there will occur a sequence of interference pulses of amplitude u_0, equal to the cutoff level of the UPCh, regardless of the form of the useful signal. It is obvious that in this case goniometers using the method of scanning the directional pattern, and also the method of scanning with compensation or IAC, working on circuits without formation of sum and difference signals will absolutely malfunction. In all these cases the output signal of the goniometer circuits will not contain information about angular coordinates of the target.

In circuits of the method of scanning with compensation and IAC in which there are formed sum and difference signals at the output of the antenna system the matter will be somewhat different. In these circuits of the method of scanning with compensation and IAC in which there are formed sum and difference signals at the output of the antenna system the matter will be somewhat different. In these circuits the difference signal with exact tracking of the target is equal to zero, and with sufficiently small errors of tracking this signal is close to zero.

Thus, one may assume that under the influence of interference the signal in the difference channel does not fall under limitation. The amplitude of the output signal of the UPCh in the difference channel is proportional to mismatch, but it does not remain constant, since the gain factor of the UPCh in the difference channel drops as a result of work of the AGC system, removing from limitation the signal in the sum channel. It is easy to calculate that the amplitude of the signal in the difference channel will decrease according to the law

$$u(t) = u_{\text{н}}\left[K_0 - bK_1(u_{\text{орр}} - E_3)\left(1 - e^{-\frac{t}{T_0}}\right)\right] \tag{10.17.19}$$

(with the AGC circuit depicted in Fig. 2.20). After turning off of interference the output signal of the UPCh to the difference channel disappears. Therefore at the output of the circuit there will appear pulses of duration $t_{\text{ВКЛ}}$ and form $\mu_a \alpha/2\, u(t)u_0$, where u(t) is given by formula (10.17.19), and u_0 — cutoff level of the UPCh in the sum channel. Tracking of a target in this case will continue, however, inasmuch as the time interval between pulses of the error signal, equal to $t_{\text{ВЫКЛ}}$ is usually great, and the tracking system will pass into a mode of discrete operation.

Similar phenomena will be observed in circuits using phase methods of direction finding. Intermittent interference in these cases also, obviously, will not lead to considerable disturbances of the process of measurement angular mismatch. Actually, due to the influence of intermittent interference at the UPCh output there appear pulses with amplitude equal to the cutoff level of the UPCh; however, the phase of the filling of these pulses is not distorted during passage through UPCh. Inasmuch as useful information here is included in phase, an error signal will be produced. E.g., for IPC the signal of error will be pulses of duration $t_{\text{ВКЛ}}$ with amplitude $\alpha u_0^2 \pi d/\lambda$, where u_0 — cutoff level of the UPCh (with identical channels).

Thus, the tracking system will continue to work; however, due to the great time interval between pulses of the error signal, equal to $t_{\text{ВЫКЛ}}$ work of the tracking system will be in discrete mode. Due to the influence of such interferences dynamic properties of the tracking system change.

10.17.5. Influence of Passive Interference

On the basis of results of Chapter I we can affirm that the signal from a passive interference, received by the antenna of a goniometer, after multiplication by the reference signal is equivalent to a stationary normal process with a definite spectral density $S_{\text{П}}(\omega)$, depending on the difference of Doppler frequencies of the signal from the target and from the interference. The spectrum of interference is a sequence of expanded spectral lines, removed from each other the frequency of repetition of the signal. The width of each line is determined by irregularity of motion of the dipoles, the wavelength of the radar, the speed of the radar with respect to the cloud of interferences and the width of its directional pattern. As we already noted in § 7.14, the width of each spectral line is tens of cycles per second with

a motionless radar and can reach thousands of cycles per second with a moving radar. Therefore with good approximation in coherent goniometers passive interference is equivalent to white noise with spectral density

$$N_{\Pi} = \frac{P_c}{2\Delta f_c} \frac{\sigma_{\Pi}}{\sigma_{Ц}} f(\Delta\omega_{Д}), \tag{10.17.20}$$

where $\sigma_{Ц}$ — reflecting surface of the target;

σ_{Π} — reflecting surface of the interference in the resolution volume of the radar;

$\Delta\omega_{Д}$ — difference of Doppler frequencies of the signal from the target and from interference;

$f(\omega)$ — form of the spectral line of interference ($f(0) = 1$).

As soon as passive interference is reduced to equivalent white noise its influence on accuracy of goniometers can easily be accounted for. In goniometers with one antenna (i.e., in goniometers with scanning of the directional pattern or phase center, and also with flat scanning of the pattern) for calculation of accuracy during passive interferences we can directly use the formulas derived in the preceding paragraphs, only the signal-to-noise ratio should be replaced by the signal-to-interference ratio

$$h_{\Pi} = \frac{\sigma_{Ц}}{\sigma_{\Pi}} \frac{1}{f(\Delta\omega_{Д})}. \tag{10.17.21}$$

The magnitude of $\sigma_{Ц}/\sigma_{\Pi}$ is usually considerably less than one. Acceptable signal-to-interference ratios occur only with sufficiently large frequency separations of the signal and interference $\Delta\omega_{Д}$, when $f(\Delta\omega_{Д}) \ll 1$. With coincidence of the speeds of the target and interference $\Delta\omega_{Д} \approx 0$ and $f(\Delta\omega_{Д}) \approx 1$; therefore the signal-to-interference ratio in this case will be minute.

In the case of goniometers with two or more antennas the possibility of use during passive interferences of the formulas for accuracy derived for the case of natural noises is not clear. The fact is that the directional patterns receive reflected signals from the same reflectors; therefore interferences at the output of the antennas are correlated. However, it is possible to show that in the practically most interesting cases this correlation does not lead to change of the shown formula. Let us consider, for instance, a goniometer with IAC. Let us designate signals from the passive interference at the output of the antennas by $n_{\Pi 1}(t)$ and $n_{\Pi 2}(t)$. If we assume the cloud of reflectors uniform in a sufficiently wide angular range, and the directional patterns of antennas identical, interferences $n_{\Pi 1}(t)$ and $n_{\Pi 2}(t)$ are correlated and have identical variances σ_{Π}^2.

If at the output of the antennas there are formed sum and difference signals, interferences in the sum and difference channels will be equal to:

$$n_{\text{п}+}(t) = n_{\text{п}_1}(t) + n_{\text{п}_2}(t),$$
$$n_{\text{п}-}(t) = n_{\text{п}_1}(t) - n_{\text{п}_2}(t).$$

It is easy to see that the sum and difference interferences are not correlated. Actually,

$$\overline{n_{\text{п}+}(t)\, n_{\text{п}-}(t)} = \overline{n_{\text{п}_1}(t)^2} - \overline{n_{\text{п}_2}(t)^2} = 0.$$

Consequently, accuracy of IAC circuits with formation of sum and differnce signals in the presence of passive interference can be estimated by the formulas derived for the case of natural noises with replacement of the signal-to-noise ratio by the signal-to-interference ratio (10.17.21). A circuit without formation of sum and difference signals with identical channels, as we know, is identical to a circuit with formation of such signals. Consequently, accuracy of circuits without formation of sum and difference signals with passive interferences is expressed by the formulas derived for the case of natural noises.

Everything said, obviously, also pertains to the method of scanning with compensation and to the IPC method.

Thus it is rather easy to approximately allow for the influence of passive interferences: under the influence of passive interference goniometer circuits behave the same way as with natural noise with some new spectral density, which is expressed through parameters of the interference. More exact analysis is complicated, although in principle it is a clear problem. We will not continue our study of these questions.

§ 10.18. Conclusion

In the preceding sections we synthesized and studied in detail optimum radar goniometers. Basic attention was allotted to synthesis and analysis of radio channels of goniometer devices. Namely, here there is manifested to the greatest extent the specific character of the problem of measurement of angles, in many respects differing in initial prerequisites and results of the conducted investigation from that presented in chapters devoted to other radar meters. To a considerably lesser extent this specific character affects smoothing circuits of meters, since the structure of smoothing circuits does not depend on the method of encoding the measured parameter in the radar signal, but is determined, basically, by laws of change of the parameter.

Synthesized circuits of optimum radio channels of tracking goniometers were found close in basic features to circuits described in the literature, created on the basis of technical and physical intuitions. An exception are the circuits of optimum radio channels of goniometers using the method of pattern scanning or scanning with compensation. For these methods of direction finding synthesized circuits ensure considerable or even total compensation of the harmful influence of amplitude fluctuations of the signal and somewhat differ from known circuits.

Analysis, which was allotted considerable attention, had as it purpose to reveal the criticality of circuits to various deviations from optimality in their structure. We showed, in particular, that imperfectness of heterodyning (imperfectness of inversion of amplitude and phase modulations of the received signal) is always equivalent to some decrease of the signal-to-noise ratio. In circuits for the method of scanning with compensation, IAC and IPC nonidentity of channels leads to very harmful consequences. Here there is established the evident advantage of circuits with formation of sum and difference signals, which turned out to be least critical to nonidentity of channels.

Let us enumerate the most important problems concerning radar goniometers not touched on or insufficiently illuminated in the present chapter and requiring further investigation.

The first problem is the problem of synthesis of an optimum radar goniometer when the method of direction finding is not assigned, and there are assigned, e.g., only the dimensions and geometric shape of the aperture of the antenna array.

In this chapter this problem was solved for antennas of phased grids type. For the case when the aperture of the antenna array is an arbitrary planar region, the problem of synthesis of an optimum goniometer, including processing of the field in the assigned aperture, has not yet been solved. The essence of this problem and difficulties in its formulation were discussed in detail in § 10.13.

Another important problem not yet solved is the problem of synthesis of optimum radar goniometers for the case when the time of correlation of fluctuations of the radar signal is comparable to the time of correlation of angular shifts of the target. This problem is very important, inasmuch as the shown case is often encountered in practice. However, in the way of solution of this problem there are considerable difficulties of a mathematical character.

We note further a number of problems concerning smoothing circuits of goniometers. Here it is necessary to produce systematic study of a priori statistics of

angular shifts of the target and to synthesize optimum smoothing circuits. It is interesting also to investigate the influence of inaccuracies in assignment of a priori statistics of angular shifts on accuracy of the goniometers.

Finally, very far from complete solution are problems connected with investigation of nonlinear regimes of tracking goniometers. Here it would be desirable to obtain more exact expressions for error of tracking with large mismatch and for an average time to breakoff of tracking with real smoothing circuits of tracking goniometers.

The enumerated problems present doubtless practical interest. However, even the problems already solved permit us to handle many questions of radar goniometry, in particular, to correctly select in each case the structure of the goniometer and to sensibly make various technical simplifications, estimating their effect beforehand.

CHAPTER XI

MEASUREMENT OF ANGULAR COORDINATES WITH AN INCOHERENT SIGNAL

§ 11.1. Introductory Remarks

In radar practice incoherent pulse radiation is widely used. Let us remember that according to the terminology of Chapter I we called incoherent pulse signals with random initial phases of the high-frequency filling in each pulse.

Use of incoherent signals, as was shown in the preceding chapters, provides, in general, less possibilities both in radar detection of targets and also in measurement of coordinates. However, in many cases the loss due to incoherence is not essential, and generation of such signals is a technically simpler task than generation of coherent pulse signals. Therefore, application of incoherent signals is often fully justified and desirable. In the present chapter we will pursue a systematic study of radar goniometers using incoherent signals, or more simply, of incoherent goniometers.

In distinction from a coherent signal, statistical properties of an incoherent signal reflected from a target are very complicated. The functional of the distribution of probabilities of a fluctuating incoherent signal cannot be calculated in general. In connection with this, we also cannot synthesize the circuit of optimum incoherent goniometers. Therefore, we shall be limited here to an approximate approach to synthesis of optimum circuits, similar to that which we used, for instance, in Chapter VIII.

In contrast to the problem of synthesis, analysis of incoherent circuits can be performed, in general, without any essential limitations. As a result of such analysis in the present chapter we find accuracies of different circuits of

incoherent goniometers, we investigate dependences of these accuracies on different parameters, we compare real accuracy with potential. Consideration is conducted consecutively for all the methods of angular direction finding described in § 10.2.

Thus, in its structure and actual content the present chapter is close to the preceding, which was devoted to coherent goniometers. Further we shall widely use the designations of the preceding chapter.

§ 11.2. Optimum Radio Channel of Incoherent Goniometers

In examining the problem of synthesis of the optimum radio channel of incoherent goniometers we, as in the preceding chapter, start from a generalized method of direction finding, consisting in reception of a signal on several antennas with scanning directional patterns and phase centers. Preserving all the designations introduced in § 10.3, we can record output signals of the antennas of the considered goniometer in the form

$$\begin{gathered} y_i(t)=\sqrt{P_{ci}}U_{ai}(t,\ \alpha)\,u_a(t)\,\{a(t)\cos[\omega_0 t+\psi(t)+ \\ +\Phi_i(t,\ \alpha)+\delta(t)]+b(t)\sin[\omega_0 t+\psi(t)+\Phi_i(t,\ \alpha)+ \\ +\delta(t)]\}+\sqrt{N_{0i}}\,n_i(t)=\sqrt{P_{ci}}\operatorname{Re}U_i(t,\ \alpha)\,E(t)\times \\ \times u(t)\,e^{i\omega_0 t+i\delta(t)}+\sqrt{N_{0i}}n_i(t), \end{gathered} \tag{11.2.1}$$

where $\delta(t)$ takes constant random and independent values in every period of repetition of the signal, uniformly distributed in interval $(0,2\pi)$, and $U_a(t)$ and $\psi(t)$ – periodic functions with period T_r, being the intraperiod amplitude and phase modulations of the incoherent signal.

The functional of the distribution of probabilities of an incoherent signal for certain cases (for instance, for a small signal-to-noise ratio) was calculated in Chapters V and VI. A peculiarity of the considered problem is the presence of several dependent signals, for which it is necessary to obtain a joint functional of the probability density. We repeat in brief the reasonings of Chapters V and VI in reference to this more general problem. The conditional functional of distribution of probabilities of signals (11.2.1) with assigned phases $\delta(t)$ has form [see (10.3.8)]:

$$\begin{gathered} P[y_1(\tau),\ y_2(\tau),\ \ldots,\ y_n(\tau),\ \alpha\,|\,\delta(t)]= \\ =K\exp\left\{-\frac{1}{2}\int_0^T\int_0^T \mathbf{y}^+(t_1)\,\mathbf{W}(t_1,\ t_2)\,\mathbf{y}(t_2)\,dt_1\,dt_2\right\}= \\ =K\exp\left\{-\frac{1}{2}\sum_{k,l=1}^{m}\int_{(k-1)T_r}^{kT_r}\int_{(l-1)T_r}^{lT_r}\sum_{i,j=1}^{n}W_{ij}(t_1,\ t_2)\,y_i(t_1)\times\ y_j(t_2)\,dt_1\,dt_2\right\}, \end{gathered} \tag{11.2.2}$$

where α — angular coordinate of the target;

m — number of periods of repetition of the signal in the interval of observation (0, T) (all designations here are the same as in § 10.3).

If in (11.2.2) we substitute for $W_{ij}(t_1, t_2)$ expression (10.3.14) (with replacement of $\psi(t)$ by $\psi(t) + \delta(t)$), functional (11.2.2) can be reduced to form

$$P[y_1(t), y_2(t), \ldots, y_n(t), \alpha|\delta(t)] = \\ = K' \exp\left\{-\frac{1}{2}\sum_{k,l=1}^{m} v_{kl} \sum_{i,j=1}^{n} \frac{\sqrt{P_{ci}P_{cj}}}{N_{0i}N_{0j}} \times \right. \\ \left. \times \operatorname{Re} U_i(kT_r, \alpha) U^*_j(lT_r, \alpha) f_{ki} f^*_{lj} e^{i(\delta_k - \delta_l)}\right\}, \tag{11.2.3}$$

where

$$K' = Ke^{-\frac{1}{2}\int_0^T\left(\sum_{i=1}^{n}\frac{y_i(t)^2}{N_{0i}}\right)dt}; \quad v_{kl} = v(kT_r, lT_r);$$

$$f_{ki} = \int_{(k-1)T_r}^{kT_r} u(t) y_i(t) e^{i\omega_0 t} dt, \tag{11.2.4}$$

and δ_k, k = 1, 2, ..., m are values of phase $\delta(t)$ in different periods of repetition of the signal.

For production of an unconditional functional of distribution of the considered signal it is necessary in (11.2.3) to average phases $\delta_1, \delta_2, \ldots, \delta_m$. This averaging in general is hard to produce. However, for instance, with a sufficiently small signal-to-noise ratio it is possible to expand functional (11.2.3) in a series of powers of this ratio and, being limited to the first two terms of the obtained expansion, to produce averaging. Here we will obtain

$$P[y_1(t), y_2(t), \ldots, y_n(t)|\alpha] \approx \\ \approx K' \exp\left\{\frac{1}{2}\sum_{k=1}^{m}\left|\sum_{i=1}^{n}\frac{\sqrt{P_{ci}}}{N_{0i}} U_i(kT_r|\alpha) f_{ki}\right|^2\right\}. \tag{11.2.5}$$

Here we also used the circumstance that by virtue of (10.3.15) for a small signal-to-noise ratio $v(t_1, t_2) \approx \rho(t_1 - t_2)$, so that $v_{kk} \approx 1$.

It is possible to show that the unconditional functional of probability density of an incoherent signal will have the same form in certain other cases, for instance in the case of rapid fluctuations of the signal, when pulses fluctuate independently. Here the signal-to-noise ratio can be anything. Subsequently, we start from expression (11.2.5) for the functional of the probability density of the signal.

Proceeding from this expression, it is possible to construct the operation of the optimum radio channel of an incoherent goniometer. In fact, obviously, we have

$$\frac{\partial \ln P[y_1(t), y_2(t), \ldots, y_n(t)|\alpha]}{\partial\alpha} = \mathrm{Re}\sum_{k=1}^{m}\left(\sum_{i=1}^{n}\frac{\sqrt{P_{ci}}}{N_{0i}}U_i(kT_r, \alpha)f_{ki}\right)\times$$

$$\times\left(\sum_{i=1}^{n}\frac{\sqrt{P_{ci}}}{N_{0i}}U'^{*}_i(kT_r, \alpha)f^{*}_{ki}\right),$$

where the stroke denotes, as always, the derivative with respect to α (here we used the fact that $\ln K'$, as it is easy to show (see § 10.2), does not depend on α). It follows from this that operation of the optimum radio channel consists of formation of signal $z(t)$, for which

$$z(kT_r) = \mathrm{Re}\sum_{j=1}^{n}\sqrt{\frac{P_{c1}}{P_{cj}}}q_jU_j(kT_r, \hat{\alpha})\times$$

$$\times\int_{(k-1)T_r}^{kT_r}u(t)y_j(t)e^{i\omega_0 t}\,dt\sum_{j=1}^{n}\sqrt{\frac{P_{c1}}{P_{cj}}}q_jU'^{*}_j(kT_r, \hat{\alpha})\times$$

$$\times\int_{(k-1)T_r}^{kT_r}u^{*}(t)y_j(t)e^{-i\omega_0 t}\,dt, \qquad (11.2.6)$$

where $q_j = \frac{P_{cj}T_r}{2N_{0j}}$ – ratio of signal energy for the period of repetition at the output of the j-th antenna to the spectral density of the corresponding noise (the applied normalization, as one shall see from what follows, is very convenient);

$\hat{\alpha}$ – measured value of the angular coordinate (in the case of goniometers with a tracking antenna, as in the preceding chapters, $\hat{\alpha}$ should be replaced by zero).

Further processing consists of simple summation, or as it is said, incoherent accumulation of quantities $z(kT_r)$.

Let us consider the question of the circuit realization of optimum operation (11.2.6). Here in the present section we limit our consideration to goniometers with a tracking antenna, for which the optimum circuit has the simplest form. Considering that $U_{aj}(t, 0) = 1$ [see (10.3.1)], we can rewrite operation (11.2.6) in the following form:

$$z(kT_r)=\mathrm{Re}\sum_{j=1}^{n}\sqrt{\frac{P_{c1}}{P_{cj}}}q_j\int\limits_{(k-1)T_r}^{kT_r}u(t)\,y_j(t)\,e^{i\omega_0 t}\,dt\times$$

$$\times\sum_{j=1}^{n}\sqrt{\frac{P_{c1}}{P_{cj}}}q_j\,[U'_{aj}(kT_r,\ 0)-i\Psi'(kT_r,\ 0)]\times$$

$$\times\int\limits_{(k-1)T_r}^{kT_r}u^*(t)\,y_j(t)\,e^{-i\omega_0 t}\,dt. \qquad (11.2.7)$$

One of the variants of the block diagram of a device realizing operation (10.2.7) [sic] is shown in Fig. 11.1 (we shall not stop to discuss the transformation of (11.2.7) to real form, on the basis of which there is constructed the block diagram, since this was done in sufficient detail in the preceding chapter). In this circuit signals are first normalized to the same level (normalization is necessary in the case of different gains of antennas and a different level of noises in the channels); then they are heterodyned with simultaneous multiplication by the law of intraperiod amplitude modulations and with inversion of phase modulation. After this the signals are filtered. The filter must have pulse response

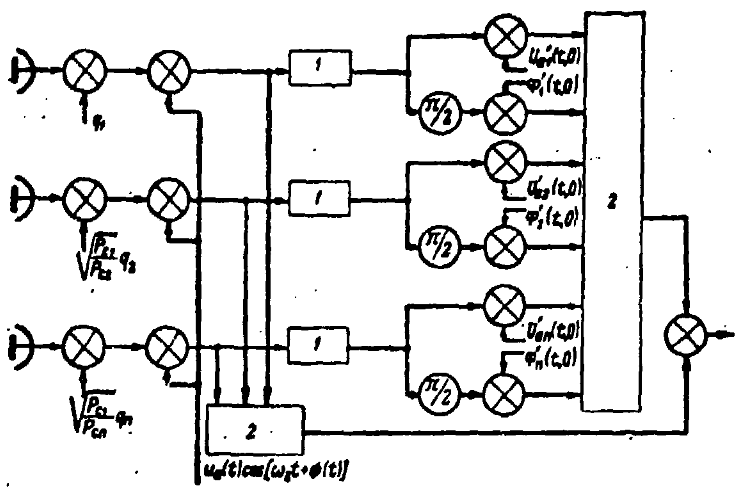

Fig. 11.1. Optimum circuit of the radio channel of an incoherent radar goniometer with a generalized method of direction finding. 1 — optimum filters with pulse response (11.2.8); 2 — adders.

$$h(t)=\begin{cases}\cos\omega_{\text{пр}}t, & 0<t<T_r,\\ 0, & t>T_r,\end{cases} \qquad (11.2.8)$$

i.e., must realize clearing of the accumulated voltage at the end of the period of repetition of the signal. The low-frequency equivalent of this filter is an integrator with clearing. When the signal has a large off-duty factor, i.e., the duration of the signal pulse $\tau_и \ll T_r$, as the indicated filter we can use any filter with passband $\Delta f_ф$ which satisfies relationship

$$\frac{1}{T_r}\ll\Delta f_ф\ll\frac{1}{\tau_и}. \qquad (11.2.9)$$

The signal from the output of each such filter enters two channels: in one channel it is multiplied by a function varying according to the law of scanning of the directional pattern of the corresponding antenna; in the other after a shift in phase of $\pi/2$ it is multiplied by a function varying according to the law of scanning of the phase center of this antenna. After these operations all signals are added and proceed to the phase detector. As the reference voltage of the phase detector there is used the sum of signals taken directly from the output of the filters.

This circuit may, of course, be modified. For instance, multiplication by functions varying according to laws of scanning could have been carried out before filtration. However, these modifications are nonessential. More essential is modification connected with other interpretation of the operation of taking integrals over the period of repetition in (.2.7). Namely, it is possible to carry out the operation of heterodyning in the absence of modulation of the heterodyne voltage, and then pass the signal through a filter with pulse response

$$h(t)=\mathrm{Re}\,u(-t)\,e^{i\omega_{np}t}. \tag{11.2.10}$$

From the output voltage of the filter it is necessary then to separate only the values corresponding to discrete moments of time kT_r. This can be carried out by gating the output voltage of the filters by gate pulses of sufficiently small duration, with period T_r. Gating may also, obviously, be carried out after all operations of the discriminator.

A filter of type (11.2.10) is called "matched with the form of the signal" or "shortening." Processing using such a filter is called "filtrational," while processing in the initial interpretation is called "correlation." These terms already were introduced in Chapter VIII, and we shall not discuss them in detail here. The block diagram of filtrational processing differs from the circuit of Fig. 11.1 insignificantly: the heterodyne signal should have the form $\cos \omega_{np} t$, and filters should have pulse response (11.2.10); at the output of the circuit there should be a gating unit. In other respects the circuits coincide.

Thus, the basic difference between an incoherent circuit and the corresponding coherent one (see Fig. 10.7) is that in the incoherent circuit each period of the signal is processed separately, and results are totaled. In the coherent circuit, due to the presence of narrow-band filters there occurs joint processing of many periods of the signal.

More detailed study of synthesized circuits we postpone to subsequent sections, where we shall consider concrete methods of direction finding.

Thus, with very general assumptions with respect to the method of direction finding we performed approximate synthesis of optimum circuits of incoherent goniometers. The purpose of further investigation is a detailed and systematic study of quasi-optimum circuits from the point of view of their real accuracy. In particular very interesting is the study of such circuits with a large signal-to-noise ratio, since their optimality in this case is not clear a priori. Basic attention during the study of different circuits we will allot to variants with correlation processing, since in radar goniometers circuits with correlation processing are the most common. However, here we shall not limit our consideration to filters carrying out integration of the signal during the period of repetition: characteristics of filters as far as possible will be assumed arbitrary.

Study of quasi-optimum circuits in the above-described aspect, as already repeatedly noted, is very interesting and important from the practical point of view, since only such study will allow us to solve the question of the possibility of sufficiently good approximation of properties of optimum circuits.

§ 11.3. Method of Scanning the Directional Pattern

Let us turn to a study of incoherent goniometers using the method of scanning the directional pattern. The optimum circuit of the radio channel of such a goniometer can easily be obtained after certain concretizations from the general circuit of Fig. 11.1. For this it is necessary to set $n = 1$ (one antenna), $\Phi(t, \alpha) = 0$ (the phase center of the antenna is fixed), and express $U_a(t, \alpha)$ by formula (10.2.2). The optimum circuit can easily be reduced to the form depicted in Fig. 11.2. This circuit is known; however, the conducted synthesis permits us to more exactly formulate the requirements for the circuit from the point of view of its optimality: exact processing of intraperiod modulation of the signal and use of a filter whose low-frequency equivalent is an integrator with clearing. The influence on accuracy of all possible deviations from optimality in the circuit will be studied in detail in the present section.

Before passing to analysis of real accuracy of the synthesized circuit, we note that the circuit of Fig. 11.2 is theoretical and reflects only the fundamental operations performed on the signal. The practical variant of this circuit, depicted in Fig. 11.3, will, of course, differ somewhat from its theoretical prototype.

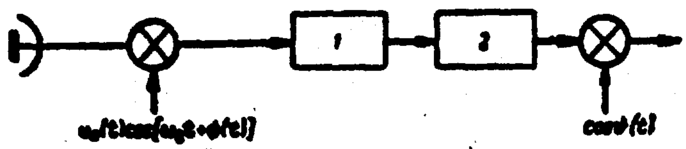

Fig. 11.2. Optimum circuit of the radio channel of an incoherent goniometer with pattern scanning: 1 — optimum filter with pulse response (11.2.8); 2 — square-law detector.

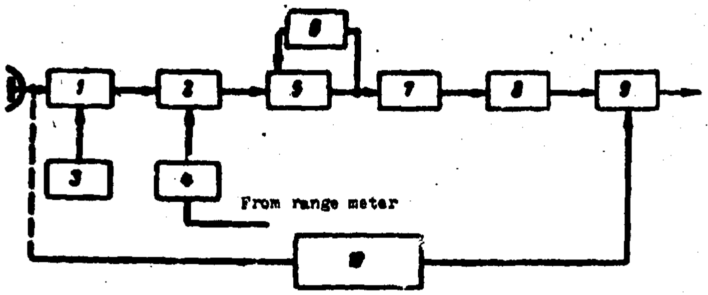

Fig. 11.3. Practical circuit of the radio channel of an incoherent goniometer with pattern scanning: 1 — high-frequency amplifier-mixer; 2 — amplitude modulator; 3 — heterodyne oscillator; 4 — generator of gate pulses; 5 — UPCh; 6 — system of automatic gain control; 7 — amplitude detector; 8 — pulse detector; 9 — phase detector; 10 — GON (reference voltage generator).

The difference is caused by the necessity of introducing certain elements, necessary during practical use of the circuit. Such an element in the first place is a system of automatic gain control [AGC]. It is introduced, as we already said, to maintain a constant level of the received signal, thanks to which the amplifier in the circuit always works in linear conditions. Furthermore, in the incoherent circuit of the method of pattern scanning after the amplitude detector in practice there usually is used a so-called pulse detector. The discharge time of the capacitor in the circuit of this detector is selected sufficiently large (so that the capacitor was discharged before the moment of arrival of the following pulse, there frequently is used forced discharge of this capacitor). Use of a pulse detector in such circuits permits us to obtain a very high transmission factor of the radio channel with minimum technical expenditures (otherwise at the output of the circuit it would be necessary to substitute an amplifier with a rather high gain factor).

As we already said in the preceding chapter, an AGC system in the first approximation influences only the slope of the discrimination characteristic of the circuit, and, consequently, such characteristics of accuracy as equivalent spectral density can be calculated without taking into account the AGC system. The influence of the AGC system on the slope of the discrimination characteristic of the circuit and on accuracy of incoherent goniometers as a whole we shall consider later, in § 11.7.

The pulse detector, as calculations show [53], approximately is equivalent to a normal amplifier. Consequently, from the point of view of calculation of accuracy

of the goniometer and study of the dependence of accuracy on the signal-to-noise ratio the pulse detector is a link such as can be omitted in general.

Thus, let us turn to analysis of accuracy of the circuit of Fig. 11.2. Calculation of accuracy characteristics of this circuit we shall produce in comparative detail, so that we can subsequently omit analogous calculations. Let us note that these calculations are very close to those with which we dealt in examining coherent circuits.

The received y(t) in the considered circuit has the form

$$y(t)=\sqrt{P_c}\,\mathrm{Re}\,U(t,\alpha)\,E(t)\,u(t)\,e^{i\omega_0 t+i\delta(t)}+\sqrt{N_0}\,n(t),$$

where $U(t, \alpha)$ is given by formula (10.2.2), and the remaining parameters were introduced earlier [see the explanation with formula (11.2.1)].

At the output of the circuit, obviously, we have signal

$$z(t)=\cos\vartheta(t)\left|\int_{-\infty}^{t}\dot{h}(t-\tau)\,y(\tau)\,v(\tau)\,e^{i\omega_0\tau}d\tau\right|^2, \tag{11.3.1}$$

where $\dot{h}(t)$ – complex pulse response envelope of the filter in the circuit;

$v(t)$ – complex signal amplitude envelope of the heterodyne oscillator.

Ideally

$$\dot{h}(t)=\begin{cases}1, & 0<t<T_r,\\ 0, & t>T_r,\end{cases}$$

[see formula (11.2.8)], and $v(t)$ should coincide with the complex signal amplitude envelope $u(t)$. However, in order to allow for possible imperfectnesses of filtration and processing of intraperiod modulation of the signal, we shall consider $\dot{h}(t)$ and $v(t)$ arbitrary. We introduce only the assumption that the passband of the filter $\Delta f_{\phi} \gg \frac{1}{T_r}$, which always occurs in incoherent goniometers. We shall place no further limitations on characteristics of the filter.

Let us calculate the mean value $\overline{z(t)}$, necessary for calculation of the slope of the discrimination characteristic and systematic error. Obviously,

$$\begin{gathered}\overline{z(t)}=\cos\vartheta(t)\int_{-\infty}^{t}\int_{-\infty}^{t}\dot{h}(t-t_1)\,\dot{h}^*(t-t_2)\,\overline{y(t_1)y(t_2)}\times\\ \times v(t_1)\,e^{i\omega_0 t_1}v^*(t_2)e^{-i\omega_0 t_2}\,dt_1dt_2=\\ =\cos\vartheta(t)\int_{-\infty}^{t}\int_{-\infty}^{t}\dot{h}(t-t_1)\,\dot{h}^*(t-t_2)\{P_c\rho(t_1-t_2)\times\\ \times\mathrm{Re}\,U(t_1,\alpha)U^*(t_2,\alpha)\,u(t_1)\,u^*(t_2)\times\\ \times e^{i\omega_0(t_1-t_2)+i[\delta(t_1)-\delta(t_2)]}+N_0\delta(t_1-t_2)\}\times\\ \times v(t_1)\,v^*(t_2)\,e^{i\omega_0(t_1-t_2)}dt_1dt_2.\end{gathered} \tag{11.3.2}$$

We use the broad-banded nature of the filter, so that with respect to functions $\rho(t)$ and $U(t, \alpha)$ pulse response $\dot{h}(t)$ can be considered a δ-function. Here we have (averaging also the rapidly oscillating functions under the sign of the integrals)

$$\overline{z(t)} = \cos\theta(t) P_a |U(t, \alpha)|^2 \frac{1}{2} \int_{-\infty}^{t} \int_{-\infty}^{t} \dot{h}(t-t_1) \dot{h}^*(t-t_2) \times$$
$$\times u(t_1) u^*(t_2) v^*(t_1) v(t_2) dt_1 dt_2 +$$
$$+ N_0 \cos\theta(t) \int_{-\infty}^{t} |\dot{h}(t-\tau) v(t)|^2 d\tau.$$

Averaging now $\overline{z(t)}$ in time and reducing by usual methods the integrals over time to integrals over frequency, we obtain

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \overline{z(t)}\, dt = \frac{\mu_a \alpha P_c}{4\pi} \int_{-\infty}^{\infty} |H(i\omega)|^2 |F(i\omega)|^2 d\omega,$$

where

$$F(i\omega) = \frac{1}{\sqrt{T_r}} \int_0^{T_r} u(t) v^*(t) e^{i\omega t} dt, \qquad (11.3.3)$$

and $H(i\omega)$ — frequency response of the filter (more exactly, of its low-frequency equivalent).

It follows from this that systematic error in the considered circuit is absent, since

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \overline{z(t)}\, dt \Big|_{\alpha=0} = 0.$$

The slope of the discrimination characteristic turns out to be equal to

$$K_д = \frac{d}{d\alpha} \lim_{T\to\infty} \frac{1}{T} \int_0^T \overline{z(t)}\, dt \Big|_{\alpha=0} = \frac{\mu_a P_c}{4\pi} \int_{-\infty}^{\infty} |H(i\omega)|^2 |F(i\omega)|^2 d\omega. \qquad (11.3.4)$$

For equivalent spectral density it is easy to analogously obtain the following:

$$S_{экв} = \frac{1}{K_д^2} \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \int_{-\infty}^{\infty} \overline{z(t) z(\tau)}\, d\tau \Big|_{\alpha=0} =$$
$$= \frac{1}{K_д^2} \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \int_{-\infty}^{\infty} d\tau \cos\theta(t) \cos\theta(\tau) \times$$
$$\times \int_{-\infty}^{t} \int_{-\infty}^{t} \int_{-\infty}^{\tau} \int_{-\infty}^{\tau} \dot{h}(t-t_1) \dot{h}^*(t-t_2) \dot{h}(\tau-\tau_1) \dot{h}^*(\tau-\tau_2) \times$$
$$\times \overline{y(t_1) y(t_2) y(\tau_1) y(\tau_2)}\, v(t_1) v^*(t_2) \times$$
$$\times v(\tau_1) v^*(\tau_2) e^{i\omega_0(t_1 - t_2 + \tau_1 - \tau_2)} dt_1 dt_2 d\tau_1 d\tau_2. \qquad (11.3.4')$$

Producing simple, but rather bulky calculations and limiting ourselves for simplicity to the case of uniform conical scanning, we finally can obtain

$$S_{\text{экв}} = \frac{1}{4\mu_a^2}\tilde{S}_0(\Omega) + \frac{T_r}{2\mu_a^2 q^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|H(i\omega_1)H(i\omega_2)|^2 \times$$
$$\times\{2\,\mathrm{Re}\,qF(i\omega_1)F^*(i\omega_2)V^*(i\omega_1+i\omega_2)+$$
$$+|V(i\omega_1+i\omega_2)|^2\}\,d\omega_1 d\omega_2\left[\int_{-\infty}^{\infty}|H(i\omega)|^2|F(i\omega)|^2 d\omega\right]^{-2}, \tag{11.3.5}$$

where

$$\tilde{S}_0(\Omega) = \int_{-\infty}^{\infty}\rho(t)^2 e^{-i\Omega t}dt; \tag{11.3.6}$$

$\rho(t)$ — correlation function of fluctuations of the signal, normalized so that $\rho(0) = 1$;

q — ratio of energy of the signal for the period of repetition and the spectral density of noise;

Ω — angular frequency of scanning;

$$V(i\omega) = \int_0^{T_r}|v(t)|^2 e^{i\omega t}dt. \tag{11.3.7}$$

When $\rho(t) = e^{-2\Delta f_c |t|}$

$$\tilde{S}_0(\Omega) = \frac{1/2\,\Delta f_c}{1+\left(\frac{\Omega}{4\Delta f_c}\right)^2}. \tag{11.3.8}$$

Let us study the found results in more detail. The simplest formulas are obtained in the case of a rather narrow-band filter whose passband satisfies relationship (11.2.9). In this case under the sign of integrals in (11.2.9) it is possible to set $|H(i\omega)|^2 = 2\pi\delta(\omega)\Delta f_ф$, where $\Delta f_ф$ — effective width of the passband of the filter (gain of the filter is assumed to be one). Here we easily obtain the following results. The slope of the discrimination characteristic is equal to

$$K_д = \frac{1}{2}\mu_a P_0 T_r \varkappa_1 \Delta f_ф, \tag{11.3.9}$$

where

$$\varkappa_1 = \left|\frac{1}{T_r}\int_0^{T_r}u(t)v^*(t)\,dt\right|^2$$

This coefficient was introduced already in Chapter X [see formula (10.5.9)]. It allows for imperfectness of processing of modulation of the signal. For equivalent spectral density we obtain the very simple formula

$$S_{экв} = \frac{1}{2\mu_a^2}\tilde{S}_\varphi(\Omega) + \frac{T_r}{2\mu_a^2}\,\frac{\mathrm{Re}\,2q\,|F(0)|^2 V^*(0) + |V(0)|^2}{q^2\,|F(0)|^4} = $$
$$= \frac{1}{2\mu_a^2}\tilde{S}_\varphi(\Omega) + \frac{T_r}{2\mu_a^2}\,\frac{2q' + 1}{q'^2}, \tag{11.3.10}$$

where

$$q' = \frac{P_c T_r}{2N_0}\,\frac{\left|\frac{1}{T_r}\int_0^{T_r} u(t)\,v^*(t)\,dt\right|^2}{\frac{1}{T_r}\int_0^{T_r} |v(t)|^2\,dt} = \frac{P_c T_r}{2N_0}\,\frac{\varkappa_1}{\varkappa_2}. \tag{11.3.11}$$

Ratio $\varkappa_1/\varkappa_2$, as we saw in the preceding chapter, is always smaller than one and equal to 1 only with ideal processing of modulation of the signal. Thus, imperfectness of processing of modulation with a narrow-band filter is equivalent to decrease of the signal-to-noise ratio.

From formula (11.3.10) it is clear that even with total elimination of noises there exists a residual value of equivalent spectral density

$$S_{экв} = \frac{\tilde{S}_\varphi(\Omega)}{2\mu_a^2}, \tag{11.3.12}$$

caused by fluctuations of the signal. With increase of the frequency of scanning the error caused by this factor disappears, and we obtain

$$S_{экв} = \frac{T_r}{2\mu_a^2}\left(\frac{2}{q'} + \frac{1}{q'^2}\right). \tag{11.3.13}$$

Let us consider asymptotic cases of high and low noise levels. With high noise levels

$$S_{экв} = \frac{T_r}{2\mu_a^2 q'^2}. \tag{11.3.14}$$

With ideal processing of signal modulation, when $q' = q$, this formula gives, obviously, the potential value of equivalent spectral density for a small signal-to-noise ratio, since the analyzed circuit is optimum for this case. It is therefore of interest to compare (11.3.14) with the equivalent spectral density of an optimum coherent circuit with a small signal-to-noise ratio. With approximation of the spectrum of the signal by formula (10.3.26) and for the case of uniform conical scanning it is easy to find that the equivalent spectral density of error in the optimum coherent circuit [formula (10.4.10)] is less than the equivalent spectral density in the incoherent circuit by a factor of $1/4T_r\Delta f_c$. Let us emphasize once

again that this occurs only for small signal-to-noise ratios.

Consider now the case of large signal-to-noise ratios. In this case

$$S_{\text{экв}} = \frac{T_r}{\mu_a^2 q}.$$

Let us compare the equivalent spectral density for the given case with the optimum spectral density of a coherent circuit.

For large signal-to-noise ratios $S_{\text{опт}}$ for the coherent circuit we take from formula (10.4.11). Here it is easy to find that the ratio of the equivalent spectral density of the considered incoherent circuit and the optimum spectral density of the coherent circuit (in the case of high frequencies of scanning) is equal to

$$\frac{S_{\text{экв}}}{S_{\text{опт}}} = \frac{T_r}{\mu_a^2 q} \cdot \mu_a^2 h \Delta f_c = 1.$$

Thus, for large values of the signal-to-noise ratio the considered incoherent circuit (with ideal processing of signal modulation) ensures the same accuracy as an optimum coherent circuit. From this we can conclude that the circuit of Fig. 11.2 realizes the potential accuracy of the method of pattern scanning also for a large signal-to-noise ratio, i.e., is a good approximation of the optimum circuit in the whole range of changes of the signal-to-noise ratio.

We repeat that everything said pertains to the case when the frequencies of scanning are sufficiently great as compared to the width of the spectrum of signal fluctuations. With low frequencies of scanning this circuit gives a component of error (11.3.12) which does not depend on the signal-to-noise ratio, and for a sufficiently large value of the signal-to-noise ratio, apparently, differing from the optimum.

Investigation of formula (11.3.5) with an arbitrary width of the filter passband is better produced for some concrete approximation of the form of modulation of the sounding and heterodyne signals and of the frequency response of the filter. Let us consider, for instance, the case of a pulse signal without phase modulation with pulses of Gaussian form:

$$u_a(t) = 1.15 \sqrt{\frac{T_r}{\tau_и}}\, e^{-2.8\left(\frac{t}{\tau_и}\right)^2}, \tag{11.3.15}$$

where $\tau_И$ — duration of the pulse at level 0.5 [the coefficient is selected from condition of normalization (10.3.4)].

Gate pulses we consider to have the same form, but with different duration:

$$v_s(t) = e^{-2.8\left(\frac{t}{\tau_s}\right)^2}. \tag{11.3.16}$$

We approximate the frequency response of the filter by function

$$|H(i\omega)|^2 = e^{-2.8\left(\frac{\omega}{2\pi\Delta f_{\text{ф}}}\right)^2}. \tag{11.3.17}$$

We designate

$$\Delta f_{\text{согл}} = \frac{5.6}{2\pi}\cdot\frac{1}{\tau_{\text{и}}} \approx \frac{0.9}{\tau_{\text{и}}}.$$

A filter with bandwidth $\Delta f_{\text{ф}} = \Delta f_{\text{согл}}$ frequently is called matched. We assume, further, that

$$x = \frac{\Delta f_{\text{ф}}}{\Delta f_{\text{согл}}} = 1.11\tau_{\text{и}}\Delta f_{\text{ф}},\quad y = \frac{\tau_{\text{и}}}{\tau_s}. \tag{11.3.18}$$

Calculation by formula (11.3.4) with such approximations gives the following result:

$$S_{\text{экв}} = \frac{1}{2\mu_s^2}\widetilde{S}_s(\Omega) + \frac{T_r}{2\mu_s^2}\left\{\frac{(1+y^2)(1+x^2+y^2)}{q^2\cdot 4y\sqrt{x^2+y^2}} + \right.$$
$$\left. + \frac{2}{q}\cdot\frac{(1+y^2)(1+x^2+y^2)}{\sqrt{[x^2+2(1+y^2)][x^2+2y^2(1+x^2+y^2)]}}\right\}. \tag{11.3.19}$$

With high frequencies of scanning the first term can be disregarded. Here $S_{\text{экв}}$ attains if minimum value $S_{\text{мин}}$ at $x = 0$, $y = 1$. This value coincides with (11.3.13), where it is necessary to set $q' = q$. However, already with small deviation of y from 1 the picture changes rather sharply: the minimum of $S_{\text{экв}}$ starts to be reached at $x \approx 1$. This one may see well from graphs of the dependence of $S_{\text{экв}}/S_{\text{мин}}$ on q, shown for different values of y and x (Fig. 11.4).

Thus, with nonideal gating the optimum filter is a filter, close to the matched one. With ideal gating the optimum is a narrow-band filter, integrating the signal in the period of repetition. However, all optima here lie rather closely to one another.

Let us note the very simple formulas obtained from (11.3.19) in limiting cases. For sufficiently wide gate pulses, when $y \ll 1$, we have

$$S_{\text{экв}} = \frac{1}{2\mu_s^2}\widetilde{S}_s(\Omega) + \frac{T_r}{2\mu_s^2}\left\{\frac{1+x^2}{4xyq^2} + \frac{2}{q}\cdot\frac{1+x^2}{x\sqrt{2+x^2}}\right\}. \tag{11.3.20}$$

This formula with considerable widening of the band of the filter, i.e., when $x \gg 1$, takes the following very simple form

$$S_{\text{экв}} = \frac{1}{2\mu_s^2}\widetilde{S}_s(\Omega) + \frac{T_r}{2\mu_s^2}\left(\frac{0.28\Delta f_{\text{ф}}\tau_s}{q^2} + \frac{2}{q}\right). \tag{11.3.21}$$

We shall now discuss the question of parametric fluctuations. The spectral density of parametric fluctuations in general is difficult to calculate, in view of the very great cumbersomeness of computations. We will give results of calculation of the spectral density of parametric fluctuations for the case of high frequencies of scanning, when the component of error caused by equivalent spectral density is rather small. In this case we have the very simple relationship

$$S_{пар} = \tilde{S}_0(0), \tag{11.3.22}$$

where $\tilde{S}_0(\Omega)$ is determined by formula (11.3.6). In particular, with approximation $\rho(t) = e^{-2\Delta f_c |t|}$ we obtain

$$S_{пар} = {}^1/_2 \Delta f_c.$$

Thus, the spectral density of parametric fluctuations does not depend on the form of modulation of the sounding signal and is inversely proportional to the width of the spectrum of signal fluctuations.

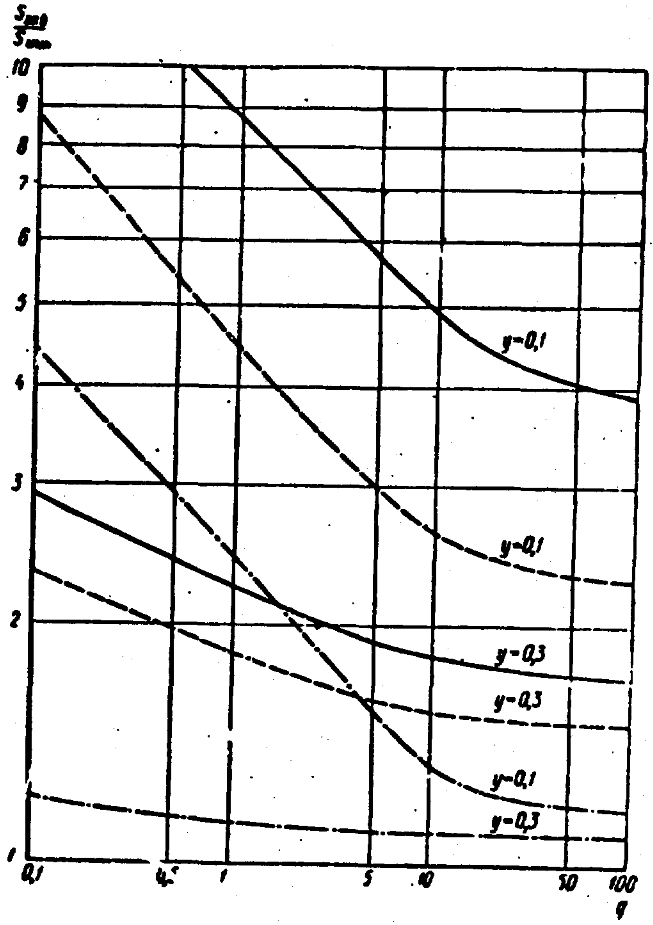

Fig. 11.4. Graph of the dependence of $S_{ЭКВ}/S_{МИН}$ on the signal-to-noise ratio q for the circuit of the method of pattern scanning: ——— x = 0.1; — — — x = 0.3; —·—·— x = 1.

§ 11.4. Methods of Scanning with Compensation and IAC

11.4.1 Optimum Circuits and Their Characteristics

Optimum circuits for the method of scanning with compensation are easily obtained from the general circuit of Fig. 11.1, where n = 2 (two directional patterns) and $\Phi_i(t, \alpha) = 0$ (their phase centers are fixed and coincide). Function $U_{ai}(t, \alpha)$ must here be expressed by formula (10.2.5) for the method of scanning with compensation or by formula (10.2.8) for IAC [the latter one coincides with (10.2.5), if there we set $\vartheta_1(t) = 0$, $\vartheta_2(t) = \pi$]. If we consider

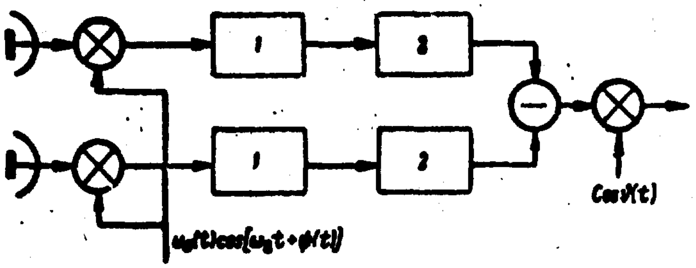

Fig. 11.5. Variant of the optimum circuit of the radio channel of an incoherent goniometer for the method of scanning with compensation: 1 — optimum filters with pulse response (11.2.8); 2 — square-law detectors.

the amplifications and gain factors of the directional patterns, and also noises in channels identical, and consider the laws of scanning connected by relationship (10.6.2) ($\vartheta_1(t) = \vartheta(t) = \vartheta_2(t) + \pi$), we can present the optimum circuit for the method of scanning with compensation in one of two forms, depicted in Figs. 11.5 and 11.6. By analogy with the terminology of Chapter X the circuit of Fig. 11.5 we shall subsequently call a circuit with subtraction, and the circuit of Fig. 11.6 — a circuit with multiplication of signals. Circuits for IAC have, obviously, precisely the same form, only it is necessary to set $\vartheta(t) = 0$, i.e., multiplication by cos $\vartheta(t)$ at the output of the circuits will be absent. The obtained circuits are very similar to the corresponding coherent circuits, synthesized in Chapter X.

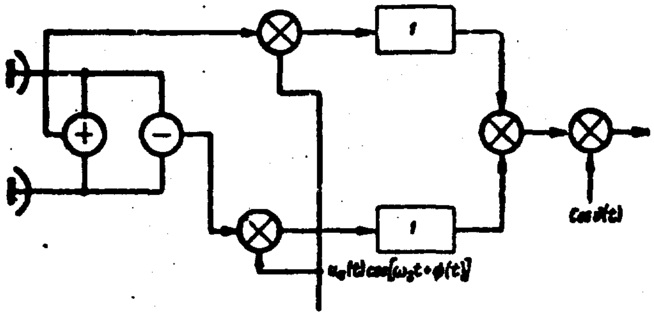

Fig. 11.6. Variant of the optimum circuit of the radio channel of an incoherent goniometer for the method of scanning with compensation. 1 — optimum filters with pulse response (11.2.8).

11.4.2. Investigation of Circuits with Multiplication of Signals

Let us investigate characteristics of synthesized circuits. It is obvious that both considered variants of circuits (with multiplication and subtraction of signals) with ideal construction give identical accuracy. The difference between these circuits will appear only in their unequal criticality with respect to nonidentity of channels. Investigation of this question is very interesting from the practical point of view.

In view of cumbersomeness of calculations we shall produce a simplified allowance for nonidentity of channels of the circuits, seeking basically to find the physics of the influence of nonidentity of different parameters and obtain simple calculating formulas.

Let us consider first the circuit with multiplication of signals. The output signal of the circuit of Fig. 11.6 has, obviously, form (10.7.1) or (10.7.2) (the latter when $\vartheta(t) = 0$). The received signals are equal to

$$y_i=\sqrt{P_{ci}}\,\mathrm{Re}\,U_i(t,\alpha)\,E(t)\,e^{i\omega_0 t+i\theta(t)}+\sqrt{N_{0i}}\,n_i(t), \tag{11.4.1}$$

where $U_i(t, \alpha)$ are expressed by formulas (10.2.5) or (10.2.8).

We assume that the width of the bands of the filters in the circuit satisfies relationship (11.2.9). We assume that the frequency responses of the filters are identical; however between the channels there is a certain constant phase shift $\Delta\varphi$. Obviously, this type of nonidentity is the most essential for circuits with multiplication of signals.

Omitting intermediate computations, we give the final results, identical for circuits of the method of scanning with compensation and IAC. The slope of the discrimination characteristic turns out to be equal to

$$K_{Д}=\frac{\mu_{a}P_{c\Sigma}T_{r}^{2}x_{1}}{2}\cos\Delta\varphi\,\Delta f_{\Phi}, \tag{11.4.2}$$

where $P_{c\,\Sigma}$ — total power of signals; the remaining designations are the same as in formula (11.3.9).

From this it is clear that in the presence of phase shift between channels the slope of the discrimination characteristic decreases. With very great nonidentity of channels slope $K_{Д}$ can fall to zero or even become negative. The circuit in this case, obviously, will not work.

Systematic error in the considered circuit will always be absent. This is also understandable from very simple reasonings: if the target is in the equisignal

direction, the signal in the difference channel will be equal to zero. Consequently, the signal of error also will be equal to zero, regardless of identity or nonidentity of the channels.

For the equivalent spectral density of error in the circuit with multiplication of signals we obtain

$$S_{\text{экв}} = \frac{T_r}{\mu_s^2 \cos^2 \Delta\varphi}\left(\frac{1}{q_\Sigma'} + \frac{1}{q_\Sigma'^2}\right), \tag{11.4.3}$$

where q'_Σ is determined by formula (11.3.11) (with replacement of P_C by $P_{C\,\Sigma}$).

From this we see that with nonidentity of channels there also occurs increase of the equivalent spectral density of error. This formula differs somewhat from the analogous formula (11.3.13) for the method of pattern scanning.

Let us assume now that the filter has a very wide band, so that there does not occur pure integration of the signal. Here we shall limit ourselves to the case of identical channels. Results will be analogous to those which we had for the method of pattern scanning: the slope of the discrimination characteristic is determined by formula (11.3.3) (with replacement of P_C by $P_{C\,\Sigma}$), and equivalent spectral density is determined by an expression somewhat differing from (11.3.4'):

$$S_{\text{экв}} = \frac{T_r}{\mu_s^2} \cdot \frac{1}{q_\Sigma^2\left[\int\limits_{-\infty}^{\infty} |H(i\omega)|^2 |F(i\omega)|^2 d\omega\right]^2} \times$$

$$\times \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} |H(i\omega_1) H(i\omega_2)|^2 [\operatorname{Re} q_\Sigma F(i\omega_1) F^*(i\omega_2) V^*(i\omega_1 + i\omega_2) +$$

$$+ |V(i\omega_1 + i\omega_2)|^2]\, d\omega_1 d\omega_2. \tag{11.4.4}$$

Here there is no term caused by nonlinear transformation of the useful signal, which testifies to compensation of the influence of amplitude fluctuation of the signal. Furthermore, the term proportional to $1/q'_\Sigma$ is decreased by a factor of 2. All designations in formula (11.4.4) are the same as in formula (11.3.4'). If $|H(i\omega)|^2 = 2\pi\Delta f_\phi\, \delta(\omega)$, formula (11.4.4) passes into (11.4.3), where $\Delta\varphi = 0$.

With approximations (11.3.15), (11.3.16), (11.3.17) and (11.3.18) for $S_{\text{экв}}$ we obtain the expression

$$S_{\text{экв}} = \frac{T_r}{\mu_s^2}\left[\frac{1}{q^2}\,\frac{(1+\nu^2)(1+x^2+\nu^2)}{4\nu\sqrt{x^2+\nu^2}} + \right.$$

$$\left. + \frac{2}{q}\,\frac{(1+\nu^2)(1+x^2+\nu^2)}{\sqrt{[\nu^2+2(1+\nu^2)][x^2+2\nu^2(1+x^2+\nu^2)]}}\right]. \tag{11.4.5}$$

insignificantly differing from (11.3.19). All qualitative conclusions with respect to the influence on $S_{\text{экв}}$ of various factors remain the same as before.

11.4.3. Investigation of Circuits with Subtraction of Signals

In circuits with subtraction of signals there already is a certain difference between the method of scanning with compensation and IAC.

The signal of error at the output of the circuit of Fig. 11.5 has the form (10.7.6) in the case of scanning with compensation (where $\tau_з = 0$ in view of the broad-band nature of the filters) or form (10.7.12) in the case of IAC. The received signals, as before, are expressed by formula (11.4.1). For these circuits the most essential type of nonidentity is nonidentity of gain factors of the channels.

Let us consider the case of a rather narrow-banded filter. Proceeding from the expression for the output signal, after simple transformations we obtain for the method of scanning with compensation the following results.

The slope of the discrimination characteristic

$$K_{\text{д}} = \frac{P_{c\Sigma}\mu_a T_r^2 x_1}{4\mu_a}\Delta f_{\Phi}(K_1^2 + K_2^2), \tag{11.4.6}$$

where K_1 and K_2 — gain factors of the channels.

Equivalent spectral density (in the case of uniform conical scanning)

$$S_{\text{экв}} = \frac{1}{8\mu_a^2}\cdot\frac{(1-k)^2}{1+k^2}\tilde{S}_0(\Omega) + \frac{T_r}{2\mu_a^2}\left(\frac{1}{q'_\Sigma} + \frac{1}{q'^2_\Sigma}\right), \tag{11.4.7}$$

where $k = K_1^2/K_2^2$ — ratio of squares of gain factors of the channels; the remaining designations are the same as before.

Systematic error in the considered case is absent.

From formula (11.4.7) it is clear that in the case of nonidentity of channels there exists a component of error caused by nonlinear transformation of the signal and not canceled even with complete elimination of noises. We had an analogous component of error in the case of the method of pattern scanning (11.3.5). With growth of the frequency of scanning this component of error disappears. With identical channels, when $k = 1$, this component of error is absent.

For the IAC method calculations do not differ at all from calculations for the method of scanning with compensation. Results are somewhat different. Gain factor $K_{\text{д}}$ is expressed by the same formula (11.4.6). However, now there will exist systematic error equal to

$$\Delta = \frac{1}{\mu_0}\frac{1-k}{1+k}\left(1 + \frac{1}{q'_\Sigma}\right). \tag{11.4.8}$$

Systematic error monotonically grows with growth of nonidentity of gain of the channels. The signal-to-noise ratio also affects systematic error.

For equivalent spectral density in the case of IAC we obtain the following expression:

$$S_{\text{экв}}=\frac{T_{\text{р}}}{\mu_{\Sigma}^{2}}\left[\frac{(1-k)^{2}}{4(1+k^{2})}+\frac{1}{q'_{\Sigma}}+\frac{1}{q_{\Sigma}'^{2}}\right]. \tag{11.4.9}$$

Thus, here there also exists a residual value of error, caused by amplitude fluctuations of the signal. In the case of identical channels, as one should have expected, formulas (11.4.9) and (11.4.7) coincide, i.e., the methods of scanning with compensation and IAC give identical accuracy, regardless of the circuit of the radio channel, if the channels of the latter are completely identical. This occurs, naturally, also for arbitrary frequency responses of the filters; then for circuits with subtraction accuracy will be characterized already by the known formula (11.4.4).

Comparing circuits of the method of scanning with compensation and with IAC, we note that any of the circuits with compensation with two channels can provide measurement of angular coordinates in two planes. Circuits with IAC for measuring two angular coordinates with switching will contain three channels (one with the sum signal and two with different signals). The IAC circuit with subtraction should already contain 4 channels (two channels for measurement of the angle in one plane and two for the other). Thus, simplest are circuits of the method of scanning with compensation, then follows the IAC circuit with multiplication of signals and, finally, the IAC circuit with subtraction of signals.

From the point of view of criticality of the circuits to various nonidentities it is possible to note the following. Circuits with multiplication are critical basically to nonidentity of phase-frequency responses, where circuits of the method of scanning with compensation and IAC are equivalent. Circuits with subtraction are basically critical to nonidentity of gain of the channels. Here the circuit of the method of scanning with compensation has an obvious advantage over the IAC circuit, especially in the case of high frequencies of scanning: at high frequencies of scanning nonidentity of gain in the circuit of the method of scanning with compensation does not affect accuracy; in the IAC circuit it always leads to the appearance of a component of equivalent spectral density, not depending on the signal-to-noise ratio, and also appearance of systematic error.

§ 11.5. Method of Instantaneous Phase Comparison of Signals

Of phase methods of direction finding with an incoherent signal we can use only the method of instantaneous phase comparison of signals (IPC). The method of scanning the phase center with an incoherent signal is not useful for measurement of angular coordinates. The IPC method can be used with an incoherent signal and will give accuracy of the same order as, for example, IAC.

The optimum circuit for IPC can be obtained from the general circuit of Fig. 11.1, if in it we set $n = 2$, $U_{ai}(t, \alpha) = 1$, and introduce $\Phi_i(t, \alpha)$ according to formula (10.2.9).

It is easy to see that there are two identical variants of IPC circuits, depicted in Figs. 11.7 and 11.8: a circuit with formation of sum and difference signals and a circuit without formation of these. In structure these circuits are the same as for a coherent signal; however characteristic of the filters already are substantially different.

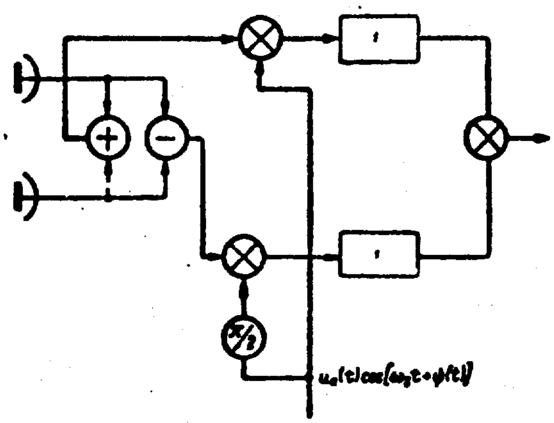

Fig. 11.7. Variant of the optimum circuit of the radio channel of an incoherent goniometer with IPC. 1 — optimum filters with pulse response (11.2.8).

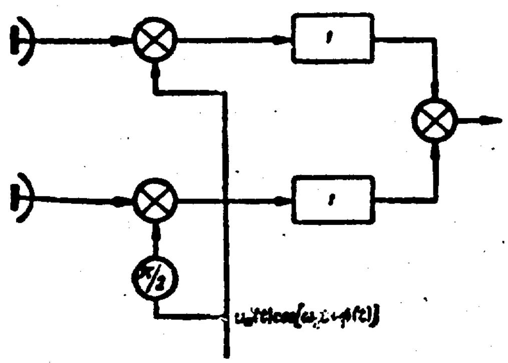

Fig. 11.8. Variant of the optimum circuit of the radio channel of an incoherent goniometer with IPC. 1 — optimum filters with pulse response (11.2.8).

Let us investigate the synthesized circuits. We consider first the case of sufficiently narrow-banded filters, carrying out exact integration of every period of the signal. We consider that channels of the circuits are nonidentical: between channels there is a constant phase shift $\Delta\varphi$. Nonidentity of gains also can be considered; however, it does not influence accuracy of the considered circuits. As also in the case of a coherent signal, the circuit of Fig. 11.8 for IPC is completely equivalent to the circuit with multiplication for IAC in the sense that

accuracy of the circuits are expressed by absolutely identical formulas; only the role of the gain factor of the directional pattern in the case of IPC is played by $\pi d/\lambda$. This is eady to prove by calculating the signal of error at the output of the circuit of Fig. 11.8 and finding the characteristic of accuracy of the circuit. Calculation shows that the slope of the discrimination characteristic will be expressed by the known formula (11.4.4), and $S_{\text{экв}}$ is expressed by formula (10.4.5), where μ_a must be replaced by $\pi d/\lambda$. Systematic error is equal to zero.

For a circuit without formation of a sum and difference signal results are substantially different. Although the slope of the discrimination characteristic in this case is expressed by formula (11.4.4) with replacement of μ_a by $\pi d/\lambda$, here there will already exist systematic error

$$\Delta=\frac{1}{2\pi d/\lambda}\operatorname{tg}\Delta\varphi. \tag{11.5.1}$$

Equivalent spectral density, as calculation shows, is expressed in the following way:

$$S_{\text{экв}}=\frac{T_r}{2\left(\frac{\pi d}{\lambda}\right)^2}\operatorname{tg}^2\Delta\varphi+\frac{T_r}{\left(\frac{\pi d}{\lambda}\right)^2}\cos^2\Delta\varphi\left(\frac{1}{q'_\Sigma}+\frac{1}{q'^2_\Sigma}\right). \tag{11.5.2}$$

From (11.5.2) it is clear that with nonidentical channels there exists a component of error which is not canceled with total elimination of noises, i.e., caused exclusively by fluctuations of the signal. From this we see that the circuit with IPC without formation of the sum and difference signals is considerably more critical to nonidentity of channels than a circuit with formation of the sum and difference signals, so that the latter circuit is preferable. With identical channels, when $\Delta_\Phi = 0$, both circuits are identical. Without dwelling on the calculation, we note that in the case of filters whose bandwidth is comparable with the broadened spectrum of modulation of the signal accuracies of both circuits are identical and are expressed by formula (11.4.4), in which it is necessary to replace μ_a by $\pi d/\lambda$.

We shall now stop to compare the considered methods of direction finding. It is easy to see that with a large signal-to-noise ratio and high frequencies of scanning (for the method of pattern scanning) accuracies of the considered methods of direction finding with ideal construction of the radio channel are expressed by the same formula

$$S_{\text{экв}}=T_r/c^2q,$$

where $c^2 = \mu_a^2$ (for methods of scanning the directional pattern, scanning with compensation and IAC) and $c^2 = \left(\frac{\pi d}{\lambda}\right)^2$ (for the IPC method). Comparison of methods

here is very easy to carry out, and results are the same as in the coherent case: the method of pattern scanning is inferior to methods of IAC and scanning with compensation in view of a worse signal-to-noise ratio; with decrease of the frequency of scanning accuracy of the method of pattern scanning becomes still lower. Methods of IAC and scanning with compensation are equivalent in accuracy. For comparison of IPC and IAC it is necessary to compare μ_a^2 and $\left(\frac{\pi d}{\lambda}\right)^2$; in particular, for a square aperture of an antenna system of assigned dimensions use of the methods of IAC and IPC is equivalent.

§ 11.6. Method of Flat Scanning of the Directional Pattern

We shall investigate incoherent circuits of the method of flat scanning of the directional pattern (method of tracking by pulse packs). For this method the incoherent circuit will essentially differ from the coherent. From general expression (11.2.6) it is easy to obtain the optimum operation for the pack method in the following form:

$$z(kT_r)=\left|\int_{(k-1)T_r}^{kT_r} u(t)y(t)e^{i\omega_0 t}\,dt\right|^2 U_a(kT_r,\hat{\alpha})U_a(kT_r,\hat{\alpha}).$$

Substituting for modulation $U_a(kT_r, \hat{\alpha})$ its value from (10.2.6) for the method of flat scanning, we obtain

$$z(kT_r)=\left|\int_{(k-1)T_r}^{kT_r} u(t)y(t)e^{i\omega_0 t}\,dt\right|^2 \sum_n g\left[\Omega(kT_r-\hat{\tau}_3+nT_\text{п})\right]g'\left[\Omega(kT_r-\hat{\tau}_3+nT_\text{п})\right], \tag{11.6.1}$$

where $\hat{\tau}_3 = \hat{\alpha}/\Omega$ — measured value of delay of packs of the signal;

$g(\varphi)$ — form of the directional pattern (for power);

Ω — angular velocity of motion of the pattern over the sector;

T_r — period of the sector scan.

Note that for every k in formula (11.6.1) there is only one term of the sum standing there with number

$$n=\left[\frac{kT_r-\tau_3}{T_r}\right],$$

where [x] signifies the integer nearest to x.

The block diagram of the device realizing operation (11.6.1) is presented in Fig. 11.9. Here, after heterodyning, filtration and square detection, the signal is multiplied by gate pulses having form $g(\Omega t)\,g'(\Omega t)$, delay of which is controlled. These gate pulses are bipolar, which is equivalent to multiplication by single-pole

gate pulses and further subtraction. The circuit of Fig. 11.9 is well-known. Our analysis will help one to establish optimum values of parameters of this circuit.

Let us turn to investigation of characteristics of the synthesized circuit. We assume, as before, that there is used a filter with possband $\Delta f_{\Phi} \gg 1/T_r$ and arbitrary frequency response, and that the heterodyne signal is not matched with the received signal. Furthermore, we consider that the strobes by which the output signal is multiplied have a certain derivative form $f(t)$. We introduce the hypothesis that the packs fluctuate harmonously. Allowance for distortion of packs resulting from fluctuations of the signal is an extraordinarily difficult and cumbersome problem.

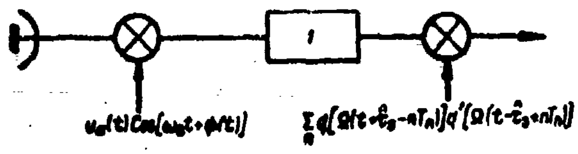

Fig. 11.9. Optimum circuit of the radio channel of a goniometer using the method of flat scanning. 1 — optimum filter with pulse response (11.2.8).

The signal of error at the output of the circuit of Fig. 11.9 has the form

$$z(t)=\left|\int_{-\infty}^{t} h(t-\tau)\,y(\tau)\,v(\tau)\,e^{i\omega_0\tau}\,d\tau\right|^2 \sum_n f(t-\tau_з+nT_п), \qquad (11.6.2)$$

where the received signal $y(t)$ is equal to

$$y(t)=\sqrt{P_c}\sum_n g[\Omega(t-\tau_з+nT_п)]\times$$
$$\times \operatorname{Re} E(t)\,u(t)\,e^{i\omega_0 t+i\theta(t)}+\sqrt{N_0}\,n(t).$$

Here $\tau_з = \alpha/\Omega$; α — angular coordinate of the target. The remaining designations are the same as before.

To ease calculations we assume that function $f(t)$ is odd. Otherwise calculations and results become very cumbersome.

Averaging expression (11.6.2) over the ensemble and over time, we obtain

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \overline{z(t)}\,dt=\frac{P_c}{4\pi}\int_{-\infty}^{\infty}|H(i\omega)|^2|F(i\omega)|^2\,d\omega\times\frac{1}{T_п}\int_0^{T_п} g[\Omega(t-\tau_з)]^2 f(t-\hat{\tau}_з)\,dt$$

$$=\frac{P_c}{4\pi}\int_{-\infty}^{\infty}|H(i\omega)|^2|F(i\omega)|^2 d\omega\cdot\frac{1}{T_п}\int_0^{T_п} g(\Omega t)^2 f(t-\Delta\tau_з)\,dt,$$

where $\Delta\tau_з = \hat{\tau}_з - \tau_з$.

From this it is easy to find the expression for the slope of the discrimination characteristic:

$$K_{д}=\frac{\partial}{\partial\Delta\tau_{0}}\lim_{T\to\infty}\frac{1}{T}\int_{0}^{T}\overline{z(t)}\,dt=$$

$$=\frac{P_{c}}{4\pi}\int_{-\infty}^{\infty}|H(i\omega)|^{2}|F(i\omega)|^{2}d\omega\cdot\frac{1}{T_{п}}\int_{0}^{T_{п}}g(\Omega t)^{2}f'(t)\,dt=$$

$$=\frac{P_{c}C_{0}}{4\pi}\int_{-\infty}^{\infty}|H(i\omega)|^{2}|F(i\omega)|^{2}d\omega, \tag{11.6.3}$$

where

$$C_{0}=\frac{1}{T_{п}}\int_{0}^{T_{п}}g(\Omega t)^{2}f'(t)\,dt. \tag{11.6.4}$$

For the equivalent spectral density of error we obtain the following expression:

$$S_{экв}=\frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}|H(i\omega_{1})|^{2}|H(i\omega_{2})|^{2}[C_{1}|V(i\omega_{1}+i\omega_{2})|^{2}+ +2\operatorname{Re}q C_{2}F(i\omega_{1})F^{*}(i\omega_{2})V^{*}(i\omega_{1}+i\omega_{2})]\,d\omega_{1}d\omega_{2}}{q^{2}C_{0}^{2}\left[\int\limits_{-\infty}^{\infty}|H(i\omega)|^{2}|F(i\omega)|^{2}d\omega\right]^{2}}, \tag{11.6.5}$$

where

$$C_{1}=\frac{1}{T_{п}}\int_{0}^{T_{п}}f(t)^{2}\,dt, \tag{11.6.5'}$$

$$C_{2}=\frac{1}{T_{п}}\int_{0}^{T_{п}}g(\Omega t)^{2}f(t)^{2}\,dt.$$

Formula (11.6.5) is very general and from it it is difficult to perceive the laws of change of $S_{экв}$ depending on different factors. We expand formula (11.6.5), introducing suitable approximations for parameters in it. Considering the filter sufficiently narrow-banded so that relationship (11.2.9) is carried out, we have

$$S_{экв}=T_{r}\frac{C_{1}+2q'C_{2}}{C_{0}q'^{2}}, \tag{11.6.6}$$

where q' is expressed by formula (11.3.11).

We use now a Gaussian approximation for the directional pattern (10.10.26). Here

$$g(\Omega t)=1{,}15\sqrt{\frac{T_{п}}{\tau_{н}}}\,e^{-2{,}8\left(\frac{t}{\tau_{н}}\right)^{2}},$$

where $\tau_{\Pi} = \Delta\varphi/\Omega$ — duration of a pack of the signal at level 0.5. We shall approximate gate pulse f(t) by function

$$f(t) = te^{-2,8\left(\frac{t}{\tau_c}\right)^2},$$

i.e., gate pulses of form f(t) are obtained by differentiation of Gaussian gate pulses having duration τ_c at level 0.5. With these assumptions it is easy to calculate that

$$C_0 = \frac{1}{(1+0,5x^2)^{3/2}},$$

$$C_1 \approx 0,07\frac{\tau_c^3}{T_{\text{п}}},$$

$$C_2 = 0,09\frac{\tau_{\text{п}}^2}{(1+x^2)^{5/2}},$$

where $x = \tau_{\Pi}/\tau_c$.

Substituting these results in formula (11.6.6), we obtain

$$S_{\text{экв}} = 0,07\frac{T_{\text{п}}\Delta\varphi^2}{\Omega^2 q_{\text{п}}'^2}\left[\frac{\gamma}{x^3} + q'_{\text{п}}\frac{2,56}{(1+x^2)^{5/2}}\right](1+0,5x^2)^3, \qquad (11.6.7)$$

where $\gamma = \frac{\tau_{\Pi}}{T_r}$ — number of periods of repetition of the signal in a pack (number of pulses in a pack),

$$q'_{\text{п}} = q'\frac{T_{\text{п}}}{T_r} = \frac{P_{\text{п}}T_{\text{п}}}{2N_0}$$

is the ratio of energy of the signal for the period of repetition of packs to the spectral density of noise.

In formula (11.6.7) factor $1/\Omega^2$ is taken for calculation of the spectral density and error of measurement of delay of the signal. To determine error of measurement of an angle we need to drop factor $1/\Omega^2$, which we shall do subsequently.

Let us study the formula for $S_{\text{экв}}$ in greater detail.

For small signal-to-noise ratios

$$S_{\text{экв}} = 0,07\frac{T_{\text{п}}\Delta\varphi^2\gamma}{q_{\text{п}}'^2}\left(\frac{1+0,5x^2}{x}\right)^3.$$

The minimum value of $S_{\text{экв}}$ in this case occurs, obviously, at $x = \sqrt{2} \approx 1.4$, and is equal

$$S_{\text{мин}} = 0,2\frac{T_{\text{п}}\Delta\varphi^2\gamma}{q_{\text{п}}'^2}. \qquad (11.6.8)$$

If we also consider processing of signal modulation ideal, so that $q'_{\Pi} = q_{\Pi}$, then $S_{\text{мин}}$ will coincide with the minimum possible value of equivalent spectral

density for high noise levels, since with the given assumptions the analyzed circuit is optimum. It is curious to compare, therefore, formula (11.6.8) with formula (10.10.28), giving the minimum possible value of equivalent spectral density $S_{\text{опт}}$ for the method of flat scanning with a small signal-to-noise ratio and a coherent signal. This comparison gives the following:

$$\frac{S_{\text{опт}}}{S_{\text{экв}}}=\frac{0{,}2q_{п}^{2}}{c^{2}k^{2}T_{п}\Delta f_{c}\left(\sum\limits_{k=-\infty}^{\infty}\rho(kT_{п})^{2}\right)T_{п}\Delta\varphi^{2}\gamma}\approx\frac{0{,}2}{c^{2}\Delta\varphi^{2}\gamma\left(\sum\limits_{k=-\infty}^{\infty}\rho(kT_{п})^{2}\right)}.$$

Hence, in the case of rigidly correlated packs, when $T_{п}\ \Delta f_{c} \ll 1$, we have (with a Gaussian approximation of the directional pattern and exponential approximation of the correlation function of fluctuation)

$$\frac{S_{\text{опт}}}{S_{\text{экв}}}\approx\frac{0{,}3T_{п}\Delta f_{c}}{\gamma}.$$

From this formula it is clear that for small values of the signal-to-noise ratio an incoherent goniometer gives considerable worsening of accuracy as compared to a coherent one. The worsening is inversely proportional to the number of packs in the interval of correlation of the signal, and proportional to the number of pulses in a pack. Physically, both circumstances are understandable.

In the case of independently fluctuating packs, when $T_{п}\ \Delta f_{c} \gg 1$, we have

$$\frac{S_{\text{опт}}}{S_{\text{экв}}}=\frac{0{,}07}{\gamma},$$

i.e., here there naturally occurs lowering of accuracy, proportional to the number of pulses of the signal in a pack.

Let us consider the case of large values of the signal-to-noise ratio. Here, from formula (11.6.7) we obtain

$$S_{\text{экв}}=0{,}186\,\frac{T_{п}\Delta\varphi^{2}}{q'_{п}}\left[\frac{1+0{,}5x^{2}}{\sqrt{1+x^{2}}}\right]^{2}.$$

It is easy to see that the obtained value of $S_{\text{экв}}$ monotonically drops with decrease of x, reaching its minimum at x = 0. Thus, for low noise levels gate pulses must be taken as wide as possible.

For an arbitrary signal-to-noise ratio there will exist a certain optimum value of x. The graph of the dependence of optimum values of $x^{2}_{\text{опт}}$ on $q'_{п}$ for different γ is shown in Fig. 11.10. From this graph one may graphically see what the optimum relationship of durations of packs and gate pulses $x_{\text{опт}}$ should be for different values of the signal-to-noise ratio.

In Fig. 11.11 there is shown the dependence of ratio $S_{ЭКВ}/S_{МИН}$ on the signal-to-noise ratio $q'_{П}$, where $S_{МИН}$ is the equivalent spectral density at optimum x. From Fig. 11.11 it is clear that when x = 1.4 the equivalent spectral density $S_{ЭКВ}$ is very close to its minimum value in the whole range of signal-to-noise ratios. At x = 0.5 or x = 4 quantity $S_{ЭКВ}$ is considerably increased.

On this we complete our study of the radio channel of incoherent goniometers, which together with antennas comprise the discriminator of the goniometer system, and we turn to investigation of accuracy of tracking goniometers.

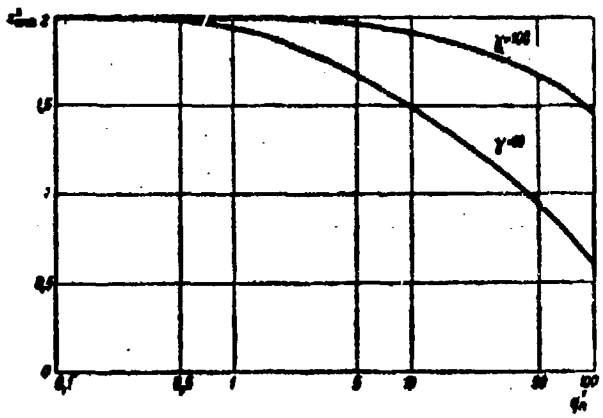

Fig. 11.10. Graph of the dependence of optimum values of $x^2_{ОПТ}$ on the signal-to-noise ratio $q'_{П}$.

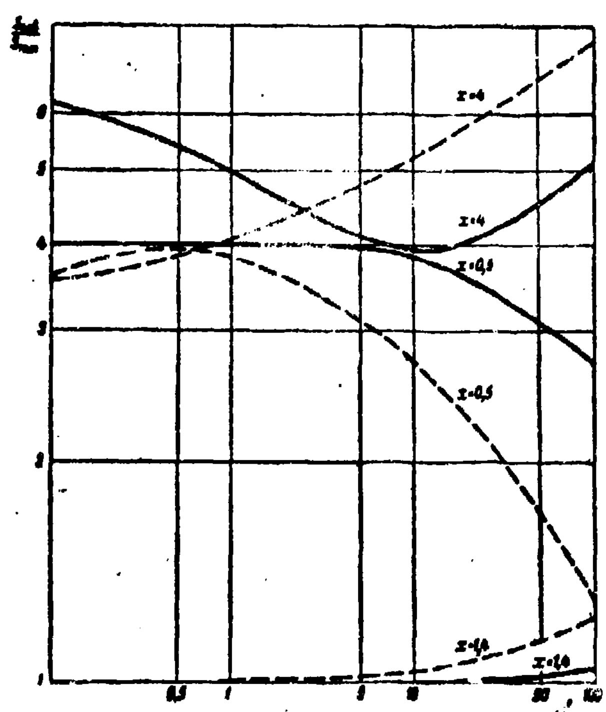

Fig. 11.11. Graph of the dependence of $S_{ЭКВ}/S_{МИН}$ on signal-to-noise ratio $q'_{П}$ for the circuit of the method of flat scanning: ——— γ = 100; – – – – γ = 10.

§ 11.7. Analysis of Accuracy of Incoherent Goniometers

Let us calculate errors of measurement of angular coordinates of a target by tracking radar goniometers using an incoherent signal. These calculations contain nothing new in principle as compared with the coherent case: it is necessary only in formulas § 10.12 to replace the characteristics of the radio channel for the coherent case by characteristics for the incoherent case. In view of this all laws relating to accuracy of measurement of angles fixed in § 10.12 are preserved and do not need repeated discussion. Therefore, we shall limit ourselves in the present chapter basically to consideration of illustrative examples.

First of all we must consider the influence of an AGC system. As we know, under the influence of an AGC system there is established a certain transmission factor of the radio channel, depending on the signal-to-noise ratio. There are no peculiarities here as compared to coherent goniometers. The system of automatic gain control in single-channel incoherent circuits is closed from the output of the filter of the radio channel. In the case of multichannel radio channels the system of automatic gain control is closed either from the output of one of the filters, or the input voltage of the AGC will be formed by means of summation of the output voltages of the filters. In all cases, as it is easy to see, the transmission factor of the radio channel will be expressed by formula (10.13.1), which now is best reduced to form

$$K_{д}=\frac{K_{д0}}{1+\frac{\Delta f_{y}T_{r}}{q}}=\frac{K_{д0}}{1+\frac{y}{q}}, \tag{11.7.1}$$

where q – ratio of energy of the signal for the period of repetition to the spectral density of noise at the input of the AGC system;

Δf_{y} – effective bandwidth of the amplifier covered by the AGC loop.

Let us turn to consideration of examples of calculation of errors of measurement in tracking incoherent goniometers. Let us stop on calculation of fluctuating error. Assume we have a goniometer with IAC, whose radio channel is constructed according to the circuit of Fig. 11.6. (circuit with multiplication of signals) with identical channels. We assume that smoothing circuits are linear with constant parameters and have transfer function $H_{I}(p)$ (10.12.3).

Fluctuating error in such a goniometer will be expressed, obviously, by formula

$$\Delta^{2}_{\phi ш}=2\Delta f_{эф}S_{вкв}=\frac{T_{r}}{\mu^{2}_{ш}}\left(\frac{1}{q'_{\Sigma}}+\frac{1}{q'^{2}_{\Sigma}}\right)\omega_{0}\frac{\frac{1}{1+\frac{y}{q'_{\Sigma}}}+\frac{1}{2}\beta_{c}}{4-\beta^{2}_{c}}.$$

We use here formula (11.4.7) for $S_{\text{ОКВ}}$ and (10.12.18) for $\Delta f_{\text{эф}}$. The graph of the dependence of $\sigma^2_{\text{фл}}$ on q'_{Σ} for different values of y is shown in Fig. 11.12 (for $\psi_0 = 30^{\circ}$). As can be seen from this figure, $\sigma^2_{\text{фл}}$ rather sharply drops with growth of the signal-to-noise ratio. Here the best results are obtained for large y, i.e., for a wide passband of the filter. However, the dependence on y is very weak, especially for large signal-to-noise ratios.

Dynamic errors during processing of random or nonrandom inputs are expressed, obviously, by the same formulas as in the coherent case [see formulas (10.12.21) and (10.12.25)]. Inasmuch as the slope of the discrimination characteristic $K_{\text{д}}$ in incoherent circuits depends on the signal-to-noise ratio q exactly as it depended on h in coherent circuits, dynamic errors will be the same, only in the corresponding expressions it is necessary instead of h to place q.

Thus, all the results of the preceding chapter with respect to dynamic errors are identically transferable to the considered case, so that additional consider[illegible] required. [illegible] also that an incoherent goniometer is completely like the coherent one with respect to nonlinear phenomena occurring during tracking of targets, inasmuch as the given discrimination characteristics of incoherent goniometers simply coincide with discrimination characteristics of coherent goniometers.

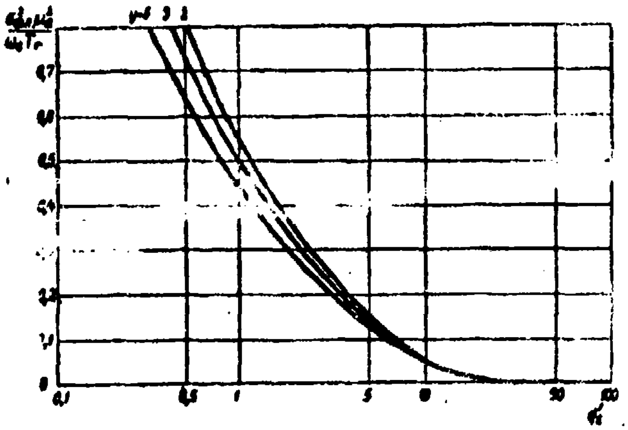

Fig. 11.12. Example of the dependence of mean square fluctuating error $\sigma^2_{\text{фл}}$ of an incoherent goniometer on the signal-to-noise ratio q'_{Σ}.

§ 11.8. Influence of Interferences on Incoherent Goniometers

We turn to a study of the influence of interferences on incoherent goniometers [70]. As compared with the coherent case there are no essential peculiarities here. True, the passbands of filters in the radio channels of incoherent goniometers are considerably wider than in coherent goniometers. Therefore, equivalence of certain interferences to white noise, fixed for coherent goniometers, here occurs with greater strains. We shall consider these questions in greater detail.

11.8.1. Influence of Broad-Band Interferences on Side Lobes

Active interferences usually have a spectrum width considerably exceeding the spectrum width of the sounding signal. Consequently, even for radio channels of incoherent goniometers such interferences can be considered equivalent to white noise. If a broad-band interference influences a side lobe, it does not carry information and is equivalent simply to natural noises of the receiver. Therefore, the presence of such interference can be sufficiently accurately accounted for by means of introduction of a new spectral density of noise, equal to $N_0 + N_\Pi$, where N_Π — spectral density of interference at the input of the radar receiver. It is easy to see that the spectral density of interference N_Π will be expressed by absolutely the same formulas as for the coherent case: for noise interference we have formula (10.13.1); for pulse chaotic, formula (10.13.2).

Accuracy of incoherent goniometers is given here by formulas derived in the preceding sections with replacement of signal-to-noise ratio $q = \frac{P_c T_r}{2N_0}$ by

$$q_\Pi = \frac{P_c T_r}{2(N_0 + N_\Pi)}.$$

Hence for weak interferences (and weak natural noises), when $q_\Pi \gg 1$, noise immunity of coherent and incoherent goniometers against the considered interference is identical. For strong interferences, when $q_\Pi \ll 1$, incoherent goniometers will have an equivalent spectral density of error, larger than the coherent by a factor of approximately $\frac{1}{T_r \Delta f_c}$ i.e., losses here may be very considerable.

11.8.2. Influence of Broad-Band Interferences From the Target

Broad-band interference radiated from a target also is equivalent to white noise. However this interference carries in itself angular information, by which it essentially differs from interference having effect on side lobes. In view of the considerable excess of interference from the target over the signal, the latter is usually suppressed through the AGC system, and tracking is conducted only by the interference. Let us calculate the accuracy of incoherent goniometers during work on broad-band interference from the target. We must not directly use formulas of the preceding sections giving accuracy of measurement by a fluctuating signal here, since in deriving these formulas the width of the spectrum of fluctuations was assumed considerably narrower than the passband of the filters. For broad-band interference we have the reverse case, so that calculation here should be performed anew.

Let us consider first the method of pattern scanning. Calculation of accuracy here can be produced proceeding from formulas (11.3.4) and (11.3.4'), if we replace the correlation function of signal fluctuations $P_{c}\rho(t - s)$ by $N_{\pi}\,\delta(t - s)$, and also set $u(t) = 1$, $N_0 = 0$ (i.e., natural noises are absent, and instead of the signal there acts white noise). With these replacements calculations are essentially simplified, and it is easy to obtain the following:

$$S_{\text{экв}} = \frac{T_r}{2\mu_a^2 \Delta f_{\Phi}^2 V(0)^2} \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |H(i\omega_1) H(i\omega_2)|^2 \times$$

$$\times |V(i\omega_1 + i\omega_2)|^2 \, d\omega_1 d\omega_2,$$

where $V(i\omega)$ is given by formula (11.3.7);

$H(i\omega)$ is frequency response of the filter;

Δf_{Φ} is its effective passband;

T_r is period of repetition of the signal.

If the filter band is so narrow that it integrates gate pulses proceeding to it, then, considering $|H(i\omega)|^2 = 2\pi\Delta f_{\Phi}\,\delta(\omega)$, we obtain

$$S_{\text{экв}} = \frac{T_r}{2\mu_a^2}. \tag{11.8.1}$$

This formula coincides with formula (10.13.4), occurring for the coherent circuit, if we set $\Delta f_{\Phi} = \frac{1}{T_r}$. If the filter is so broad-banded that it passes gate pulses with minute distortion, then instead of (11.8.1) we have an essentially different result. Considering in this case

$$|V(i\omega)|^2 = 2\pi\delta(\omega) \int_0^{T_r} |v(t)|^4 \, dt,$$

we obtain

$$S_{\text{экв}} = \frac{T_r}{2\mu_a^2 \Delta f_{\Phi}^2 V(0)^2} \int_0^{T_r} |v(t)|^4 dt \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(i\omega)|^4 \, d\omega.$$

With a square approximation of gate pulses and the frequency response of the filter we have

$$S_{\text{экв}} = \frac{T_r}{2\mu_a^2 \Delta f_{\Phi} \tau_c}, \tag{11.8.2}$$

where τ_c — duration of a gate pulse.

The absolute value of error when tracking a target by broad-band interference radiated from it can easily be found from the derived formulas. For instance, at

$T_r = 10^{-3}$ sec, $\mu_a = 0.2$ 1/deg by (11.8.1) we find $S_{\text{экв}} \approx 0.01$ deg^2/cps.

Let us consider now the influence of broad-band interference from the target on goniometers using methods of scanning with compensation, IAC and IPC. The rules here are the same as for coherent goniometers. When radio channels have identical channels, tracking by interference occurs without fluctuating error (if we do not consider parametric fluctuations). With nonidentical channels in circuits with subtraction of signals there appear fluctuating and sometimes systematic errors. If we assume the filters in the circuits have a sufficiently narrow band so that gate pulses proceeding to them are integrated, the formula for equivalent spectral densities of errors can be obtained from the corresponding formulas for the coherent case (see § 10.13) by replacement of $\Delta f_{\text{ф}}$ by $1/T_r$. Systematic errors have absolutely identical form for coherent and incoherent circuits.

Thus, for circuits of the method of scanning with compensation and IAC with subtraction of signals we have

$$S_{\text{экв}} = \frac{T_r}{4\mu_a^2}\left(\frac{1-k^2}{1+k^2}\right)^2,$$

where k — ratio of squares of gain factors in the channels.

Systematic error exists only for IAC and is expressed by formula (10.13.18).

Analogously for an IPC circuit without formation of sum and difference signals, with nonidentical channels we have equivalent spectral density

$$S_{\text{экв}} = \frac{3T_r}{4\left(\frac{\pi d}{\lambda}\right)^2}\operatorname{tg}^2\Delta\varphi,$$

where $\Delta\varphi$ — difference of phases of signals at the output of the channels, caused by nonidentity of channels.

Systematic error, as before, is expressed by formula (10.13.21). Absolute values of errors here are of the same order as errors during work on a fluctuating signal with considerable excess of signal over noise. From what has been said we can conclude that interference radiated from a target for goniometer channels of radars does not present danger, and tracking by such interference occurs with fully permissible errors. It is necessary, of course, to mention that for more sure tracking of interference in a number of cases we should take certain special measures: widening of the gate pulses (with broad-band filters) and cutoff of noise automatic gain control (ShARU), if we have it (since ShARU would reduce the level of interference to the level of natural noises). These measures depend on concrete conditions.

11.8.3. Influence of Active Interferences with Narrow-Band Amplitude Modulation

Let us consider the question of the influence of active interferences with special narrow-band amplitude modulation, having a spectrum near the frequency of scanning, on goniometers using methods of pattern scanning. The calculations and results here absolutely do not differ from what we had for coherent goniometers, and equivalent spectral density is expressed by the known formula (10.13.25). Errors here, as shown in § 10.13, may be very considerable.

Analogously for the method of scanning with compensation we can obtain results completely coinciding with results for coherent goniometers: in a circuit with multiplication of signals fluctuating error during tracking by amplitude-modulated interference is equal to zero if we disregard parametric fluctuations; the spectral density of parametric fluctuations with identical channels is given by formula (10.3.27). The same thing takes place in a circuit with subtraction of signals with identical channels. With nonidentical channels in a circuit with subtraction $S_{\text{эКВ}}$ is different from zero and is expressed by formula (10.3.26). Error here depends on the degree of nonidentity of channels and usually is rather small. Consequently, interference with narrow-band amplitude modulation is very dangerous for radars using the method of pattern scanning and practically does not affect radars using the method of scanning with compensation.

The influence of powerful intermittent interferences, exciting in the radar receiver prolonged transients, was qualitatively studied for the coherent case in §10.13.4. Everything presented there is completely transferable to incoherent circuits. Quantitatively, of course, there will be certain differences, due to difference of the parameters of coherent and incoherent circuits; however, the influence of intermittent interferences on goniometer devices practically does not yield to theoretical analysis, and the problem of obtaining quantitative characteristics of the influence of such interferences, as also in the coherent case, has not been studied.

With respect to passive interference it is possible to say the following. It is known that noise immunity of incoherent radars with respect to passive interference is very low if there is not provided special measures of protection from this interference (see, for instance, Chapters V, VII and Paragraph 8.10.4). Incoherent radar goniometers are not an exception in this respect. The influence of passive interference on incoherent goniometers practically always lead to sharp increase

of errors of measurement and breakoff of tracking.

Theoretical analysis of the influence of passive interferences on incoherent goniometers therefore is deprived of practical interest, and we shall not study it.

§ 11.9. Conclusion

In the two preceding chapters we studied radar goniometers using coherent and incoherent signals. What conclusions can be drawn with respect to comparison of these two cases? Comparison of circuits of optimum radio channels of coherent and incoherent goniometers shows that these circuits are similar in general. Their structure in both cases is identical, and the whole difference reduces only to difference of the characteristics of the filters. Filters of coherent circuits are comparatively narrow-banded and ensure integration of the signal during the time of its correlation; filters of incoherent circuits usually are considerably broader-banded and ensure integration of signal only for the period of its repetition.

We stop now to discuss the relationship of accuracy of coherent and incoherent goniometers. The preceding analysis showed that with large signal-to-noise ratios accuracies of incoherent and coherent goniometers completely coincide, i.e., with a large signal-to-noise ratio the form of the signal does not affect accuracy of goniometer. With small signal-to-noise ratios incoherent goniometers have considerable losses of accuracy as compared to coherent. The ratio of the equivalent spectral densities of coherent and incoherent circuits for small signal-to-noise ratios has the order $T_r \Delta f_c$, i.e., lowering of accuracy due to incoherence of the signal is approximately proportional to the number of periods of repetition of the signal in the interval of its correlation. From this there ensures lower noise immunity of incoherent goniometers as compared to coherent against interferences similar to natural noises of receivers (broad-band interferences in side lobes of the directional pattern of the antenna). The influence of such interference is equivalent to decrease of the signal-to-noise ratio in the goniometer, which will lead in incoherent goniometers to greater decrease of accuracy than in coherent goniometers.

A very great deficiency of incoherent goniometers, greatly narrowing the domain of their application, is their very low immunity against passive interferences. Here coherent goniometers have a decisive advantage over incoherent ones.

We shall now discuss certain questions concerning incoherent goniometers which require further investigation. In the first place here one should place the problem of exact synthesis of optimum goniometers using incoherent radiation.

This problem was solved by us only with certain rather limiting assumptions. Exact solution of this problem is of considerable interest. The remaining unsolved questions concerning incoherent goniometers are basically the same as were formulated in the concluding section of the preceding chapter: synthesis of goniometers without preliminary assignment of the method of direction finding, synthesis of goniometers with comparable times of correlation of angular shifts of the target and of the reflected signal, study of statistical regularities of angular shifts of the target and synthesis of optimum smoothing circuits, investigation of nonlinear phenomena in tracking incoherent goniometers.

CHAPTER XII

JOINT MEASUREMENT OF SEVERAL COORDINATES

§ 12.1 Introduction

In Chapters VI-XI we considered separately measurement of each radar coordinate. Theoretically this is permissible either when finding approximate solutions or when all remaining coordinates besides the measured one are constant or change in a certain way. Therefore, of simultaneously great practical and theoretical interest are questions of analysis and synthesis of joint meters of several coordinates. Only complicated theoretical consideration can show how exact methods of analysis and synthesis developed in idealized premises of one measured coordinate are.

With development of radar technology questions of multi-dimensional measurement become more and more urgent. Thus, in early surveillance radars they usually measured two parameters, corresponding to bearing and range. For a long period they used incoherent radars of varying assignment, measuring range, bearing and elevation. With the appearance of coherent techniques as a fourth parameter there began to appear radial speed. Finally, at present they predict the appearance of n-dimensional radar installations in which among the measured quantities there are included angular velocities, radial acceleration, characteristics of the form and fluctuations of the signal, and so forth.

Inasmuch as joint meters are known in radar practice, the theory of joint measurement, as in Chapter VI, is reasonably started from analysis of joint meters of assigned structure. This in brief is done in § 12.2. However, a whole series of questions, for instance, concerning interconnections of codings of measured coordinates in the signal, can be comprehended with difficulty on the basis of analysis alone, without finding adequate potential characteristics of this

interconnection. Therefore basic attention in this chapter is paid to questions of synthesis of joint meters, to which we devote §§ 12.3-12.6. We synthesize complete circuits of optimum joint meters (§ 12.3). First we take Gaussian distribution of all parameters, but make no concrete assumptions about the structure of input signals and correlation properties of the parameters. We consider also the method of synthesis with Markovian parameters. We separately study questions of synthesis of optimum discriminators (§ 12.4) and smoothing circuits (§ 12.5). We give several examples of construction of joint meters (§ 12.6). During selection of these examples the authors were guided by considerations of their applied interest and maximum detection of the specifics of joint measurement.

Theoretical propositions mentioned below are far from complete and can be considered the direction for new investigations. However, for instance, measurement of several coordinates of a single target can be considered on the basis of the theory mentioned below fairly rigorously and in detailed form.

§ 12.2. Analysis of Multi-Dimensional Tracking Meters

Multi-dimensional meters known in practice usually are constructed in the form of multi-dimensional tracking systems. In the literature there exists a whole series of works on tracking systems with a great number of tracked quantities (e.g., [54]). However, the specific character of radar meters, of which we talked in Chapter VI, forces us to examine from a new point of view questions of analysis of tracking systems in application to meters. A number of results of analysis considering the appearing peculiarities are given below.

12.2.1. Basic Features of Construction of Circuits and Components of Measuring Errors

Generally, a multi-dimensional tracking meter for l parameters can be presented by the block diagram of Fig. 12.1. In it there are marked two basic types of elements — discriminators and smoothing circuits with drives. Discriminators whose outputs are numbered from 1 to l detect mismatch between the input and output values of some parameters and are nonlinear radio devices for processing one or several input signals. A peculiarity of the partial discriminators in a multi-dimensional meter is the fact that to each of them there proceed measured values of all l coordinates, and not only that by which the error signal is detected. This is explained by the fact that radar coordinates are coded in the useful component of the input mixture in such a connected form that during processing of the signal by narrowly-selective devices, close to optimum, disturbance of selection with respect to any

parameter makes it impossible to detect information immediately on all parameters.

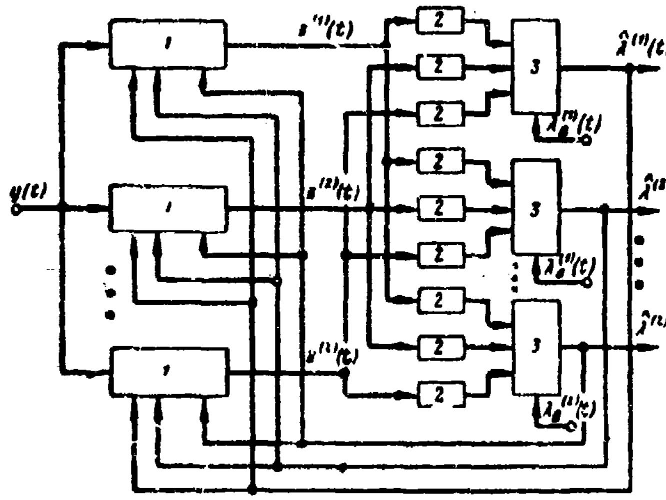

Fig. 12.1. Multi-dimensional meter: 1 — partial discriminators; 2 — linear smoothing circuits; 3 — adders.

Smoothing circuits carrying out accumulation of primary data are usually built from linear elements separately for all l coordinates. However, we also meet smoothing circuits with cross couplings. Most frequently these couplings are realized in the form of inertialess devices of recalculation of coordinates immediately at the output of the discriminator, but we can meet more complicated cases. In the general plan we have the matrix of smoothing circuits depicted in Fig. 12.1. The matrix has l inputs and l outputs, and coupling between any input and output is possible. Usually the matrix is linear, but it can also consist of inertial nonlinear elements. Nonlinear circuits have not yet received wide use in practice, and subsequently will not be studied.

Besides discriminators and smoothing circuits in meters there sometimes is provided introduction through adders of additional quantities, intended to compensate known components of the measured parameters. In order to have before oneself graphic examples of practical realization of joint meters on the part of the discriminators, it is sufficient to combine a set of circuits similar to those studied in Chapters VII-XII and pertaining to measurement of various coordinates. Here it is necessary to anticipate selection in all circuits with respect to all coordinates, and technically to combine identical elements which appear.

We introduce a system of designations similar to that adopted in the one-dimensional analysis, starting with the case of one input mixture:

$$y[t;\ \lambda^{(1)}(t), \ldots, \lambda^{(l)}(t)] = y[t;\ \boldsymbol{\lambda}(t)],$$

where $\{\lambda^{(1)}(t), \ldots, \lambda^{(l)}(t)\} = \boldsymbol{\lambda}(t)$ – set of measured parameters (coordinates).

At the output of each discriminator there will be formed a function of time (usually in the form of voltage)

$$z^{(\alpha)}[t;\ \boldsymbol{\varepsilon}(t)]\ (\alpha = 1, \ldots, l),$$

where $\boldsymbol{\varepsilon}(t) = \{\varepsilon^{(1)}(t), \ldots, \varepsilon^{(l)}(t)\}$ – current mismatches between $\boldsymbol{\lambda}(t)$ and measured values of parameters $\hat{\boldsymbol{\lambda}}(t) = \{\hat{\lambda}^{(1)}(t), \ldots, \hat{\lambda}^{(l)}(t)\}$, so that

$$\varepsilon^{(\alpha)}(t) = \lambda^{(\alpha)}(t) - \hat{\lambda}^{(\alpha)}(t)\ \ (\alpha = 1, \ldots, l).$$

The dependence in general of each of the output voltages of the discriminator on mismatch with respect to all parameters is a purely multi-dimensional effect. It is not worth explaining that, as in the one-dimensional case, values of $\boldsymbol{\lambda}(t)$ and $\boldsymbol{\varepsilon}(t)$ determine input and output voltage of discriminators only in the statistical sense: concrete realizations of $y[t;\ \boldsymbol{\lambda}(t)]$ and $z^{(\alpha)}[t;\ \boldsymbol{\varepsilon}(t)]$ are determined by input fluctuating disturbances.

The number of input mixtures m in general can be arbitrary. It usually is not connected with the number of measured quantities. In conditions of several mixtures recording of discriminator inputs alone is complicated, i.e., we consider set $\mathbf{y}(t) = \{{}^{(1)}y(t), \ldots, {}^{(m)}y(t)\}$. Nevertheless, the remaining designations as well as the method of analysis of meters are kept constant.

Let us stop to classify different components of errors of measurement. These components are basically the same as in the one-dimensional case. One of the components is fluctuating errors due to internal noises of the receiver, natural and organized interferences, and fluctuations of the reflected radar signal. A peculiarity of the multi-dimensional case is that similar errors detected at the output of different partial discriminators sometimes turn out to be correlated. The reason for this is not so much the common source of disturbances for the discriminators (corresponding component of the input mixture) as the similar method of encoding certain coordinates in the signal observed sometimes. A thorough understanding of this effect will be possible only on the basis of the findings of § 12.3.

In multi-dimensional meters there are also dynamic errors due to change of the measured quantities themselves and fluctuating and systematic errors introduced by equipment for processing of signals. However we shall consider only fluctuating and dynamic errors. With respect to the character of fluctuating disturbances,

which sometimes are conveniently related to immaterial randomly varying parameters of the signal, we shall limit ourselves to the assumption that all of them vary considerably faster than the measured variables.

12.2.2. Characteristic of Discriminators

In conditions of rapid random disturbances the output voltage of every discriminator $z^{(\alpha)}[t, \varepsilon(t)]$ can be presented in the form of two parts: the mean

$$a^{(\alpha)}(\varepsilon, t)=\overline{z^{(\alpha)}(t, \varepsilon)} \tag{12.2.1}$$

and white noise with (two-way) spectral density

$$S^{(\alpha\alpha)}(\varepsilon, t)= =\int_{-\infty}^{+\infty}\overline{[z^{(\alpha)}(t, \varepsilon)-\overline{z^{(\alpha)}(t, \varepsilon)}][z^{(\alpha)}(t+\tau, \varepsilon)-\overline{z^{(\alpha)}(t+\tau, \varepsilon)}]}\,d\tau, \tag{12.2.2}$$

equal to the spectral density at low frequencies of the fluctuating component of function $z^{(\alpha)}(t, \varepsilon)$. Averaging in (12.2.1) and (12.2.2) is produced over the complete ensemble of fluctuations, and quantities ε here are considered "frozen" in accordance with the assumption of slowness of their variation. The first peculiarity of the multi-dimensional case is the presence of functional dependence of $a^{(\alpha)}(\varepsilon, t)$ and $S^{(\alpha\alpha)}(\varepsilon, t)$ on all mismatches. We return to this circumstance below. A second peculiarity is the necessity of introduction besides (12.2.2) of characteristics of crosscorrelation coupling of fluctuating components at the output of different partial discriminators. Coupling between the α-th and β-th output voltages is characterized by mutual spectral density

$$S^{(\alpha\beta)}(\varepsilon, t)=S^{(\beta\alpha)}(\varepsilon, t)=\int_{-\infty}^{+\infty}\overline{[z^{(\alpha)}(t, \varepsilon)-\overline{z^{(\alpha)}(t, \varepsilon)}][z^{(\beta)}(t+\tau, \varepsilon)-\overline{z^{(\beta)}(t+\tau, \varepsilon)}]}\,d\tau, \tag{12.2.3}$$

which, depending upon character of the coupling, can be positive, negative or zero. The last case is observed in the absence of correlation coupling of the considered disturbances, by no means signifying their complete independence.

Thus, we have a square symmetric matrix $(l \times l)$ of characteristics of fluctuating disturbances, in which in all there are $l(l + 1)/2$ independent elements. The statistically equivalent form of recording output voltages of the discriminator is

$$z_{\varepsilon}^{(\alpha)}(t, \varepsilon)=a^{(\alpha)}(\varepsilon, t)+\sqrt{S^{(\alpha\alpha)}(\varepsilon, t)}\,\zeta^{(\alpha)}(t), \tag{12.2.4}$$

where $\zeta^{(\alpha)}(t)$ — white noises with unit spectral density, the interconnection of which is established by relationship

$$\int_{-\infty}^{+\infty}\overline{\zeta^{(\alpha)}(t)\zeta^{(\beta)}(t+\tau)}\,d\tau=\frac{S^{(\alpha\beta)}(\varepsilon, t)}{\sqrt{S^{(\alpha\alpha)}(\varepsilon, t)\,S^{(\beta\beta)}(\varepsilon, t)}}.$$

Functions $a^{(\alpha)}(\boldsymbol{\varepsilon}, t)$ we shall call discrimination characteristics, and $S^{(\alpha\beta)}(\boldsymbol{\varepsilon}, t)$ – fluctuation characteristics. We shall find them by means of analysis of passage of the signal and noises through a discriminator with fixed mismatches $\boldsymbol{\varepsilon}$.

The form of discrimination and fluctuation characteristics in an l-dimensional parameter space is very complicated. We shall first of all give a qualitative description of these dependences in that particular case when each α-th output of the discriminator on the average is determined by mismatch with respect to the α-th parameter, and all other mismatches have an effect only if their magnitude is great, determining several scale factors. Simultaneously we shall consider absent correlation couplings between noises at the output of the discriminators, i.e., matrix $\|S^{(\alpha\beta)}(\boldsymbol{\varepsilon}, t)\|$ is diagonal. In these conditions the α-th discrimination characteristic for fixed $\varepsilon^{(\beta)}(\beta \neq \alpha)$ is described by an odd function of $\varepsilon^{(\alpha)}$ of the same form as in the one-dimensional case. Values of all other mismatches basically affect the scale of the curve. The scale is maximum at $\varepsilon^{(\beta)} = 0$ $(\beta \neq \alpha)$ and varies little for small $|\varepsilon^{(\beta)}|$. For large values of $|\varepsilon^{(\beta)}|$, $a^{(\alpha)}(\boldsymbol{\varepsilon}, t)$ asymptotically seeks zero independently of the value of $\varepsilon^{(\alpha)}$. For the case of two parameters the first discrimination characteristic in plane $(\varepsilon^{(1)}, \varepsilon^{(2)})$ is graphically presented in Fig. 12.2.

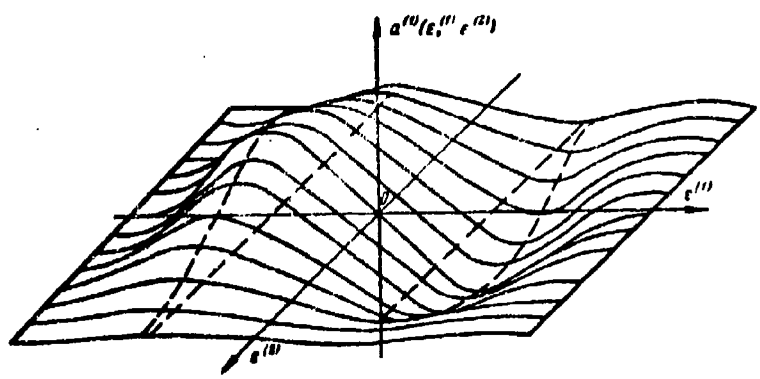

Fig. 12.2. Discrimination characteristic of a two-dimensional meter.

Fluctuation characteristic $S^{(\alpha\alpha)}(\boldsymbol{\varepsilon}, t)$ in the same conditions with fixed $\varepsilon^{(\beta)}$ $(\beta \neq \alpha)$ is described by a symmetric curve of a form which is known from the one-dimensional case. With growth of other mismatches the double-humped structure of the considered sections is smoothed and $S^{(\alpha\alpha)}(\boldsymbol{\varepsilon}, t)$ for any values of $\varepsilon^{(\alpha)}$ changes into a hyperplane, parallel to the coordinate hyperplane $\boldsymbol{\varepsilon}$. Fluctuation characteristic $S^{11}(\varepsilon^{(1)}, \varepsilon^{(2)})$ on plane $(\varepsilon^{(1)}, \varepsilon^{(2)})$ is shown in Fig. 12.3.

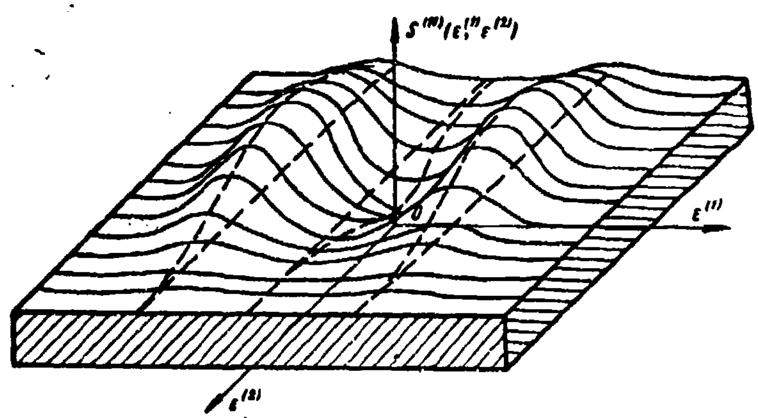

Fig. 12.3. Fluctuation characteristic of a two-dimensional meter.

The illustration helps us to comprehend the law that increase of mismatch $\varepsilon^{(\beta)}$ $(\beta \neq \alpha)$ leads to the same change of the discrimination $a^{(\alpha)}(\boldsymbol{\varepsilon})$ and fluctuation $S^{(\alpha\alpha)}(\boldsymbol{\varepsilon})$ characteristics, constructed as a function of $\varepsilon^{(\alpha)}$, as decrease of the signal-to-noise ratio. This is understandable inasmuch as increase of mismatch for other coordinates leads to power losses, fully equivalent to decrease of the mean input signal level.

When all mismatches in the considered particular case are small, instead of general dependences (12.2.4) it is permissible to use expansions of characteristics of discriminators

$$a^{(\alpha)}(\boldsymbol{\varepsilon}) = a_0^{(\alpha)} + K_{\text{Д}}^{(\alpha)}\varepsilon^{(\alpha)} + \sum_{\beta,\gamma} Q^{(\beta\gamma)}\varepsilon^{(\beta)}\varepsilon^{(\gamma)} + \ldots,$$
$$S^{(\alpha\alpha)}(\boldsymbol{\varepsilon}) = S_0^{(\alpha\alpha)} + S_1^{(\alpha\alpha)}\varepsilon^{(\alpha)} + S_2^{(\alpha\alpha)}(\varepsilon^{(\alpha)})^2 + \ldots,$$

where

$$a_0^{(\alpha)} = a^{(\alpha)}(0);\quad K_{\text{Д}}^{(\alpha)} = \frac{\partial a^{(\alpha)}(0)}{\partial \varepsilon^{(\alpha)}};\quad Q^{(\beta\gamma)} = \frac{1}{2}\,\frac{\partial^2 a^{(\alpha)}(0)}{\partial \varepsilon^{(\beta)}\partial \varepsilon^{(\gamma)}};$$
$$S_0^{(\alpha\alpha)} = S^{(\alpha\alpha)}(0);\quad S_1^{(\alpha\alpha)} = \frac{\partial S^{(\alpha\alpha)}(0)}{\partial \varepsilon^{(\alpha)}};\quad S_2^{(\alpha\alpha)} = \frac{1}{2}\,\frac{\partial^2 S^{(\alpha\alpha)}(0)}{\partial \varepsilon^{(\alpha)2}}.$$

In conditions of symmetry of the circuit $a_0^{(\alpha)}$, $S_1^{(\alpha\alpha)}$ are usually equal to zero. Quantities $K_{\text{Д}}^{(\alpha)}$ are called gain factors of the discriminators. Coefficients $Q^{(\beta\gamma)}$ show the influence of other mismatches, in certain approximation leading to decrease of the scale factor. During investigation of accuracy of measurements by virtue of the smallness of mismatches it is often sufficient to use formulas

$$a^{(\alpha)}(\boldsymbol{\varepsilon}) \approx K_{\text{Д}}^{(\alpha)}\varepsilon^{(\alpha)},$$
$$S^{(\alpha\alpha)}(\boldsymbol{\varepsilon}) \approx S_0^{(\alpha\alpha)} + S_2^{(\alpha\alpha)}\varepsilon^{(\alpha)2}, \qquad (12.2.5)$$

which are a simple repetition of one-dimensional (6.2.9).

Thus, in the case of small errors, when linear approximations are valid, the separate analysis of meters of separate parameters given in preceding chapters is fully permissible.

An example of a multi-dimensional meter which satisfies the formulated conditions is a joint meter of range and speed in a coherent radar. If, for instance, we are interested in the range discriminator, it contains narrow-band filters (Figs. 7.7, 7.13) which are tuned from the output of the speed channel. With small errors of measurement of speed the signal frequency is within the limits of the frequency response of the filter.

In a number of cases the described dependences of $a^{(\alpha)}(\varepsilon)$ and $S^{(\alpha\beta)}(\varepsilon)$ on ε are invalid. Roughly speaking, this distinction consists of zero shift of the α-th discriminator in the presence of mismatch for the β-th parameter ($\beta \neq \alpha$). Changes of dependences appear also with large mismatches. Inasmuch as general description, even qualitative, is hampered here, we shall only discuss the case of small mismatches. The generalization of formulas (12.2.5) then will be

$$a^{(\alpha)}(\varepsilon) = a_0^{(\alpha)} + \sum_{\beta=1}^{l} K_{\text{Д}}^{(\alpha\beta)} \varepsilon^{(\beta)}, \tag{12.2.6}$$

$$S^{(\alpha\beta)}(\varepsilon) = S_0^{(\alpha\beta)} + \sum_{\gamma=1}^{l} S_{\gamma}^{(\alpha\beta)} \varepsilon^{(\gamma)} + \sum_{\gamma,\delta=1}^{l} S_{\gamma\delta}^{(\alpha\beta)} \varepsilon^{(\gamma)} \varepsilon^{(\delta)}, \tag{12.2.7}$$

where

$$\left.\begin{aligned} & a_0^{(\alpha)} = a^{(\alpha)}(0);\ K_{\text{Д}}^{(\alpha\beta)} = \frac{\partial a^{(\alpha)}(0)}{\partial \varepsilon^{(\beta)}}; \\ & S_0^{(\alpha\beta)} = S^{(\alpha\beta)}(0);\ S_{\gamma}^{(\alpha\beta)} = \frac{\partial S^{(\alpha\beta)}(0)}{\partial \varepsilon^{(\gamma)}};\ S_{\gamma\delta}^{(\alpha\beta)} = \frac{1}{2} \frac{\partial^2 S^{(\alpha\beta)}(0)}{\partial \varepsilon^{(\gamma)} \partial \varepsilon^{(\delta)}}. \end{aligned}\right\} \tag{12.2.8}$$

In conditions of symmetry of circuits of the partial discriminators, $a_0^{(\alpha)}$ and $S_{\gamma}^{(\alpha\beta)}$ disappear, and there remain these simpler dependences:

$$a^{(\alpha)}(\varepsilon) = \sum_{\beta=1}^{l} K_{\text{Д}}^{(\alpha\beta)} \varepsilon^{(\beta)}, \tag{12.2.9}$$

$$S^{(\alpha\beta)}(\varepsilon) = S_0^{(\alpha\beta)} + \sum_{\gamma,\delta=1}^{l} S_{\gamma\delta}^{(\alpha\beta)} \varepsilon^{(\gamma)} \varepsilon^{(\delta)}. \tag{12.2.10}$$

Coefficient $K_{\text{Д}}^{(\alpha\beta)}$ we call the gain factor of the α-th discriminator for the β-th parameter. Quantities $S_0^{(\alpha\beta)}$ characterize fluctuating components not depending on mismatch, and $S_{\gamma\delta}^{(\alpha\beta)}$ – components proportional to $\varepsilon^{(\gamma)}$ ($\gamma = 1, \ldots, l$). Formulas (12.2.9) and (12.2.10) correspond to the following presentation of output voltage of discriminators:

$$z_э^{(\alpha)}(t, \varepsilon) = \sum_{\beta=1}^{l} K_д^{(\alpha\beta)} \eta^{(\beta)}(t) + \sum_{\alpha,\beta=1}^{l} [K_д^{(\alpha\beta)} + \varkappa^{(\alpha\beta)}(t)] \varepsilon^{(\beta)}(t), \tag{12.2.11}$$

where $\eta^{(\beta)}$ and $\varkappa^{(\alpha\beta)}(t)$ $(\alpha, \beta = 1, \ldots, l)$ — random, uncorrelated, functions; $\eta^{(\beta)}(t)$ — white noises having the dimensionality of the measured parameters with a matrix of spectral densities $\mathbf{S}_{ЭКВ}$, determined by recalculation of matrix $\mathbf{S}_0 = \|S_0^{(\alpha\beta)}\|$ with the help of the matrix of gain factors of discriminators $\mathbf{K}_д = \|K_д^{(\alpha\beta)}\|$ by the formula

$$\mathbf{S}_{экв} = \| S_\eta^{(\alpha\beta)} \| = \mathbf{K}_д^{-1} \mathbf{S}_0 \mathbf{K}_д^{-1+}. \tag{12.2.12}$$

For proof of formula (12.2.12) it is sufficient to multiply vector $\mathbf{K}_д\boldsymbol{\eta}$, representing noises at the discriminator output, by row $(\mathbf{K}_д\boldsymbol{\eta})^+$, average, and, passing to spectra, use the designation for the spectral matrix of vector $\boldsymbol{\eta}$:

$$\mathbf{S}_0 = \int \overline{\mathbf{K}_д \boldsymbol{\eta}(t) (\mathbf{K}_д \boldsymbol{\eta}(t+\tau))^+} d\tau = \mathbf{K}_д \int \overline{\boldsymbol{\eta}(t) \boldsymbol{\eta}^+(t+\tau)} d\tau \mathbf{K}_д^+ = \mathbf{K}_д \mathbf{S}_{экв} \mathbf{K}_д^+.$$

From this (12.2.12) is obtained by two matrix multiplications. Subsequently we shall call $\mathbf{S}_{ЭКВ}$ the matrix of equivalent spectral densities of a multi-dimensional discriminator.

Functions $\varkappa^{(\alpha\beta)}(t)$ in (12.2.11) are white noises with mutual spectral densities

$$\| S_{\gamma\delta}^{(\alpha\beta)} \| = \| \int_{-\infty}^{+\infty} \overline{\varkappa^{(\alpha\gamma)}(t) \varkappa^{(\beta\delta)}(t+\tau)} d\tau \|. \tag{12.2.13}$$

Subsequently we will not consider the influence of parametric fluctuation $\varkappa^{(\gamma\delta)}(t)$ on accuracy of measurement, concentrating our attention on noises $\eta^{(\alpha)}(t)$. Their level, naturally, is determined by the intensity of interfering signals at the input, not carrying information about the measured quantities. Basically, for $\eta^{(\alpha)}(t)$ there are observed the same laws as in the one-dimensional case.

For illustration of the method of using the obtained relationships we shall consider the fairly simple example* of a discriminator for two parameters with interconnected encoding in the signal.

Let us assume that the signal has a pulse structure with fluctuations of amplitude E and phase φ, independent between periods. It is received against a background of white noise $n(t)$. Pulses have intrapulse linear frequency modulation (LChM). The realization of the received mixture of the signal with noise in one period has the form

*Example belongs to G. F. Bugachev.

$$y(t) = \mathrm{Re}\,[E\, e^{i\varphi} u_a(t-\tau)\, e^{iV(t-\tau)^2/2 + i(\omega_0+\omega_Д)t}] + n(t),$$

where V – slope of LChM;

τ – delay of regular modulation;

$\omega_Д$ – Doppler frequency shift.

The last two quantities are also the measured parameters.

The circuit of a joint discriminator of quantities τ and $\omega_Д$ is depicted in Fig. 12.4. Signal $y(t)$ is fed to mixer 1, where convolution of spectrum LChM is effected by means of multiplication by the voltage of heterodyne 2

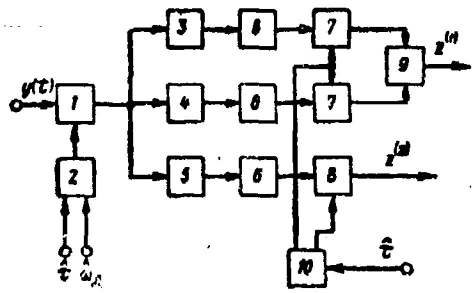

Fig. 12.4. Discriminator of $\omega_Д$ and τ: 1 – mixer; 2 – controlled heterodyne oscillator; 3, 4 – bandpass filters tuned to frequencies $\omega_1 \pm \delta$; 5 – bandpass filter tuned to frequency ω_1; 6 – amplitude detectors; 7 – gated amplifiers; 8 – time discriminator; 9 – comparison circuit; 10 – generator of gate pulses.

$$\mathrm{Re}\,[e^{iV(t-\hat{\tau})^2/2 + i(\omega_0+\omega_1+\hat{\omega}_Д)t}],$$

where $\hat{\tau}$, $\hat{\omega}_Д$ – measured values of parameters.

Simultaneously there is produced transfer of oscillations to a frequency close to intermediate ω_1. Then there is produced bandpass filtration in filters 3, 4, 5, of which filter 5 is exactly tuned to ω_1, and filters 3, 4 have detuning of $\mp\delta$. Results of filtration are detected in detectors 6. In the upper two channels of Fig. 12.4 in elements 7 there is produced gating; the middle of the gate pulse coincides with $\hat{\tau}$. The difference of the two formed voltages, formed in element 9, is the first output voltage of the joint discriminator $z^{(1)}$. This voltage is sensitive to frequency shift of voltage at the output of the mixer.

In the lower channel of Fig. 12.4 after the detector there is time discriminator 8. Its action reduces to multiplication by odd function $l(t)$, whose zero also coincides with $\hat{\tau}$, and subsequent accumulation (integration) of signals.

Consequently, there will be formed the second output voltage $z^{(2)}$, sensitive primarily to time delay τ.

According to the above results for determination of matrix $\Sigma_{ЭКВ}$ of errors of measurement of τ and $\omega_Д$ with respect to a unit pulse input it is necessary to calculate the matrix of gain factors of the discriminator

$$K_Д = \left\| \begin{matrix} \frac{\partial \overline{z^{(1)}}}{\partial \Delta_\omega} & \frac{\partial \overline{z^{(1)}}}{\partial \Delta_\tau} \\ \frac{\partial \overline{z^{(2)}}}{\partial \Delta_\omega} & \frac{\partial \overline{z^{(2)}}}{\partial \Delta_\tau} \end{matrix} \right\|_{\substack{\Delta_\tau = 0 \\ \Delta_\omega = 0}} \tag{12.2.14}$$

where $\Delta_\omega = \omega_Д - \hat{\omega}_Д$, $\Delta_\tau = \tau - \hat{\tau}$, and the matrix of mean squares of fluctuating components of outputs is

$$\Sigma = \left\| \begin{matrix} \overline{(z^{(1)})^2} & \overline{z^{(1)}z^{(2)}} \\ \overline{z^{(1)}z^{(2)}} & \overline{(z^{(2)})^2} \end{matrix} \right\|_{\substack{\Delta_\tau = 0 \\ \Delta_\omega = 0}}. \tag{12.2.15}$$

Analogously to (12.2.12) matrix Σ_{eq} is equal to

$$\Sigma_{экв} = K_Д^{-1} \Sigma K_Д^{+-1}. \tag{12.2.16}$$

Passing to calculation, for simplicity we assume that the bandpass filters are matched with amplitude modulation, i.e., their pulse response envelope is the inversion in time of function $u_a(t)$, that amplitude detectors 6 are square-law, and that amplifiers 7 carry out ultranarrow gating. With some selection of overall gain factors in channels, which do not affect final results, we have:

$$\frac{\partial \overline{z^{(1)}}}{\partial \Delta_\omega}\bigg|_{\substack{\Delta_\tau = 0 \\ \Delta_\omega = 0}} = \mathcal{E}\,\mathrm{Re}\left\{ i \int t_1 u_a^2(t_1)\, e^{-i\dot{\Omega} t_1}\, dt_1 \int u_a^2(t_2)\, e^{-i\dot{\Omega} t_2}\, dt_2 \right\},$$

$$\frac{\partial \overline{z^{(1)}}}{\partial \Delta_\tau}\bigg|_{\substack{\Delta_\tau = 0 \\ \Delta_\omega = 0}} = \mathcal{E}\,\mathrm{Re}\left\{ \left[\int u_a'(t)\, u_a(t)\, e^{-i\dot{\Omega} t}\, dt + iV \int t u_a^2(t)\, e^{-i\dot{\Omega} t}\, dt \right] \int u_a^2(\tau)\, e^{i\dot{\Omega}\tau}\, d\tau \right\},$$

$$\frac{\partial \overline{z^{(2)}}}{\partial \Delta_\omega}\bigg|_{\substack{\Delta_\tau = 0 \\ \Delta_\omega = 0}} = 0,$$

$$\frac{\partial \overline{z^{(2)}}}{\partial \Delta_\tau}\bigg|_{\substack{\Delta_\tau = 0 \\ \Delta_\omega = 0}} = \frac{\mathcal{E}}{2} \int l(t) \left[\int u_a'(\tau_1)\, u_a(\tau_1 - t)\, d\tau_1 \times \int u_a(\tau_2)\, u_a(\tau_2 - t)\, d\tau_2 \right] dt, \tag{12.2.17}$$

where $\mathcal{E}$ — mean energy of a sending.

From formula (12.2.17) it is clear that the matrix of gain factors is nondiagonal; mismatch for τ causes the appearance of a constant component of both output voltage of the discriminator. This circumstance is easily explained. In fact, the heterodyned oscillation carries information about time shift of regular modulation both in the amplitude envelope and also in the frequency.

It is interesting that matrix Σ turns out to be diagonal:

$$\overline{(z^{(1)})^2} = N_0^2 \left\{ q \left[\int u_a^2(t)\, e^{-i\dot{\Omega} t}\, dt \right]^2 \left[1 - \mathrm{Re} \int u_a^2(t)\, e^{2i\dot{\Omega} t}\, dt \right] + \frac{1}{2} \left\{ 1 - \left[u_a^2(t)\, e^{2i\dot{\Omega} t}\, dt \right]^2 \right\} \right\},$$

$$\overline{(z^{(2)})^2} = \frac{N_0}{2} \left\{ q \iint l(t_1)\, l(t_2) \left[\int u_a(\tau - t_1)\, u_a(\tau - t_2)\, d\tau \right] dt_1 dt_2 + \right.$$

$$\left. + \frac{1}{2} \iint l(t_1)\, l(t_2) \left[\int u_a(\tau - t_1)\, u_a(\tau - t_2)\, d\tau \right]^2 dt_1 dt_2 \right\}, \tag{12.2.18}$$

$$\overline{z^{(1)}z^{(2)}} = 0,$$

where N_0 — spectral density of noise;

$q = \frac{\mathscr{E}}{2N_0}$ — signal-to-noise ratio in one period (see Chapter VI).

Now let us take Gaussian approximations for $u_a(t)$ and $l(t)$:

$$u_a(t) = c_1 \exp\left\{-\frac{\pi}{2}(t/\tau_и)^2\right\},$$

$$l(t) = c_2 t \exp\left\{-\frac{\pi}{2}(t/\tau_c)^2\right\},$$

where c_1, c_2 — constants;

$\tau_и$ — pulse duration;

τ_c — equivalent duration of gates of the time discriminator.

After we integrate according to formulas (12.2.17) and (12.2.18) and carry out operations (12.2.16) on relationships (12.2.14) and (12.2.15) elements of matrix $\Sigma_{экв}$ will be quantities:

$$\left.\begin{aligned}
\Sigma^{(\tau\tau)}_{экв} &= \frac{\tau_и^2}{\pi}(1+m^2)^{3/2}\left[\frac{1}{q}+\frac{1}{q^2}\left(\frac{1+m^2}{1+2m^2}\right)^{3/2}\right],\\
\Sigma^{(\tau\omega)}_{экв} &\equiv \Sigma^{(\omega\tau)}_{экв} = V\Sigma^{(\tau\tau)}_{экв},\\
\Sigma^{(\omega\omega)}_{экв} &= V^2\Sigma^{(\tau\tau)}_{экв} + \frac{\pi}{2\tau_c^2}\left(\frac{\operatorname{sh} n^2}{n^2}\right)\left[\frac{1}{q}+\frac{1}{q^2}\frac{\operatorname{sh} 2n^2}{2\operatorname{sh} n^2}\right],
\end{aligned}\right\} \qquad (12.2.19)$$

where

$$m^2 = (\tau_c/\tau_и)^2; \quad n^2 = (\tau_и\delta)^2/2\pi.$$

Note that matrix $\Sigma_{экв}$ in distinction from Σ is not diagonal, which is a consequence of the interconnection of encoding of parameters.

Elements (12.2.19) depend on the signal-to-noise ratio q, pulse duration τ_0, the slope of LChM V and two parameters of the discriminator, m and n. It is easy to prove that $\Sigma_{экв}$ becomes diagonal only in the absence of frequency modulation. Remember that according to Chapter I the principal axes of the ellipsoid, which is the section of the autocorrelation function of signal, as $V \to 0$ are oriented along axes of coordinates τ and ω. This is a graphic expression of the absence of interdependence between the parameters of time delay and frequency shift of the input signal.

12.2.3. Accuracy of Measurement

During the analysis of the work of multi-dimensional tracking meters there arise the same problems as in the one-dimensional case. In general it is necessary to consider a nonlinear problem. This, in principle, is possible on the basis of the technique of the Fokker-Planck equation, the multi-dimensional generalization of which is known in the literature [20]. If, however, the level of input noises is

low, during analysis of accuracy linearization of the meter is permissible, which we shall realize assuming smallness of parametric fluctuations. We assume that the matrix of smoothing circuits is linear, but not necessarily with constant parameters, so that its input and output quantities are connected by matrix relationship

$$\hat{\lambda}(t)=\int_{t_0}^{t}\mathbf{h}(t,\ \tau)\,\mathbf{z}(\tau)\,d\tau+\lambda_{\text{в}}'(t), \tag{12.2.20}$$

where $\mathbf{h}(t,\ \tau) = \|h^{(\alpha\beta)}(t,\ \tau)\|$ – matrix of pulse responses of smoothing circuits;

$\boldsymbol{\lambda}_{\text{B}}(t)$ – column vector of functions introduced for compensations of known components of $\boldsymbol{\lambda}(t)$.

Substituting in (12.2.20) according to (12.2.11) for $\mathbf{z}(t)$ the column vector

$$\mathbf{z}(t)=\mathbf{K}_{\text{д}}\boldsymbol{\varepsilon}(t)+\mathbf{K}_{\text{д}}\boldsymbol{\eta}(t) \tag{12.2.21}$$

and considering that $\hat{\boldsymbol{\lambda}}(t) = \boldsymbol{\lambda}(t) - \boldsymbol{\varepsilon}(t)$, we have equation

$$\boldsymbol{\varepsilon}(t)+\int_{t_0}^{t}\mathbf{h}(t,\ \tau)\,\mathbf{K}_{\text{д}}\boldsymbol{\varepsilon}(\tau)\,d\tau=\lambda(t)-\lambda_{\text{в}}(t)-\int_{t_0}^{t}\mathbf{h}(t,\ \tau)\,\mathbf{K}_{\text{д}}\boldsymbol{\eta}(\tau)\,d\tau. \tag{12.2.22}$$

If we introduce the matrix of pulse responses $\mathbf{g}(t,\ \tau)$ of the closed system, considering the output quantity vector $\hat{\boldsymbol{\lambda}}(t)$, it is determined by integral equation

$$\mathbf{g}(t,\ \tau)+\int_{\tau}^{t}\mathbf{h}(t,\ s)\,\mathbf{K}_{\text{д}}\mathbf{g}(s,\ \tau)\,ds=\mathbf{h}(t,\ \tau)\,\mathbf{K}_{\text{д}}. \tag{12.2.23}$$

We introduce additionally the pulse response of the system $\mathbf{v}(t,\ \tau)$, considering mismatches the output quantities. It satisfies equation

$$\mathbf{v}(t,\ \tau)+\int_{\tau}^{h}\mathbf{h}(t,\ s)\,\mathbf{K}_{\text{д}}\mathbf{v}(s,\ \tau)\,ds=\mathbf{I}\delta(t-\tau). \tag{12.2.24}$$

It is easy to prove that the solution of equation (12.2.22) will be expressed in the form

$$\begin{aligned}\boldsymbol{\varepsilon}(t)=&-\int_{t_0}^{t}\mathbf{g}(t,\ \tau)\,\boldsymbol{\eta}(\tau)\,d\tau+\int_{t_0}^{t}\mathbf{v}(t,\ \tau)\,[\lambda(\tau)-\overline{\lambda(\tau)}]\,d\tau+\\&+\int_{t_0}^{t}\mathbf{v}(t,\ \tau)\,[\overline{\lambda}(\tau)-\lambda_{\text{в}}(\tau)]\,d\tau=\boldsymbol{\varepsilon}_{\text{фл}}(t)+\boldsymbol{\varepsilon}_{\text{дин}1}(t)+\boldsymbol{\varepsilon}_{\text{дин}2}(t).\end{aligned} \tag{12.2.25}$$

According to (12.2.25) current error is determined by the joint action of interferences $\boldsymbol{\eta}(t)$ passed through matrix of filters $\mathbf{g}(t,\ \tau)$ and of random variation of parameters $\boldsymbol{\lambda}(t) - \overline{\boldsymbol{\lambda}(t)}$ and uncompensated regular measurements $\overline{\boldsymbol{\lambda}(t)} - \boldsymbol{\lambda}_{\text{B}}(t)$ passed through the matrix of filters $\mathbf{v}(t,\ \tau)$. The first component in (12.2.25) naturally is called the set (vector) of fluctuating errors, and to components $\boldsymbol{\varepsilon}_{\text{ДИН }1}(t)$ and $\boldsymbol{\varepsilon}_{\text{ДИН }2}(t)$ we give the name vectors of dynamic errors.

The matrix of second moments of total errors of measurement is equal to

$$\Sigma_{\text{вых}}(t)=\overline{\varepsilon(t)\varepsilon^{+}(t)}=\int_{t_0}^{t}\mathbf{g}(t,\ s)\,\mathbf{S}_{\text{экв}}\mathbf{g}^{+}(t,\ s)\,ds+\int_{t_0}^{t}\int_{t_0}^{t}\mathbf{v}(t,\ s_1)\,\mathbf{R}_{\lambda}(s_1,\ s_2)\,\mathbf{v}^{+}(t,\ s_2)\,ds_1ds_2+$$
$$+\int_{t_0}^{t}\int_{t_0}^{t}\mathbf{v}(t,\ s_1)\,\Delta(s_1)\,\Delta^{+}(s_2)\,\mathbf{v}^{+}(t,\ s_2)\,ds_1ds_2, \qquad (12.2.26)$$

where we use the property of δ-correlation of interferences $\eta(t)$, ela- tion matrix $\mathbf{R}_{\lambda}(t_1,\ t_2)$ of the random part of the parameters and des $(t) = = \lambda(t) - \lambda_{B}(t)$. Certain simplifications of expression (12.2.26) are possible upon concretization of the form of smoothing circuits and the nature of change of $\lambda(t)$.

Let us assume, for instance, that random components of parameters are stationary $[\mathbf{R}_{\lambda}(t_1,\ t_2) = \mathbf{R}_{\lambda}(t_1 - t_2)]$, regular components are completely compensated, and smoothing circuits have constant parameters, i.e., $\mathbf{h}(t,\ \tau) = \mathbf{h}(t - \tau)$. The Fourier transforms from functions $\mathbf{g}(t - \tau)$ and $\mathbf{v}(t - \tau)$ by (12.2.23) and (12.2.24) are easily expressed in this case through matrices of gain factors of the discriminator $\mathbf{K}_{Д}$ and frequency responses of the smoothing circuits

$$\mathbf{H}(i\omega)=\int_{0}^{\infty}\mathbf{h}(\tau)\,e^{-i\omega\tau}\,d\tau$$

in the form

$$\mathbf{G}(i\omega)=[\mathbf{I}+\mathbf{H}(i\omega)\,\mathbf{K}_{Д}]^{-1}\,\mathbf{H}(i\omega)\,\mathbf{K}_{Д},$$
$$\mathbf{V}(i\omega)=[\mathbf{I}+\mathbf{H}(i\omega)\,\mathbf{K}_{Д}]^{-1}. \qquad (12.2.27)$$

Thus, instead of (12.2.26) we can obtain

$$\Sigma_{\text{вых}}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}[\mathbf{I}+\mathbf{H}^{*}(i\omega)\,\mathbf{K}_{Д}]^{-1}\,\mathbf{H}^{*}(i\omega)\mathbf{K}_{Д}\mathbf{S}_{\text{экв}}\mathbf{K}_{Д}^{+}\mathbf{H}(i\omega)\,[\mathbf{I}+$$
$$+\mathbf{H}(i\omega)\,\mathbf{K}_{Д}]^{-1+}\,d\omega+\frac{1}{2\pi}\int_{-\infty}^{+\infty}[\mathbf{I}+\mathbf{H}^{*}(i\omega)\,\mathbf{K}_{Д}]^{-1}\,\mathbf{S}_{\lambda}(\omega)\times[\mathbf{I}+\mathbf{H}(i\omega)\,\mathbf{K}_{Д}]^{-1+}\,d\omega, \qquad (12.2.28)$$

where $\mathbf{S}_{\lambda}(\omega)=\int_{-\infty}^{+\infty}\mathbf{R}_{\lambda}(t)\,e^{-i\omega t}dt$ -- spectral matrix of the parameters.

Studying the first component in (12.2.28), we prove that the matrix of moments of fluctuating errors, in general, cannot be expressed through the product of the matrix of spectral densities and a certain constant matrix characterizing the generalized effective passband of the system. However, we may encounter a case when we are interested only in the sum of diagonal terms of this matrix, i.e., its trace.*

*For this errors of measurement of parameters reduce to a certain unit measure, expressed in linear units of dispersion of the measured position of the object near the true position.

Then according to (12.2.28)

$$\mathrm{spur}\,\Sigma_{\text{вых}} = \mathrm{spur}\,S_{\text{экв}}\Delta + \mathrm{spur}\,\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{\lambda}(\omega)\,M(\omega)\,d\omega, \tag{12.2.29}$$

where

$$M(\omega) = \{|I + H(i\omega)K_{д}|^{2}\}^{-1},$$

and

$$\Delta = \frac{1}{2\pi}\int_{-\infty}^{+\infty} K_{д}^{+} H^{+}(i\omega)\{|I + H(i\omega)K_{д}|^{2}\}^{-1} H^{*}(i\omega)K_{д}\,d\omega \tag{12.2.30}$$

is a constant matrix which is the multi-dimensional generalization of the effective passband of the system.

Certain more complicated forms of smoothing circuits will be studied in subsequent sections, devoted to questions of synthesis.

§ 12.3. Synthesis of Multi-Dimensional Meters of Several Parameters

In Chapters VI-XI we considered questions of synthesis of meters of only one parameter. Inasmuch as radars usually measure a whole set of coordinates (or parameters of motion) of the target, there arises the question to what degree results obtained on the assumption that all other parameters besides the measured one are known can be transfered to the case of simultaneous measurement of several unknown quantities. In order to answer this question it is necessary to solve the more complicated problem of simultaneous filtration of several process-parameters, randomly varying in time, from their nonadditive mixture with interferences and noises. If the conceptual side, concerning selection of the method of synthesis, remains the same here as in the one-dimensional case (see § 6.5), concrete methods of discovering the form of the optimum operator of filtration require additional study, since they are not a simple repetition of one-dimensional methods.

12.3.1. Method of Synthesis of an Optimum Meter

First of all we shall definitize the formulation of the problem of optimum filtration of several parameters. Consider a set of m random signals ${}^{(1)}y(t)$, ..., ${}^{(m)}y(t)$, all or part of which depend on l generally interconnected, and randomly varying in time, parameters $\lambda^{(1)}(t)$, ..., $\lambda^{(l)}(t)$. We assume that at moments $t_1 < t_2 < \ldots < t_n$ of interval (t_0, t), accessible to observation, signals and parameters take values

$$\{{}^{(1)}y_1,\ {}^{(2)}y_1, \ldots, {}^{(m)}y_1; \ldots; {}^{(1)}y_n,\ {}^{(2)}y_n, \ldots, {}^{(m)}y_n\} = \mathbf{y},$$
$$\{\lambda_1^{(1)},\ \lambda_1^{(2)}, \ldots, \lambda_1^{(l)}; \ldots; \lambda_n^{(1)},\ \lambda_n^{(2)}, \ldots, \lambda_n^{(l)}\} = \boldsymbol{\lambda},$$

where by $\mathbf{y}$ and $\boldsymbol{\lambda}$ we understand block column vectors.

The joint density of distribution of $\mathbf{y}$ and $\boldsymbol{\lambda}$ has the form $P_0(\boldsymbol{\lambda})\,P(\mathbf{y}|\boldsymbol{\lambda})$, where $P_0(\boldsymbol{\lambda})$ — multi-dimensional (for moments of observation and parameters) a priori distribution, and $P(\mathbf{y}|\boldsymbol{\lambda})$ — multi-dimensional (for moments of observation, signals and parameters) likelihood function, which we shall discuss later. The task of optimum filtration is construction of the column vector of estimate $\hat{\lambda}$, in some sense the closest to the combination of true values of the parameters. As the loss function, characterizing the degree of this proximity, we first consider the quadratic form of

$$l(\boldsymbol{\lambda},\ \hat{\boldsymbol{\lambda}})=(\boldsymbol{\lambda}-\hat{\boldsymbol{\lambda}})^{+}\mathbf{B}(\boldsymbol{\lambda}-\hat{\boldsymbol{\lambda}})=\sum_{i,j=1}^{n}(\boldsymbol{\lambda}_i-\hat{\boldsymbol{\lambda}}_i)^{+}\mathbf{B}_{ij}(\boldsymbol{\lambda}_j-\hat{\boldsymbol{\lambda}}_j)=$$

$$=\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{l}B_{ij}^{(\alpha\beta)}(\lambda_i^{(\alpha)}-\hat{\lambda}_i^{(\alpha)})(\lambda_j^{(\beta)}-\hat{\lambda}_j^{(\beta)}), \qquad (12.3.1)$$

where the order of co-factors in the second and third elements of the chain of equalities corresponds to operations of matrix multiplication; sign + (raised) signifies transposition; $\mathbf{B} = \|\mathbf{B}_{ij}\|$ — block matrix with submatrix elements $\mathbf{B}_{ij} = \|B_{ij}^{(\alpha\beta)}\|$, determining the value of errors of the α-th and β-th parameters, allowed in the i-th and j-th moments, respectively. Mean risk

$$R(P_0,\ \hat{\boldsymbol{\lambda}})=\iint(\boldsymbol{\lambda}-\hat{\boldsymbol{\lambda}})^{+}\mathbf{B}(\boldsymbol{\lambda}-\hat{\boldsymbol{\lambda}})\,P(\mathbf{y}\,|\,\boldsymbol{\lambda})\,P_0(\boldsymbol{\lambda})\,d\mathbf{y}\,d\boldsymbol{\lambda} \qquad (12.3.2)$$

here is a linear combination of second moments of errors of measurement of all parameters in all moments of time. Varying $R(P_0,\ \hat{\lambda})$ for estimates $\hat{\boldsymbol{\lambda}}$, we can prove that for any non-singular matrices B the optimum estimate, as in the one-dimensional case, is the conditional mathematical expectation

$$\hat{\boldsymbol{\lambda}}=\frac{\int\boldsymbol{\lambda}P(\mathbf{y}\,|\,\boldsymbol{\lambda})\,P_0(\boldsymbol{\lambda})\,d\boldsymbol{\lambda}}{\int P(\mathbf{y}\,|\,\boldsymbol{\lambda})\,P_0(\boldsymbol{\lambda})\,d\boldsymbol{\lambda}}. \qquad (12.3.3)$$

By a method analogous to § 6.5, it is possible to prove that with symmetry of the a posteriori distribution conditional mathematical expectation is a universal optimum estimate, valid for a broader class of loss functions (namely, for functions possessing the property of symmetry). Inasmuch as direct integration of (12.3.3) in the most interesting cases cannot be accomplished, it is again necessary to solve approximately the system of equations

$$\int(\boldsymbol{\lambda}-\hat{\boldsymbol{\lambda}})\,P(\mathbf{y}\,|\,\boldsymbol{\lambda})\,P_0(\boldsymbol{\lambda})\,d\boldsymbol{\lambda}=0, \qquad (12.3.4)$$

which is another notation for (12.3.3). In order to prepare conditions for solution of this system, it is necessary to turn to study of the structure of the likelihood function.

12.3.2. Likelihood Function and Its Approximations

As also in the one-dimensional problem, for the purpose of studying the likelihood function $P(\mathbf{y}|\boldsymbol{\lambda})$ one should turn for analogies to the case of several constant parameters. Then, upon the expiration of a certain interval of observation in a parameter space, whose dimensionality is equal to the number of measured quantities l, $P(\mathbf{y}|\boldsymbol{\lambda})$ with Gaussian noises at the input can be connected by a monotonic relationship with a multi-dimensional autocorrelation function of the useful component of the input mixture (mixtures) and a certain multi-dimensional random function whose arguments are the measured quantities.

A multi-dimensional autocorrelation function can be introduced not only for parameters in the form of time delays and frequency shifts, as this is usually done in the literature (see Chapter I), but in a number case for the angular coordinates. This function in general is not equal to the product of one-dimensional autocorrelation functions for all l coordinates obtained on the assumption of knowing the other $l - 1$ coordinates. This is explained by a certain interconnection of codings of different parameters, which we shall discuss further below. Therefore, the multi-dimensional peak of $P(\mathbf{y}|\lambda)$ will be flattened in certain directions, determined by the nature of the interconnection.

If the vertex of the multi-dimensional autocorrelation function is smooth, i.e., this function has the necessary derivatives with respect to $\boldsymbol{\lambda}$, likelihood function $P(\mathbf{y}|\boldsymbol{\lambda})$ for a low level of lateral peaks of autocorrelation functions and a low level of interferences will be an l-dimensional peak of approximately Gaussian form near the true combination of values of the parameters. Flatness of the peak will be expressed in the fact that sections of constant level will be presented in the form of hyperellipsoids whose principal axes do not coincide with the coordinate axes. As also in the one-dimensional case, increase of the time of observation leads to gradual narrowing of the peak of $P(\mathbf{y}|\lambda)$ and decrease of dispersion of the position of its vertex near the combination of true values of the parameters. Here matrix $\boldsymbol{\Sigma}$ of moments of minimum errors of measurement of constant parameters is expressed through matrix $\mathbf{A}$ of mean values of second derivatives of $\ln P(\mathbf{y}|\lambda)$ in the form

$$\boldsymbol{\Sigma} = \mathbf{A}^{-1}; \quad \mathbf{A} = \left\| -\frac{\partial^2 \ln P(\mathbf{y}|\lambda)}{\partial\lambda^{(\alpha)}\partial\lambda^{(\beta)}} \right\|. \tag{12.3.5}$$

Transition to changing parameters in general leads to the same mathematical difficulties as in the one-dimensional problem. Let us assume, however, that the

rate of variation of all measured parameters is small as compared to the rate of variation of parameters belonging to the class of immaterial ones, and that the level of interferences is small. Then it is again possible to separate subintervals of observation in which the measured quantity can be considered "frozen," and immaterial parameters can be considered to vary so that for them the statistical relationship between values in the beginnings and ends of intervals is negligible.

As a supplement to the shown conditions we assume that the measured quantities differ in their physical nature and are coded dissimilarly. To clearly formulate the idea of "dissimilarity" is fairly difficult, so that we shall try to illustrate it by radar examples. Like coordinates of various targets (i.e., time delays and shifts of frequencies of signals close in form, and so forth) are coded similarly in the sense that transposition of parameters in the likelihood function leads to a situation, differing little in "likelihood" from the true situation. Unlike coordinates of one or various targets are coded dissimilarly; their transposition in the likelihood function is impermissible. (Dissimilarity, however, still does not signify disconnectedness with small deviations.)

With the initial encoding of parameters the likelihood function obtains a multipeak structure. These cases are excluded from consideration henceforth.

In the shown simplified conditions, fully sufficient, for instance, for consideration of simultaneous measurement of several coordinates of a single target, the likelihood function in the shown subinterval of observation in an l-dimensional parameter space turns out to be a single isolated peak. In the whole interval of observation it will be expressed approximately through the product of the likelihood functions in the subintervals. Its logarithm, analogously to the one-dimensional case, can be approximated by a quadratic form which is a truncated multi-dimensional Taylor series, so that

$$P(\mathbf{y}\,|\,\boldsymbol{\lambda})=P(\mathbf{y}\,|\,\boldsymbol{\lambda}_0)\exp\left\{\sum_{i=1}^{n}\sum_{\alpha=1}^{l}(\lambda_i^{(\alpha)}-\lambda_{i0}^{(\alpha)})\frac{\partial\ln P(\mathbf{y}\,|\,\boldsymbol{\lambda}_0)}{\partial\lambda_i^{(\alpha)}}+\right.$$

$$\left.+\frac{1}{2}\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{l}(\lambda_i^{(\alpha)}-\lambda_{i0}^{(\alpha)})(\lambda_j^{(\beta)}-\lambda_{j0}^{(\beta)})\frac{\partial^2\ln P(\mathbf{y}\,|\,\boldsymbol{\lambda}_0)}{\partial\lambda_i^{(\alpha)}\partial\lambda_j^{(\beta)}}\right\}, \qquad (12.3.6)$$

where $\boldsymbol{\lambda}_0=\{\lambda_{10}^{(1)},\ \ldots,\ \lambda_{10}^{(l)},\ \ldots,\ \lambda_{n0}^{(1)},\ \ldots,\ \lambda_{n0}^{(l)}\}$ — column vector, coordinates of which are assumed close to the true values of the measured parameters.

As in the one-dimensional case, two concrete expansions of type (12.3.6) are useful. The first is produced at the point of the multi-dimensional optimum

estimate of parameters $\boldsymbol{\lambda}_0 = \hat{\boldsymbol{\lambda}}$ ($\hat{\boldsymbol{\lambda}}$ – complex column vector of the considered type). Designating

$$\ln P(\mathbf{y}\,|\,\boldsymbol{\lambda}) = L(\boldsymbol{\lambda}),\ \frac{\partial L(\hat{\boldsymbol{\lambda}})}{\partial \lambda_i^{(\alpha)}} = z_i^{(\alpha)},\ \ -\frac{\partial^2 L(\hat{\boldsymbol{\lambda}})}{\partial \lambda_i^{(\alpha)} \partial \lambda_j^{(\beta)}} = A_{ij}^{(\alpha\beta)} \qquad (12.3.7)$$

and introducing matrix notation with block column vectors $\mathbf{z} = \{\mathbf{z}_1, \ldots, \mathbf{z}_n\}$, $\boldsymbol{\lambda} = \{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n\}$ and block square matrix $\mathbf{A} = \|\mathbf{A}_{ij}\|$, where a submatrix element is equal to $\mathbf{A}_{ij} = \|A_{ij}^{(\alpha\beta)}\|$, we have instead of (12.3.6) the relationship

$$P(\mathbf{y}\,|\,\boldsymbol{\lambda}) \approx P(\mathbf{y}\,|\,\hat{\boldsymbol{\lambda}}) \exp\left\{\mathbf{z}^+(\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}}) - \frac{1}{2}(\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}})^+ \mathbf{A}(\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}})\right\}, \qquad (12.3.8)$$

formally coinciding with the one-dimensional (6.6.3).

The second type of expansion corresponds to $\boldsymbol{\lambda}_0 = \check{\boldsymbol{\lambda}}$, where $\check{\boldsymbol{\lambda}}$ – the point of maximum value of $P(\mathbf{y}|\boldsymbol{\lambda})$, and has the form

$$P(\mathbf{y}\,|\,\boldsymbol{\lambda}) = P(\mathbf{y}\,|\,\check{\boldsymbol{\lambda}}) \exp\left\{-\frac{1}{2}(\boldsymbol{\lambda} - \check{\boldsymbol{\lambda}})^+ \check{\mathbf{A}}(\boldsymbol{\lambda} - \check{\boldsymbol{\lambda}})\right\}. \qquad (12.3.9)$$

Here matrix $\check{\mathbf{A}}$ is determined through second derivatives at point $\check{\boldsymbol{\lambda}}$ and practically coincides with matrix $\mathbf{A}$.

Relationships (12.3.8) and (12.3.9) will be used for synthesis of a multi-dimensional optimum meter.

12.3.3. Optimum Tracking Multi-Dimensional Meter

Let us assume that the a priori distribution of parameters is Gaussian,

$$P_0(\boldsymbol{\lambda}) = (2\pi)^{-\frac{nl}{2}} (\det \mathbf{R})^{-1/2} \exp\left\{-\frac{1}{2}(\boldsymbol{\lambda} - \bar{\boldsymbol{\lambda}})^+ \mathbf{V}(\boldsymbol{\lambda} - \bar{\boldsymbol{\lambda}})\right\}, \qquad (12.3.10)$$

where $\bar{\boldsymbol{\lambda}}$ – block column vector of mean values;

$\mathbf{R} = \|\mathbf{R}_{ij}\|$ – block $(nl \times nl)$ square matrix with submatrices $(l \times l)$ $\mathbf{R}_{ij} = \|R^{(\alpha\beta)}(t_i, t_j)\|$, elements of which $\mathbf{R}^{(\alpha\beta)}(t_i, t_j)$ show crosscorrelation of the α-th parameter at time t_i and of the β-th parameter at time t_j;

$\mathbf{V} = \mathbf{R}^{-1}$ – block square matrix, the inverse of $\mathbf{R}$.

During synthesis of an optimum meter we first use an approximation of the likelihood function in the form (12.3.8). Crossmultiplying this relationship with (12.3.10), reducing the logarithm of the resulting expression to form

$$C_0 - \frac{1}{2}(\boldsymbol{\lambda} - \boldsymbol{\lambda}_0)^+ \mathbf{C}^{-1}(\boldsymbol{\lambda} - \boldsymbol{\lambda}_0),$$

where C_0, $\boldsymbol{\lambda}_0$ and $\mathbf{C}^{-1}$ do not depend on λ, and equating $\boldsymbol{\lambda}_0$ to the optimum estimate, analogously to § 6.6 we obtain expression

$$\hat{\lambda}_n = \sum_{k=1}^{n} C_{nk}\left[z_k + \sum_{i=1}^{n} A_{ki}(\hat{\lambda}_i - \bar{\lambda}_i)\right] + \bar{\lambda}_n, \qquad (12.3.11)$$

where the quantities and the order of their location have matrix meaning, and $\|C_{ik}\| = C = [A + V]^{-1}$ – matrix $(ln \times ln)$, determined in the interval of observation. After transition to continuous functions at $t_{i+1} - t_i = \Delta \to 0$, $n \to \infty$ ($i = 1, \ldots, n$; $n\Delta = t - t_0$) relationship (12.3.11) takes form

$$\hat{\lambda}(t) = \int_{t_0}^{t} c(t, s)\left\{z(s) + \int_{t_0}^{t} A(s, \tau)[\hat{\lambda}(\tau) - \bar{\lambda}(\tau)]\, d\tau\right\} ds + \overline{\lambda(t)}, \qquad (12.3.12)$$

where

$$c(t_n, t_k) = C_{nk}, \quad z_k = \int_{t_k - \Delta/2}^{t_k + \Delta/2} z(\tau)\, d\tau;$$

$$A_{ki} = \int_{t_k - \Delta/2}^{t_k + \Delta/2} d\tau \int_{t_i - \Delta/2}^{t_i + \Delta/2} ds\, A(\tau, s). \qquad (12.3.13)$$

As above, in passages to the limit of (12.3.13) quantity Δ is considered limited from below by intervals exceeding the interval of correlation of the immaterial parameters of mixtures ${}^{(i)}y(t)$, and vector function $\mathbf{z}(t)$ is determined with accuracy to statistical equivalence. We also indicate that after transition to continuous time arguments matrices and columns in (12.3.12) and below have dimensionality $(l \times l)$ and $(l \times 1)$, respectively.

For the same reason of slowness of variation of the measured parameters function $\mathbf{A}(t, \tau)$ has filtering properties with respect to time:

$$A(t, \tau) = K(t)\,\delta(t - \tau), \qquad (12.3.14)$$

where $\mathbf{K}(t)$ – $(l \times l)$ matrix

Taking into account (12.3.14) we have the final expression for the optimum meter:

$$\hat{\lambda}(t) = \int_{t_0}^{t} c(t, s)\{z(s) + K(s)[\hat{\lambda}(s) - \bar{\lambda}(s)]\}\, ds + \overline{\lambda(t)}, \qquad (12.3.15)$$

which in expanded form should be understood as

$$\hat{\lambda}^{(\alpha)}(t) = \sum_{\beta=1}^{l} \int_{t_0}^{t} c^{(\alpha\beta)}(t, s)\left\{z^{(\beta)}(s) + \sum_{\gamma=1}^{l} K^{(\beta\gamma)}(s)\,[\hat{\lambda}^{(\gamma)}(s) - \bar{\lambda}^{(\gamma)}(s)]\right\} ds + \bar{\lambda}^{(\alpha)}(t). \qquad (12.3.16)$$

A block diagram illustrating an optimum joint filter-meter is given in Fig. 12.5. The whole set of input mixtures proceeds to nonlinear units 1 and 2. We indicate that unit 1 issues l voltages, on the average proportional to current

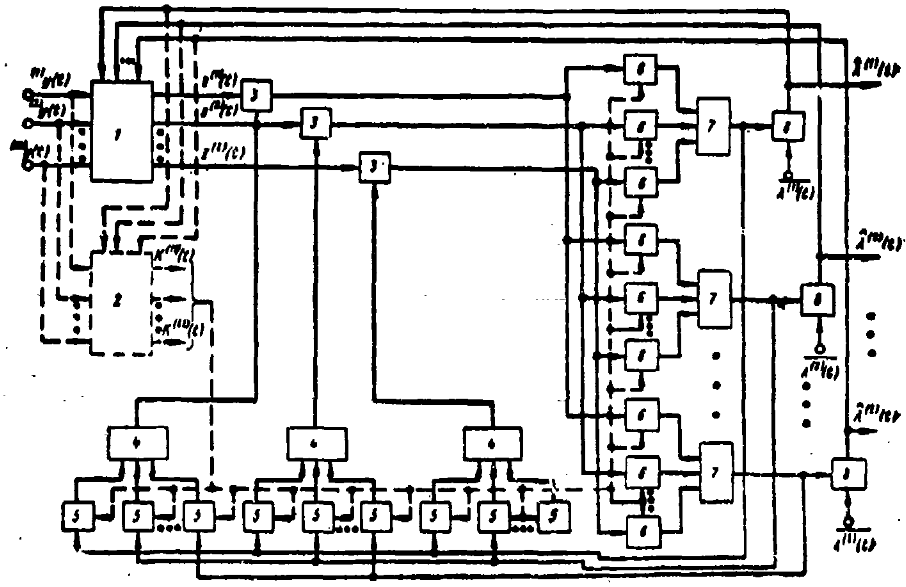

Fig. 12.5. Two-loop variant of an optimum multi-dimensional meter: 1) multi-dimensional discriminator; 2) multi-dimensional accuracy unit; 3, 4, 7, 8) adders; 5) linear filters with pulse responses $K^{(\alpha\beta)}(t, \tau)$; 6) linear filters with pulse responses $c^{(\alpha\beta)}(t, \tau)$.

mismatches between the true and measured values of parameters $\lambda^{(\alpha)}(t)$. Unit 2 issues $l(l + 1)/2$ voltages [according to the number of independent coefficients of symmetric matrix $\mathbf{K}(t)$], characterizing the current instantaneous accuracy of measurements. Therefore unit 1 is reasonably named a multi-dimensional (or joint) discriminator, and unit 2 – a multi-dimensional (or joint) accuracy unit. Output voltages of discriminators through adders 3 are fed to a ($l \times l$) matrix of smoothing filters 6, output quantities of which are combined by adders again into l groups. Thereby smoothing circuits turn out to be interconnected. After addition in 8 of output voltages of unit 6 with a priori mean values of parameters there are formed estimates of the measured parameters, which proceed to the discriminator and accuracy unit for maintaining selection. Furthermore, grouped output voltages 6 are fed to a matrix of inertialess variable-gain amplifiers 5 from whose output they proceed to adders of internal feedback 3. Consequently, there are formed, as it were, two basic multi-dimensional control loops: one using a discriminator and an accuracy unit, the other using adders at the output of the discriminators. As also in the one-dimensional case, we shall call this variant of meter double-loop. Pulse responses of filters 6 and 5 are controlled by output voltages of the

accuracy unit for compensation of unequal accuracy of separate measurements (or, which is the same, parametric fluctuations).

The other, single-loop variant of the meter corresponds to formula

$$\hat{\lambda}(t)=\int_{t_0}^{t} \mathbf{g}(t, \tau)\, z(\tau)\, d\tau+\overline{\lambda(t)} \tag{12.3.17}$$

or in expanded form

$$\hat{\lambda}^{(\alpha)}(t)=\sum_{\beta=1}^{l} \int_{t_0}^{t} g^{(\alpha\beta)}(t, \tau)\, z^{(\beta)}(\tau)\, d\tau+\overline{\lambda}^{(\alpha)}(t) \quad (\alpha=1, 2, \ldots, l). \tag{12.3.18}$$

These relationships are obtained from (12.3.15) and (12.3.16), if we solve them for the estimates. The new variant of circuit is illustrated in Fig. 12.6. It is simpler than the two-loop variant inasmuch as instead of two matrices of smoothing circuits it requires only one matrix of elements 3. Discriminator 1 and accuracy unit 2 are kept in constant form.

For matrices of pulse responses $\mathbf{c}(t, \tau)$ and $\mathbf{g}(t, \tau)$ it is easy to obtain integral matrix equations

$$\mathbf{c}(t, \tau)+\int_{t_0}^{t} \mathbf{c}(t, s)\, \mathbf{K}(s)\, \mathbf{R}(s, \tau)\, ds=\mathbf{R}(t, \tau), \tag{12.3.19}$$

$$\mathbf{c}(t, \tau)+\int_{\tau}^{t} \mathbf{c}(t, s)\, \mathbf{K}(s)\, \mathbf{g}(s, \tau)\, ds=\mathbf{g}(t, \tau), \tag{12.3.20}$$

completely analogous to the one-dimensional case.

Thus, the optimum system of measurement of several parameters is a multi-dimensional self-tuning tracking system with two nonlinear multi-dimensional units, issuing signals of errors for separate parameters, and indicators of current accuracy of measurement. The system is cl ed by a matrix (or two matrices) of linear smoothing circuits, determined by the correlation matrix of parameters and matrix $\mathbf{K}(t)$. Furthermore, in the system there is provided input of a priori mean values of parameters.

Correlation matrix $\mathbf{R}(t, \tau)$ characterizes the interconnection of the measured parameters, i.e., describes trajectory properties of the target, forming the set of measured quantities. Matrix $\mathbf{K}(t)$ determines the interconnection of codings of parameters in the signals, i.e., characterizes the process of encoding parameters of the trajectory in parameters of the signal. To the latter during reception there are added noises and interferences. With diagonalness of $\mathbf{R}$ and $\mathbf{K}$, when parameters and their codings are not connected, matrices $\mathbf{c}$ and $\mathbf{g}$ also are diagonal,

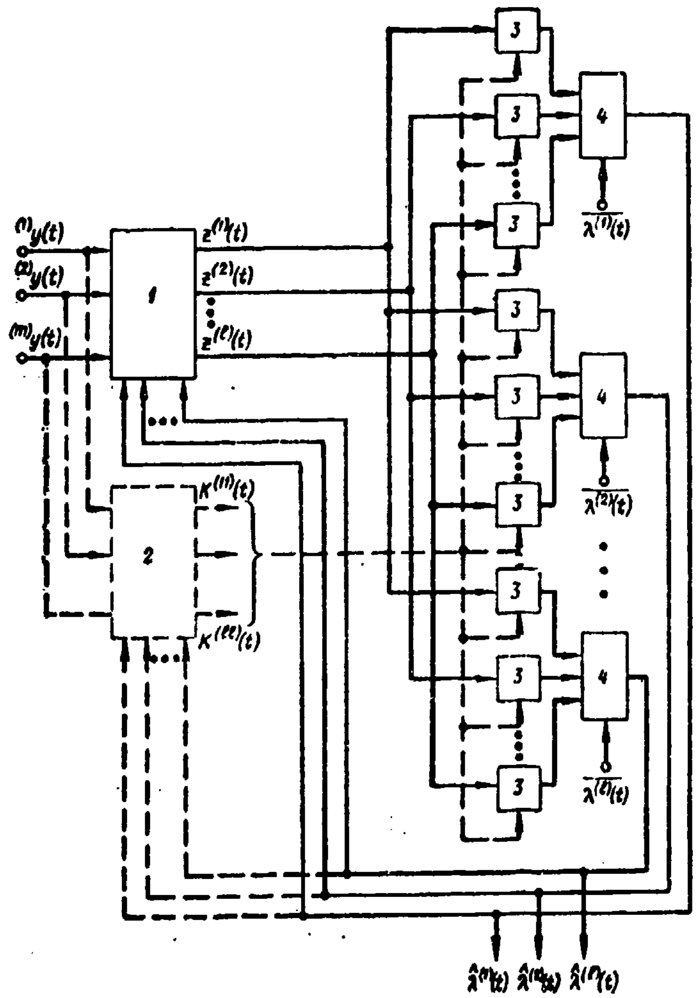

Fig. 12.6. Single-loop variant of an optimum multi-dimensional meter: 1) multi-dimensional discriminator; 2) multi-dimensional accuracy unit; 3) linear filters with pulse responses $g^{(\alpha\beta)}(t, \tau)$; 4) adders.

and smoothing circuits are considerably simplified. Common elements for subsystems of measurement of separate parameters remain only a multi-dimensional discriminator and an accuracy unit.

Finally, if the set of mixtures ${}^{(i)}y(t)$ is divided into l independent groups, depending each on only one parameter, and **R** and **K** are diagonal, the multi-dimensional measuring system is divided into a set of l of the independent one-dimensional tracking meters investigated above. Such a completely degenerate case in radar is a rare exception. Usually in the discriminator and in smoothing circuits there are circuit couplings which complicate consideration as compared to the

one-dimensional case. However, if matrix **K** is diagonal (even on average), then l outputs of the discriminator and the l outputs of the accuracy unit are a simple set of outputs of one-dimensional discriminators and accuracy units, synthesized for each parameter separately, with the distinction that derivatives of the logarithm of the likelihood function are taken at a point corresponding to the set of measured values of all l measured parameters.

Remember that in the one-dimensional case derivatives with respect to the parameter were taken at the measured value of the measured quantity, and all the other coordinates were considered accurately known. The shown circumstance is fundamental, and absence of selection for even one coordinate leads to power losses for all the others. In radar practice such a law is well-known (see § 12.2). If, however, matrix **K** is nondiagonal, then it is necessary additionally to $2l$ quantities to form in general $l(l - 1)/2$ more quantities, showing the current interconnection of errors of separate measurements (and of the very signals of mismatches, see below) for all coordinates with connected coding. It is natural that such complication leads to more complicated technical solutions.

The dependence of $\mathbf{z}(t)$ on mismatches $\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}}$ in general is nonlinear and very complicated. However, at a low level of noises this linear expansion is permissible:

$$\mathbf{z}(t) = \boldsymbol{\xi}(t) + \mathbf{K}(t)[\boldsymbol{\lambda}(t) - \hat{\boldsymbol{\lambda}}(t)], \qquad (12.3.21)$$

where

$$\boldsymbol{\xi}(t) = \{\xi^{(1)}(t), \ldots, \xi^{(l)}(t)\};$$

$$\xi_i^{(\alpha)} = \frac{\partial L(\boldsymbol{\lambda})}{\partial \lambda_i^{(\alpha)}} = \int\limits_{t_i - \Delta/2}^{t_i + \Delta/2} \xi^{(\alpha)}(\tau)\, d\tau, \qquad (12.3.22)$$

and $K(t)$ is again expressed through second derivatives with respect to $\lambda_i^{(\alpha)}$, $\lambda_j^{(\beta)}$. In view of the speed of change of immaterial parameters, noises $\boldsymbol{\xi}(t)$ according to relationship (12.3.14) are white with correlation matrix $\overline{\mathbf{K}(t)}\delta(t - \tau)$. Thereby $\overline{\mathbf{K}(t)}$ in its physical meaning is the average gain factor of the α-th (β-th) output of the discriminator with respect to mismatch of the β-th (α-th) parameter and simultaneously is the mutual spectral density of white noises of the α-th and β-th output. If we recalculate $\boldsymbol{\xi}(t)$ to the input of the discriminator, we shall have a set of quantities $\boldsymbol{\eta}(t) = \overline{\mathbf{K}(t)}^{-1}\boldsymbol{\xi}(t)$ of the dimensionality of the measured parameters. It is easy to prove that the matrix of spectral densities $\boldsymbol{\eta}(t)$ has the form $\overline{\mathbf{K}(t)}^{-1}\delta(t - \tau)$, i.e., $\overline{\mathbf{K}(t)}^{-1}$, analogously to the scalar case, has the meaning of the matrix of equivalent spectral densities of noises. This interpretation of

matrix $\overline{K(t)}^{-1}$ is the most important, inasmuch as arbitrary inertialess matrix transformation of output voltages of the discriminator changes the gain factor with respect to separate parameters and the level of output noises, but does not change the matrix of equivalent noises recalculated to the input. One should compare namely this matrix with the matrix of spectral densities obtained in the practical circuit performing the same function.

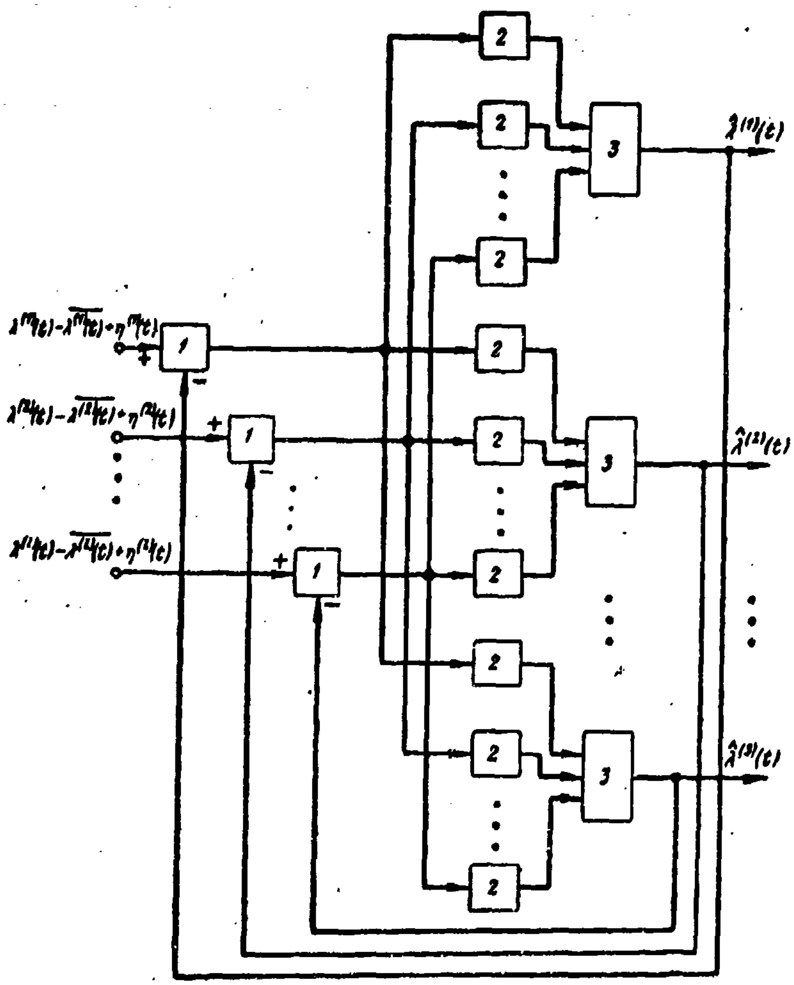

Fig. 12.7. Block diagram of an equivalent linear tracking system: 1) subtractor; 2) linear filters with pulse responses $\sum_{\beta=1}^{l} g^{(\alpha\beta)}(t, \tau) K^{(\beta\gamma)}(\tau)$; 3) adders.

In Fig. 12.7 is the block diagram of a linear tracking system, in its characteristics absolutely equivalent to the optimum meter of Fig. 12.6 with small errors of reproduction of all l parameters, when approximation (12.3.21) is valid. To the input of the equivalent system there are fed mixtures of "signals" $\lambda(t) - \overline{\lambda(t)}$

and "interferences" $\boldsymbol{\eta}(t)$. The matrix of pulse responses of smoothing circuits is equal to

$$\mathbf{c}(t, \tau)\mathbf{K}(\tau) = \left\| \sum_{\beta=1}^{l} c^{(\alpha\beta)}(t, \tau) K_1^{(\beta\gamma)}(\tau) \right\|.$$

Resultant accuracy of measurement for any variant of the circuit is determined, analogously to Paragraph 6.6.2, by relationships

$$\begin{gathered}\overline{\hat{\boldsymbol{\lambda}}(t)} = \overline{\boldsymbol{\lambda}(t)}, \ \mathbf{R}_{\text{вых}}(t, \tau) = \overline{\mathbf{c}(t, \tau)}, \\ \boldsymbol{\Sigma}_{\text{вых}}(t) = \overline{\mathbf{c}(t, t)} = \overline{\mathbf{g}(t, t)},\end{gathered} \tag{12.3.23}$$

where averaging of $\mathbf{c}(t, \tau)$ is over the ensemble of input mixtures.

In the particular case when the random component of elements of matrix $\mathbf{K}(t)$ can be ignored, accuracy units and control units for smoothing circuits disappear, nonrandom pulse responses are determined by equations (12.3.19) and (12.3.20), where $\mathbf{K}(t)$ is replaced by its mean, and during the analysis of accuracy properties of the meters there is no need for additional averaging of $\mathbf{c}(t, \tau)$.

We shall also discuss the method of obtaining functions $\mathbf{z}(t)$ and $\mathbf{K}(t)$ from the form of the likelihood functional of all the input signals. As also in the one-dimensional case, the likelihood functional, completely accounting for properties of the whole set ${}^{(i)}y(t)$, can usually be expressed in the form

$$P(\mathbf{y}(t) \mid \boldsymbol{\lambda}(t)) = \exp \int_{t_0}^{t} l(\tau, \boldsymbol{\lambda}, \mathbf{y})\, d\tau, \tag{12.3.24}$$

where $\mathbf{y}(s)$, $\boldsymbol{\lambda}(s)$ – vector functions, considered in interval $s \in (t_0, \tau)$.

Then, analogously to § 6.6

$$z^{(\alpha)}(t) = \frac{\partial}{\partial \lambda^{(\alpha)}} l(\tau; \hat{\boldsymbol{\lambda}}(t), \mathbf{y}), \tag{12.3.25}$$

$$K^{(\alpha\beta)}(t) = -\frac{\partial^2}{\partial \lambda^{(\alpha)} \partial \lambda^{(\beta)}} l(\tau, \hat{\boldsymbol{\lambda}}(t), \mathbf{y}). \tag{12.3.26}$$

If all useful components of input mixtures are incoherent pulse signals, it is necessary to produce modification of the method of synthesis, analogously to § 6.6. The optimum meter executes operations according to discrete expression

$$\hat{\boldsymbol{\lambda}}_n = \sum_{k=1}^{n} \mathbf{c}_{nk} [\mathbf{z}_k + \mathbf{K}_k (\hat{\boldsymbol{\lambda}}_k - \overline{\boldsymbol{\lambda}}_k)] + \overline{\boldsymbol{\lambda}}_n, \tag{12.3.27}$$

into which (12.3.11) passes, taking into account the diagonal nature of complex matrix $\mathbf{A}$ with respect to discrete time arguments. By $z_k^{(\alpha)}$ and $K_i^{(\alpha\beta)}$ one should understand quantities grouped in a period of repetition

$$z_k^{(\alpha)} = \frac{\partial}{\partial \lambda^{(\alpha)}} \int_{(k-1)T_r}^{kT_r} l_k(\tau, \hat{\boldsymbol{\lambda}}_k, \mathbf{y})\, d\tau, \tag{12.3.28}$$

$$K_{i}^{(\alpha\beta)}=-\frac{\partial^{2}}{\partial\lambda^{(\alpha)}\partial\lambda^{(\beta)}}\int_{(k-1)T_{r}}^{kT_{r}} l_{k}(\tau,\ \hat{\lambda}_{k},\ \mathbf{y})\,d\tau, \tag{12.3.29}$$

where l_k – functions in the expression for the likelihood functional

$$P(\mathbf{y}(t)|\lambda(t))=\exp\sum_{k}\int_{(k-1)T_{r}}^{kT_{r}} l_{k}(\tau,\ \lambda,\ \mathbf{y})\,d\tau. \tag{12.3.30}$$

Matrices of discrete smoothing circuits are determined by equations analogous to (6.6.20) and (6.6.22):

$$\left.\begin{array}{l}\mathbf{C}+\mathbf{CKR}=\mathbf{R},\\ \mathbf{C}+\mathbf{CKG}=\mathbf{G}\end{array}\right\}. \tag{12.3.31}$$

or in expanded form

$$\left.\begin{array}{l}C_{ik}^{(\alpha\beta)}+\sum\limits_{j=1}^{n}\sum\limits_{\gamma,\delta=1}^{l}C_{ij}^{(\alpha\gamma)}K_{j}^{(\gamma\delta)}R_{jk}^{(\delta\beta)}=R_{ik}^{(\alpha\beta)},\\ C_{ik}^{(\alpha\beta)}+\sum\limits_{j=k}^{n}\sum\limits_{\gamma,\delta=1}^{l}C_{ij}^{(\alpha\gamma)}K_{j}^{(\gamma\delta)}G_{jk}^{(\delta\beta)}=G_{ik}^{(\alpha\beta)}.\end{array}\right\}. \tag{12.3.32}$$

Further concretization of the problem of optimum measurement consists in finding the concrete form of the operation of formation of $\mathbf{z}(t)$ and $\mathbf{K}(t)$, the equipment realization of these operations, and establishment of the algorithm of smoothing in the linear filters.

We discuss these questions in subsequent paragraphs.

12.3.4. Optimum Nontracking Multi-Dimensional Meter

Let us consider now an optimum nontracking meter of several parameters. The nontracking variant of the meter is synthesized analogously to § 6.6. Multiplying (12.3.10) by another approximation of likelihood function (12.3.9) and transforming the logarithm of the formed expression, we have

$$\hat{\lambda}_{n}=\sum_{k=1}^{n}\mathbf{C}_{nk}\mathbf{A}_{kj}(\check{\lambda}_{j}-\overline{\lambda}_{j})+\overline{\lambda}_{n}. \tag{12.3.33}$$

After transition to continuous observation and allowing for the diagonal nature of matrix $\mathbf{A}$ for time arguments, we have finally

$$\hat{\lambda}(t)=\int_{t_0}^{t}\mathbf{c}(t,\ \tau)\,\mathbf{K}(\tau)\,[\check{\lambda}(\tau)-\overline{\lambda(\tau)}]\,d\tau+\overline{\lambda(t)}, \tag{12.3.34}$$

or in expanded form

$$\hat{\lambda}^{(\alpha)}(t)=\sum_{\gamma=1}^{l}\int_{t_0}^{t}\left\{\sum_{\beta=1}^{l}c^{(\alpha\beta)}(t,\ \tau)\,K^{(\beta\gamma)}(\tau)\right\}\times\ [\check{\lambda}^{(\gamma)}(\tau)-\overline{\lambda}^{(\gamma)}(\tau)]\,d\tau+\overline{\lambda}^{(\alpha)}(t)\ \ (\alpha=1,\ \ldots,\ l). \tag{12.3.35}$$

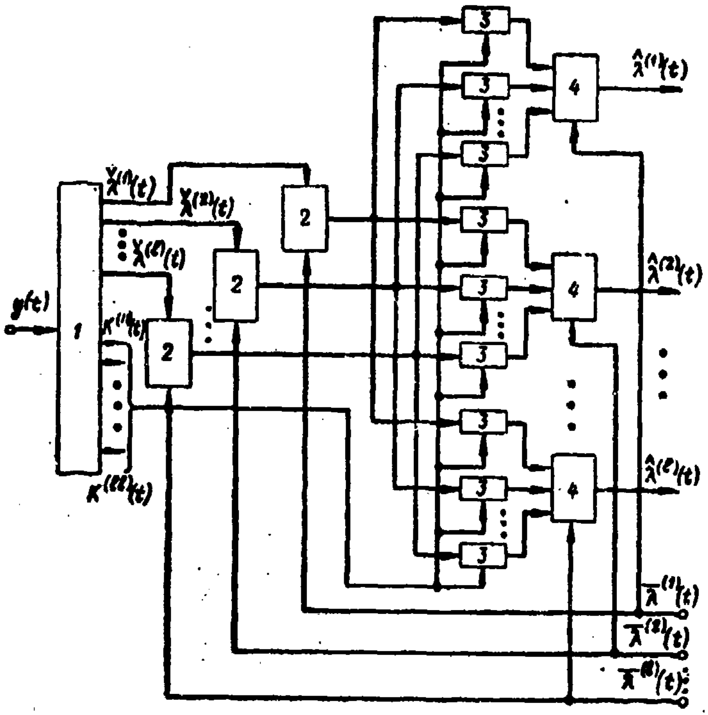

Fig. 12.8. Nontracking variant of an optimum multi-dimensional meter: 1) estimator unit; 2) subtractor; 3) linear filters with pulse responses $\sum_{\beta=1}^{l} c^{(\alpha\beta)}(t,\tau)K^{(\beta\gamma)}(\tau)$; 4) adders.

A circuit executing operations (12.3.34) is given in Fig. 12.8. As also in the one-dimensional case, here there are set off a multi-dimensional estimator unit 1 which gives the set of functions $\check{\lambda}^{(\alpha)}(t)$ and $K^{(\alpha\beta)}(t)$ and a matrix of linear filters 3, 4 with pulse responses $\sum_{\beta=1}^{l} c^{(\alpha\beta)}(t, \tau)K^{(\beta\gamma)}(\tau)$. We indicate that the estimator unit here is a very complicated device which analizes the likelihood function for all measured parameters in the whole a priori domain of their determination. As also in the one-dimensional case, the nontracking circuit somewhat weakens the requirement on the magnitude of variance of the initial distribution of error of measurement, expanding as it were the linear section of the characteristics of accuracy of measurement (12.3.23). This circumstance is very important in the stage of lock-on and transition to precision tracking. In a number of cases it will be convenient to combine tracking and nontracking circuits for separate parameters.

12.3.5. Optimum Meter of Linear Functionals of Parameters

Very often in radar along with measurement of parameters of input signals there must be formed several linear functionals of them (derivatives, expected values of parameters, integrals, and so forth). Sometimes measurement of linear functionals is the basic function of devices. Therefore the problem of optimum measurement of an arbitrary number q of linear functionals $\varphi^{(\beta)}(t)$ of l parameters $\lambda^{(\alpha)}(t)$ coded in input signals is important.

The consideration below in many respects is analogous to material of Paragraph 6.6.8 describing the case of one parameter and one functional. As before we assume Gaussian distribution of parameters $\lambda^{(\alpha)}(t)$ and consequently, also of $\varphi^{(\beta)}(t)$.

The relationship between column vectors of functionals $\varphi(t)$ and parameters $\lambda(t)$ is established by a linear matrix integral relationship

$$\boldsymbol{\varphi}(T)=\int_{T_1}^{T_2} \mathbf{F}(T,\ s)\,\boldsymbol{\lambda}(s)\,ds \qquad (12.3.36)$$

and is wholly determined by square $(q \times l)$ matrix function $\mathbf{F}(t,\ s)$.

After quantization in time by intervals Δ relationship (12.3.36) takes form

$$\boldsymbol{\varphi}=\mathbf{F}\boldsymbol{\lambda}, \qquad (12.3.37)$$

or in expanded form

$$\varphi_m^{(\alpha)}=\sum_{\beta=1}^{l}\sum_{j=N_1}^{N_2} F_{mj}^{(\alpha\beta)}\lambda_j^{(\beta)}\,(T_{1,2}=N_{1,2}\Delta+t_0,\ T=m\Delta+t_0), \qquad (12.3.38)$$

where $\boldsymbol{\varphi}$, $\boldsymbol{\lambda}$ and $\mathbf{F}$ in (12.3.37) are understood as complex columns and the matrix.

Let us assume that the input signal (signals) y(t) is accessible to observation at moments $t_1,\ t_2,\ \ldots,\ t_n$ from interval $(t_0,\ t)$, in no way attached to moment $T = t_m$, the argument of the functional φ_m. Then mean risk in the quadratic loss function with respect to errors of measurement of functionals in the m-th moment of time is recorded in the form

$$R(P_0,\ \hat{\boldsymbol{\varphi}})=\iint\sum_{\alpha,\beta=1}^{q} B^{(\alpha\beta)}(\varphi_m^{(\alpha)}-\hat{\varphi}_m^{(\alpha)})(\varphi_m^{(\beta)}-\hat{\varphi}_m^{(\beta)})\times P(y_1,\ldots,y_n\,|\,\lambda_1^{(1)},\ldots,\ \lambda_n^{(l)})\times$$
$$\times P_0(\lambda_{N_1}^{(1)},\ldots,\lambda_{N_1}^{(l)},\ldots,\ \lambda_{N_2}^{(1)},\ldots,\ \lambda_{N_2}^{(l)})\,d\boldsymbol{\lambda}\,d\mathbf{y}.$$

Varying $R(P_0,\ \hat{\boldsymbol{\varphi}})$ for estimates $\hat{\boldsymbol{\varphi}}$, we have the system of equations

$$\int(\varphi_m-\hat{\varphi}_m)\,P_{(1,n)}(\mathbf{y}|\boldsymbol{\lambda})\,P_{0\,(N_1,N_2)}(\boldsymbol{\lambda})\,d\boldsymbol{\lambda}=0, \qquad (12.3.39)$$

where subscripts in the likelihood function and the a priori distribution signify the number of moments of reading. As also for the one-dimensional case, it is easy to prove that direct formal solution of (12.3.39),

$$\hat{\varphi}_m = \frac{\sum_{k=N_1}^{N_2} F_{mk} \int \lambda_k P_{(1,n)}(\mathbf{y}|\lambda) P_{0(N_2,N_1)}(\lambda)\, d\lambda}{\int P_{(1,n)}(\mathbf{y}|\lambda) P_{0(N_2,N_1)}(\lambda)\, d\lambda} \tag{12.3.40}$$

in general is unacceptable in view of disturbance of the principle of physical realizability of the optimum operator. In other words, the linear functional F of the optimum estimate of the parameters, in general, is not the optimum estimate of linear functional F.

To find a physically realizable solution of equation (12.3.40) we again assume that for $P(\mathbf{y}|\boldsymbol{\lambda})$ approximation (12.3.8) is permissible. Expansion of ln $P(\mathbf{y}|\boldsymbol{\lambda})$ we produce at the point of the current physically realizable estimate of parameters, which it is possible to demand beforehand. Permissibility of formation and use in a meter of functionals as auxiliary functions of optimum estimates of coded parameters does not raise doubts either theoretically or practically. Substituting (12.3.8) and (12.3.10) in (12.3.39) and integrating, taking into account the diagonal nature of the matrix of second derivatives, we have

$$\hat{\varphi}_m = \sum_{N_1}^{N_2} F_{mk} \left\{ \bar{\lambda}_k + \sum_{i=1}^{n} C_{ki} [z_i + K_i (\hat{\lambda}_i - \bar{\lambda}_i)] \right\} =$$

$$= \sum_{k=1}^{n} B_{mk} [z_k + K_k (\hat{\lambda}_k - \bar{\lambda}_k)] + \sum_{k=N_1}^{N_2} F_{mk} \bar{\lambda}_k. \tag{12.3.41}$$

Here $\mathbf{C} = [\mathbf{V} + \mathbf{Q}]^{-1}$ – inverse matrix; Q – matrix of order $[i \times (N_2 - N_1)]$, determined by relationship

$$\mathbf{Q}_{ik} = \begin{cases} \mathbf{K}_i \delta_{ik} & 1 \leqslant ik \leqslant n, \\ 0 & i,k > n;\ i,\ k < 1. \end{cases} \tag{12.3.42}$$

Complex matrix **B** is determined in expanded form by relationship

$$\mathbf{B}_{mk} = \begin{cases} \sum_{i=N_1}^{N_2} \mathbf{F}_{mi} \mathbf{C}_{ik} & 1 \leqslant k \leqslant n, \\ 0 & k > n;\ k < 1. \end{cases} \tag{12.3.43}$$

From (12.3.43), taking into account values of Q_{ik} and V_{ik}, we can obtain equation

$$\mathbf{B}_{mk} + \sum_{i=1}^{n} \mathbf{B}_{mi} \mathbf{K}_i \mathbf{R}_{ik} = \sum_{i=N_1}^{N_2} \mathbf{F}_{mi} \mathbf{R}_{ik}, \tag{12.3.44}$$

relating matrix $\mathbf{B}_{ik}$ with the correlation matrix of parameters $\mathbf{R}_{ik}$ and the matrix function of the linear functional $\mathbf{F}_{ik}$.

Another presentation of solution (12.3.41) has the form

$$\hat{\varphi}_m = \sum_{k=1}^{n} \mathbf{H}_{mk}\mathbf{z}_k + \sum_{k=N_1}^{N_2} \mathbf{F}_{mk}\bar{\lambda}_k, \tag{12.3.45}$$

where $\mathbf{H}_{mk}$ is connected with $\mathbf{B}_{mk}$ by relationship

$$\mathbf{H}_{mk} = \mathbf{B}_{mk} + \sum_{i=k}^{n} \mathbf{B}_{mi}\mathbf{K}_i\mathbf{G}_{ik}, \tag{12.3.46}$$

and $\mathbf{G}_{lk}$ in turn is determined through $\mathbf{R}_{lk}$ according to equations (12.3.32), describing the optimum meter of parameters $\boldsymbol{\lambda}(\mathbf{t})$ themselves. As we see, the case of many functionals and parameters formally has great similarity with the one-dimensional case.

Transition to continuous observation gives relationship

$$\hat{\varphi}(T) = \int_{t_0}^{t} \mathbf{b}(T, s)\{\mathbf{z}(s) + \mathbf{K}(s)[\hat{\lambda}(s) - \bar{\lambda}(s)]\}\, ds + \int_{T_1}^{T_2} \mathbf{F}(T, s)\bar{\lambda}(s)\, ds, \tag{12.3.47}$$

where transition from $\mathbf{z}_k$, $\mathbf{K}_k$, $\mathbf{B}_{mk}$ and $\mathbf{F}_{lk}$ to $\mathbf{z}(t)$, $\mathbf{K}(t)$, $\mathbf{b}(T, s)$ and $\mathbf{F}(t, s)$ in the light of results of Paragraph 12.3.3 does not require explanation; $T = m\Delta + t_0$, $t = n\Delta + t_0$; and $\mathbf{b}(t_i, t_j) = \mathbf{B}_{ij}$ – matrix function, connected with $\mathbf{R}(t, \tau)$ by an integral equation analogous to (12.3.44):

$$\mathbf{b}(T, \tau) + \int_{t_0}^{t} \mathbf{b}(T, s)\mathbf{K}(s)\mathbf{R}(s, \tau)\, ds = \int_{T_1}^{T_2} \mathbf{F}(T, s)\mathbf{R}(s, \tau)\, ds. \tag{12.3.48}$$

In order not to encumber our account, we shall not give the expanded circuit of the optimum meter of functionals, constructed on the basis of relationship (12.3.47), and we only give its description. It contains the complete block diagram of the two-loop variant of an optimum meter of $\hat{\lambda}(t)$ (see Fig. 12.4). From adders of internal coupling in smoothing circuits there are made l taps to a square matrix of $(l \times q)$ special smoothing filters with pulse responses $b^{(\alpha\beta)}(t, \tau)$ $(\alpha = 1, \ldots, q;\ \beta = 1, \ldots, l)$. Output signals are combined in q groups according to the number of measured functionals, after which they are added to functions $\sum_{\beta=1}^{l} \int_{t_0}^{t} F^{(\alpha\beta)}(t, \tau)\bar{\lambda}^{(\beta)}(\tau)\, d\tau$, which are results of processing a priori mean values of parameters $\bar{\boldsymbol{\lambda}}(t)$ by matrix function $\mathbf{F}(t, \tau)$. As a result there are formed optimum estimates of functionals. As we see, as compared to the case of measurement only of $\boldsymbol{\lambda}(t)$ there are required comparatively few complications.

Another modification of the optimum circuit is obtained by means of transition to continuous observation in relationship (12.3.45):

$$\hat{\boldsymbol{\varphi}}(T)=\int_{t_0}^{t}\mathbf{h}(T,\ s)\,\mathbf{z}(s)\,ds+\int_{T_1}^{T_2}\mathbf{F}(T,\ s)\,\overline{\boldsymbol{\lambda}(s)}\,ds. \tag{12.3.49}$$

It is conveniently combined with the single-loop variant of an optimum meter of parameters $\boldsymbol{\lambda}(t)$ (Fig. 12.5). Output voltages of discriminators are fed to a matrix ($l \times q$) of smoothing circuits with pulse response $\mathbf{h}(T, s)$, satisfying a relationship analogous to (12.3.46):

$$\mathbf{h}(T,\ \tau)=\mathbf{b}(T,\ \tau)+\int_{\tau}^{t}\mathbf{b}(T,\ s)\,\mathbf{K}(s)\,\mathbf{g}(s,\ \tau)\,ds, \tag{12.3.50}$$

where $\mathbf{g}(t, \tau)$ is determined by the correlation function of the parameter according to equation (12.3.32).

For accuracy of measurement of the set of linear functionals we can obtain the following formulas:

$$\mathbf{R}_{\text{вых}}(t_1,\ t_2)=\int_{t_0}^{t}\mathbf{b}(t_1,\ s)\,\mathbf{F}^+(t_2,\ s)\,ds,$$

$$\boldsymbol{\Sigma}_{\text{вых}}(T)=\int_{t_0}^{t}\mathbf{b}(T,\ s)\,\mathbf{F}^+(T,\ s)\,ds. \tag{12.3.51}$$

Thus, for optimum measurement of linear functionals of parameters it is necessary to introduce in the optimum meter directly coded parameters of two matrices of linear filters: one, the basic one, for smoothing of data, the other for transformation of a priori mean values. Matrices have l inputs (from the number of parameters) and q outputs (from the number of functionals). Adding output quantities of these two matrices, we have optimum estimates of the functionals. Finding the first of these matrices from the concrete correlation matrix of parameters and accuracy properties of discriminator is a separate problem.

12.3.6. Optimum Meter of Markovian Parameters

During construction of a meter of several Markovian parameters we use the methods offered in § 6.9. We start from the case of discrete observation (a pulse signal), when it is possible to use the relationship for final a posteriori probabilities

$$\widetilde{P}(\boldsymbol{\lambda}_{n+1})=C^0_{n+1}P(\mathbf{y}_{n+1}|\boldsymbol{\lambda}_{n+1})\int\widetilde{P}(\boldsymbol{\lambda}_n)\,W(\boldsymbol{\lambda}_{n+1}|\boldsymbol{\lambda}_n)\,d\boldsymbol{\lambda}_n, \tag{12.3.52}$$

generalizing (6.5.24) to the vector case. Here $\boldsymbol{\lambda}_k = \{\lambda_k^{(1)}, \ldots, \lambda_k^{(l)}\}$ – column vector of the measured parameters in the k-th moment of time; $\widetilde{P}(\lambda_k)$ – its a posteriori

probability; $P(\mathbf{y}_{n+1}|\boldsymbol{\lambda}_{n+1})$ — the likelihood function of the input mixture at the $(n+1)$-th moment (period) of observation; $W(\boldsymbol{\lambda}_{n+1}|\boldsymbol{\lambda}_n)$ — multi-dimensional probability density of transition.

Assuming small a posteriori inaccuracy, we introduce the Gaussian approximation of $\tilde{P}(\boldsymbol{\lambda}_k)$ in the form

$$\tilde{P}(\boldsymbol{\lambda}_k) = (2\pi)^{-l/2}[\det \Sigma_{\text{вых}\,k}]^{-1/2} \times \\ \times \exp\left\{-\frac{1}{2}(\boldsymbol{\lambda}_k - \hat{\boldsymbol{\lambda}}_k)^+ \Phi_k (\boldsymbol{\lambda}_k - \hat{\boldsymbol{\lambda}}_k)\right\}, \qquad (12.3.53)$$

where $\Sigma_{\text{ВЫХ}\,k}$ — matrix of second moments of output errors in the k-th moment;

Φ_k — its inverse matrix;

$\hat{\lambda}_k$ — column vector of estimates of parameters.

Approximation of the likelihood function we take in one of two forms

$$P(\mathbf{y}_k|\boldsymbol{\lambda}_k) \approx P(\mathbf{y}_k|\check{\boldsymbol{\lambda}}_k)\exp\left\{-\frac{1}{2}(\boldsymbol{\lambda}_k - \check{\boldsymbol{\lambda}}_k)^+ \mathbf{K}_k(\boldsymbol{\lambda}_k - \check{\boldsymbol{\lambda}}_k)\right\}, \qquad (12.3.54)$$

$$P(\mathbf{y}_k|\boldsymbol{\lambda}_k) \approx P(\mathbf{y}_k|\mathbf{e}(\hat{\boldsymbol{\lambda}}_{k-1}))\exp\{\mathbf{z}_k^+(\boldsymbol{\lambda}_k - \mathbf{e}(\hat{\boldsymbol{\lambda}}_{k-1})) - \\ -\frac{1}{2}(\boldsymbol{\lambda}_k - \mathbf{e}(\hat{\boldsymbol{\lambda}}_{k-1}))^+ \mathbf{K}_k(\boldsymbol{\lambda}_k - \mathbf{e}(\hat{\boldsymbol{\lambda}}_{k-1}))\}. \qquad (12.3.55)$$

In the first case the expansion of $\ln P(\mathbf{y}_k|\boldsymbol{\lambda}_k)$ is conducted near the maximum likelihood estimate $\check{\boldsymbol{\lambda}}_k$, and in the second it is at point $\mathbf{e}(\hat{\boldsymbol{\lambda}}_{k-1})$, determined by extrapolation of the estimate vector from the preceding moment ($\mathbf{e}(\lambda)$ — column vector which depends on column vector $\boldsymbol{\lambda}$ with the same number of components).

It is necessary to further definitize the form of the multi-dimensional transition function. For a diffusion vector process on small intervals of time this Gaussian approximation is permissible:

$$W(\boldsymbol{\lambda}_{n+1}|\boldsymbol{\lambda}_n) = (2\pi)^{-l/2}[\det \Sigma_\lambda(\boldsymbol{\lambda}_n)]^{-1/2} \times \\ \times \exp\left\{-\frac{1}{2}(\boldsymbol{\lambda}_{n+1} - \mathbf{e}(\boldsymbol{\lambda}_n))^+ \Sigma_\lambda^{-1}(\boldsymbol{\lambda}_n)(\boldsymbol{\lambda}_{n+1} - \mathbf{e}(\boldsymbol{\lambda}_n))\right\}, \qquad (12.3.56)$$

where $\boldsymbol{\Sigma}_\lambda(\boldsymbol{\lambda}) = \|B^{(\alpha\beta)}(\boldsymbol{\lambda})\|\Delta$ — matrix of correlation moments of random variations of parameters for interval Δ, in the first approximation of proportional multi-dimensional coefficients of diffusion $B^{(\alpha\beta)}(\boldsymbol{\lambda})$; $\boldsymbol{\Sigma}_\lambda^{-1}(\boldsymbol{\lambda})$ — its inverse matrix.

The result of extrapolation to the following moment in the condition of small Δ can be expressed through multi-dimensional coefficients of drift $A^{(\alpha)}(\boldsymbol{\lambda}_n)$, again limiting ourselves to terms linear in time:

$$\mathbf{e}(\boldsymbol{\lambda}_n) = \boldsymbol{\lambda}_n + \mathbf{a}(\boldsymbol{\lambda}_n),\ \ \mathbf{a}(\boldsymbol{\lambda}) = \{A^{(1)}(\boldsymbol{\lambda})\Delta, \ldots, A^{(l)}(\boldsymbol{\lambda})\Delta\}. \qquad (12.3.57)$$

We expand $\mathbf{a}(\boldsymbol{\lambda})$ at the estimate point

$$\mathbf{a}(\boldsymbol{\lambda})=\mathbf{a}(\hat{\boldsymbol{\lambda}})+\mathbf{U}(\hat{\boldsymbol{\lambda}})(\boldsymbol{\lambda}-\hat{\boldsymbol{\lambda}})\Delta, \tag{12.3.5}$$

where $\mathbf{U}(\hat{\boldsymbol{\lambda}})=\left\| \frac{\partial A^{(\alpha)}(\hat{\boldsymbol{\lambda}})}{\partial \lambda^{(\beta)}} \right\|$ – square matrix depending on $\boldsymbol{\lambda}$.

We substitute (12.3.53), (12.3.54) and (12.3.56) taking into account (12.3.57) and (12.3.58) in (12.3.52). Then after integration and equating of logarithms of both parts of the resulting expression we obtain matrix formulas

$$\boldsymbol{\Phi}_{n+1}\hat{\boldsymbol{\lambda}}_{n+1}=[\mathbf{Q}^{+}(\hat{\boldsymbol{\lambda}}_n)\,\boldsymbol{\Phi}_n^{-1}\,\mathbf{Q}(\hat{\boldsymbol{\lambda}}_n)+\boldsymbol{\Sigma}_{\lambda}(\hat{\boldsymbol{\lambda}}_n)]^{-1}\,[\hat{\boldsymbol{\lambda}}_n+\mathbf{a}(\hat{\boldsymbol{\lambda}}_n)]+\mathbf{K}_{n+1}\check{\boldsymbol{\lambda}}_{n+1}, \tag{12.3.59}$$

$$\boldsymbol{\Phi}_{n+1}=[\mathbf{Q}^{+}(\hat{\boldsymbol{\lambda}}_n)\,\boldsymbol{\Phi}_n^{-1}\,\mathbf{Q}(\hat{\boldsymbol{\lambda}}_n)+\boldsymbol{\Sigma}_{\lambda}(\hat{\boldsymbol{\lambda}}_n)]^{-1}+\mathbf{K}_{n+1}, \tag{12.3.60}$$

where $\mathbf{Q}(\boldsymbol{\lambda}) = \mathbf{I} + \mathbf{U}(\boldsymbol{\lambda})\Delta$ – extrapolation matrix of partial derivatives of the new [(n + 1)-st] with respect to the old (n-th) values of parameters, analogous to the scalar factor Q in (6.9.6) and (6.9.7).

Relationships (12.3.59) and (12.3.60) are completely similar to (6.9.6) and (6.9.7). Thus by (12.3.59) the vector of the (n + 1)-st estimates will be formed by weighted addition of the column of n-th estimates, anticipated a step, and the column vector of the new measurements $\boldsymbol{\lambda}_{n+1}$. The matrix weight of the first term is determined by the sum of the extrapolated matrix of output errors in the preceding moment and the matrix of variances of expected variations of the parameters. The weight of the second term is equal to matrix $\mathbf{K}_n$, the physical meaning of which was explained in detail above. "Information" matrix $\boldsymbol{\Phi}_n$ (the inverse of correlation matrix $\boldsymbol{\Sigma}_{\text{BHX}\, n}$) according to (12.3.60) in every step changes due to the action of three factors – extrapolation, random changes of the parameter and new incoming information about the parameters.

When approximation of the likelihood function is conducted according to (12.3.55), relationship (12.3.59) is replaced by

$$\hat{\boldsymbol{\lambda}}_{n+1}=\hat{\boldsymbol{\lambda}}_n+\mathbf{a}(\hat{\boldsymbol{\lambda}}_n)+\boldsymbol{\Sigma}_{\text{BHX}\,(n+1)}\,\mathbf{z}_{n+1}, \tag{12.3.61}$$

where $\boldsymbol{\Sigma}_{\text{BHX}(n+1)} = \Phi_{n+1}^{-1}$ is calculated again by (12.3.60). Relationship (12.3.61) is the vector generalization of formulas of Chapter VI and is interpreted the same way.

It is comparatively simple to obtain from (12.3.59)-(12.3.61) differential equations for continuous observation. They have matrix form:

$$\frac{d}{dt}\hat{\boldsymbol{\lambda}}(t)=\mathbf{A}(\hat{\boldsymbol{\lambda}}(t))+\boldsymbol{\Sigma}_{\text{BHX}}(t)\,\mathbf{K}(t)\,[\check{\boldsymbol{\lambda}}(t)-\hat{\boldsymbol{\lambda}}(t)], \tag{12.3.62}$$

$$\frac{d}{dt}\mathbf{\Sigma}_{\text{вых}}(t)=\mathbf{B}(\hat{\boldsymbol{\lambda}}(t))-\mathbf{\Sigma}_{\text{вых}}(t)\,\mathbf{K}(t)\,\mathbf{\Sigma}_{\text{вых}}(t)+$$

$$+\mathbf{U}(\hat{\boldsymbol{\lambda}}(t))\,\mathbf{\Sigma}_{\text{вых}}(t)+\mathbf{\Sigma}_{\text{вых}}(t)\,\mathbf{U}^{+}(\hat{\boldsymbol{\lambda}}(t)), \qquad (12.3.63)$$

$$\frac{d}{dt}\hat{\boldsymbol{\lambda}}(t)=\mathbf{A}(\hat{\boldsymbol{\lambda}}(t))+\mathbf{\Sigma}_{\text{вых}}(t)\,\mathbf{z}(t), \qquad (12.3.64)$$

but physical interpretation of them practically does not differ from the scalar case. Here $\mathbf{z}(t)$, $\mathbf{K}(t)$ – set of output quantities of an optimum multi-dimensional discriminator and an estimator unit.

Above we nowhere said that all l measured parameters have to be coded in the received signal (signals); there should only be sufficient coordinates of the set of measured quantities. Parameters $\lambda^{(1)}(t)$, ..., $\lambda^{(l)}(t)$ can, for instance, be broken up into p groups, each of which characterizes one parameter actually coded in the signal $\mu^{(i)}(t)$ ($i = 1, \ldots, p$). In a particular case all $\lambda^{(1)}(t)$ ..., $\lambda^{(l)}(t)$ may be sufficient coordinates of one parameter. Thus, if distance of the target is a Markovian process of the third order, as sufficient (and necessary in the process of smoothing) coordinates of it $\lambda^{(1)}(t)$, $\lambda^{(2)}(t)$ and $\lambda^{(3)}(t)$ there can serve distance, speed and acceleration at a certain moment of the value of distance in three different moments of time (for instance, the current signal period, the preceding period, and the one preceding it). In such cases column $\mathbf{z}_n$ (or $\hat{\boldsymbol{\lambda}}_n$) and matrix $\mathbf{K}_n$ are divided into p blocks, inside which only one element is different from zero, corresponding practically to the quantity coded in the signal, and all others are equal to zero. This does not lead to any theoretical or technical difficulties during realization of the meter.

We also note that when it is permissible to replace $\mathbf{K}(t)$ by $\mathbf{K}$, and matrices $\mathbf{B}(\boldsymbol{\lambda})$ and $\mathbf{U}(\boldsymbol{\lambda})$ do not depend on parameters, behavior of the output correlation matrix $\mathbf{\Sigma}_{\text{вых}}(t)$ ceases to depend on the realization, and equation (12.3.63) can be solved beforehand in principle. As a result the operations of smoothing reduce t (12.3.62) or (12.3.64), where $\mathbf{\Sigma}_{\text{вых}}(t)$ – a certain known matrix function. This, naturally, greatly simplifies the technical problem of construction of the meter.

The structure of the closed variant of a multi-dimensional meter in approximation $\mathbf{K}(t) \approx \mathbf{K}$ is shown in Fig. 12.9. The meter consists of a multi-dimensional discriminator 1, controlled amplifiers 2 coefficients of which $\mathbf{\Sigma}^{\alpha\beta}(t)$ are issued by the unit of determination of errors 3, adders 4, integrators 5 and a unit of nonlinear converters 6 introducing the mean value of drift depending upon the measured parameters. As we see, nonlinearity in smoothing circuits remains even in this simple case.

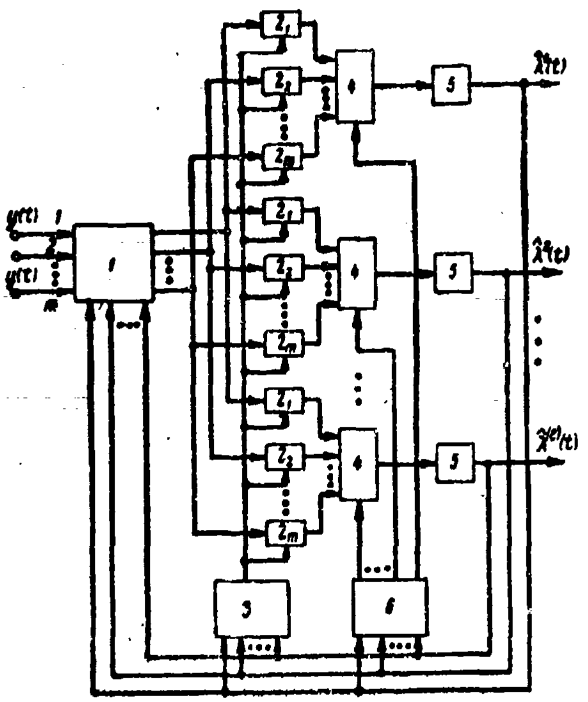

Fig. 12.9. Optimum multi-dimensional meter of Markovian parameters: 1) multi-dimensional discriminators; 2) controlled amplifiers; 3) units of determination of resultant errors; 4) adders; 5) integrators; 6) nonlinear converters.

Thus, for Markovian parameters smoothing circuits of the meter become in general nonlinear. At the same time elements of the first processing — multi-dimensional discriminator and accuracy unit — remain the same as for Gaussian parameters.

The statistical synthesis conducted above shows that a meter of several parameters (or linear functionals of them) is not a simple set of independent meters of separate parameters (functionals). Even if it is possible to detect in the joint discriminator and smoothing circuits separate elements relating to measurement of only one parameter there usually remain a whole series of elements necessary for the meter as a whole. Internal causes of the appearing couplings are the dependence of input signals immediately on several parameters, interconnection of codings of these parameters in signals and statistical couplings between the parameters themselves. Equipment couplings are realized, first, in the multi-dimensional discriminator (in the form of common input terminals, grouping of output quantities if the parameter is coded in various signals differently, and feed of the measured values of parameters immediately to all partial discriminators) and, secondly, in smoothing circuits (in the form of common input and output quantities). Concretization of the appearing couplings is given in subsequent paragraphs, where there is conducted particular synthesis of discriminators and smoothing circuits for signals and parameters with various properties.

§ 12.4. Synthesis of Multi-Dimensional Discriminators

As it was shown in § 12.3, as one of their basic elements multi-dimensional meters contain a multi-dimensional (joint) discriminator. Now it is necessary to determine general rules for finding operations of this discriminator $z^{(\alpha)}(t)$ and

its characteristics $K^{(\alpha\beta)}$ for different forms of input signals. As also for one-dimensional meters, in general consideration of discriminators at the base of our classification we place statistical properties of the signal; examples of measurement of concrete parameters will be studied later.

12.4.1. Regular Signals in White Noises

Let us assume we have m input mixtures

$$^{(i)}y(t) = {}^{(i)}v(t, \lambda(t)) + {}^{(i)}n(t) \text{ and } (i = 1, 2, \ldots m),$$

consisting of useful signals $^{(i)}v(t)$, whose form for fixed $\lambda(t)$ completely is assigned, and noises $^{(i)}n(t)$. Signals depend on l parameters $\{\lambda^{(1)}(t), \ldots, \lambda^{(l)}(t)\} = \boldsymbol{\lambda}(t)$.

Properties of noises are assigned by the correlation matrix

$$\mathbf{R}_{\text{III}}(\tau) = \overline{\|^{(i)}n(t)\,^{(j)}n(t+\tau)\|}. \tag{12.4.1}$$

Following the method of § 12.3, it is necessary to separate from the logarithm of the likelihood function, during Gaussian noises equal to

$$L(\mathbf{y}(t), \boldsymbol{\lambda}(t)) = -\frac{1}{2}\iint\limits_{t-\Delta/2}^{t+\Delta/2} [\mathbf{y}(t_1) - \mathbf{v}(t_1, \boldsymbol{\lambda}(t_1))]^+ \times \mathbf{W}_{\text{III}}(t_1, t_2)[\mathbf{y}(t_2) - \mathbf{v}(t_2, \boldsymbol{\lambda}(t_2))]\,dt_1 dt_2$$

[$\mathbf{W}_{\text{III}}(t_1, t_2)$ – function, the inverse of $\mathbf{R}_{\text{III}}(t_1, t_2)$], the integrand

$$l(\mathbf{y}, \boldsymbol{\lambda}; t) = -\frac{1}{2}[\mathbf{y}(t) - \mathbf{v}(t, \boldsymbol{\lambda}(t))]^+ \int\limits_{t-\Delta/2}^{t+\Delta/2} \mathbf{W}_{\text{III}}(t, \tau) \times [\mathbf{y}(\tau) - \mathbf{v}(\tau, \boldsymbol{\lambda}(\tau))]\,d\tau. \tag{12.4.2}$$

After differentiation of (12.4.2) with respect to $\lambda^{(\alpha)}$ we obtain two statistically equivalent notations of the α-th operation of the discriminator:

$$z^{(\alpha)}(t) = -\frac{\partial \mathbf{v}(t, \hat{\boldsymbol{\lambda}}(t))^+}{\partial \lambda^{(\alpha)}} \int\limits_{t-\Delta/2}^{t+\Delta/2} \mathbf{W}_{\text{III}}(t, \tau)[\mathbf{y}(\tau) - \mathbf{v}(\tau, \hat{\boldsymbol{\lambda}}(t))]\,d\tau, \tag{12.4.3}$$

$$z^{(\alpha)}(t) = -[\mathbf{y}(t) - \mathbf{v}(t, \hat{\boldsymbol{\lambda}}(t))]^+ \int\limits_{t-\Delta/2}^{t+\Delta/2} \mathbf{W}_{\text{III}}(t, \tau)\frac{\partial \mathbf{v}(\tau, \hat{\boldsymbol{\lambda}}(t))}{\partial \lambda}\,d\tau, \tag{12.4.4}$$

which can be interpreted in the form of the block diagrams of Figs. 12.10 and 12.11. According to (12.4.3) and Fig. 12.10 from the input mixtures there are subtracted known forms of signals with the value of parameters equal to the measured ones; then the differences are passed through a matrix of linear filters possessing rejector properties with respect to interference. Finally, grouped outputs of the matrix of filters are multiplied by the derivatives of expected forms of the signal

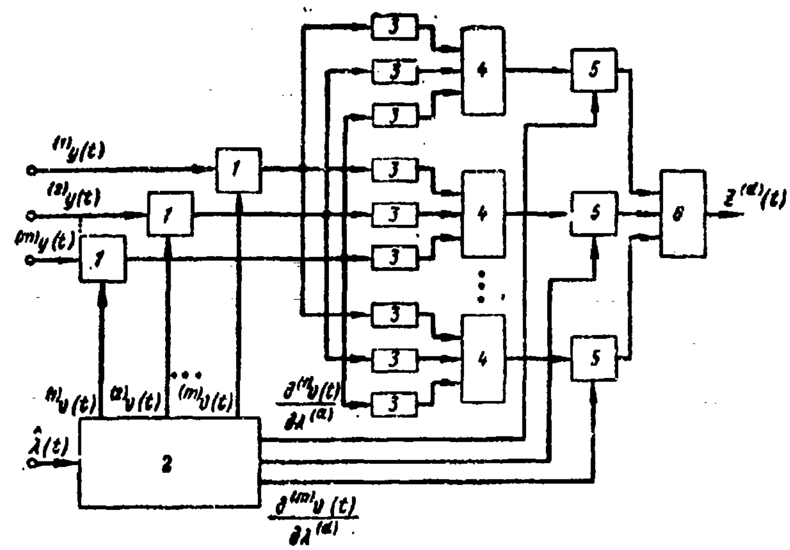

Fig. 12.10. First variant of a discriminator with regular signals: 1) subtractor; 2) units of formation of reference signals; 3) linear filters; 4, 6) adders; 5) multipliers.

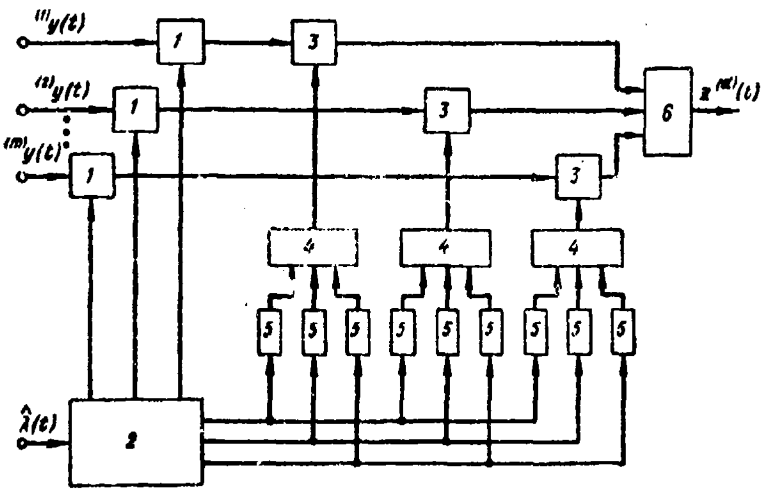

Fig. 12.11. Second variant of a discriminator with regular signals: 1) subtractor; 2) unit of formation of reference signals; 3) multipliers; 4. 6) adders; 5) linear filters.

with respect to the measured parameter $\frac{\partial^{(1)}v(t, \hat{\lambda})}{\partial\lambda^{(\alpha)}}$, and then are added. For any of the parameters common are elements up to the output of the matrix of rejector filters; different only are derivatives with respect to $\lambda^{(\alpha)}$ and thereby the final sums. The other modification (Fig. 12.10) [sic] differs from the first in that rejection of interference is carried out in circuits of feeding of derivatives,

which is absolutely equivalent in the final result.

Elements of the matrix reflecting accuracy properties of the discriminator are expressed through second derivatives of (12.4.2):

$$K^{(\alpha\beta)}=\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^T\frac{\partial}{\partial\lambda^{(\alpha)}}\mathbf{v}^+(t_1,\lambda)\,\mathbf{W}_{\text{III}}(t_1,t_2)\frac{\partial}{\partial\lambda^{(\beta)}}\mathbf{v}(t_2,\lambda)\,dt_1dt_2 \qquad (12.4.5)$$

and are determined by properties of the interference and the signals. Due to possible crosscorrelatedness of interferences in various mixtures, expressed in the nondiagonalness of matrices $\mathbf{R}_{\text{III}}(t_1, t_2)$ and $\mathbf{W}_{\text{III}}(t_1, t_2)$, for diagonalness of matrix $\mathbf{K}$ it is still insufficient that each signal depends on only one parameter. Only if the signals are broken up into l groups ${}^{(1,1)}v(t)$, ..., ${}^{(1,m_1)}v(t)$, ..., ${}^{(l,1)}v(t)$, ..., ${}^{(l,m_l)}v(t)$ (where $m_1 + m_2 + \ldots + m_l = m$), depending each on its own parameter, and noises between channels are not correlated, do we have

$$K^{(\alpha\beta)}=K^{(\alpha)}\delta_{\alpha\beta},$$

$$K^{(\alpha)}=\lim_{T\to\infty}\frac{1}{T}\int\frac{\partial}{\partial\lambda^{(\alpha)}}{}^{(\alpha)}\mathbf{v}^+(t_1,\lambda^{(\alpha)})\,\mathbf{W}_{\text{III}}^{(\alpha)}(t_1,t_2)\frac{\partial}{\partial\lambda^{(\alpha)}}\mathbf{U}(t_2,\lambda^{(\alpha)})dt_1dt_2, \qquad (12.4.6)$$

where ${}^{(\alpha)}\mathbf{v}(t) = \{{}^{(\alpha,1)}v(t), \ldots, {}^{(\alpha,m_\alpha)}v(t)\}$ – signals of the α-th group.

With a pulse signal with period T_r only the form of operation of the discriminator changes:

$$z_k^{(\alpha)}=\iint\limits_{(k-1)T_r}^{kT_r}\frac{\partial\mathbf{v}^+(t_1,\hat{\lambda}(t_k))}{\partial\lambda^{(\alpha)}}\mathbf{W}_{\text{III}}(t_1,t_2)\,[\mathbf{y}(t_2)-\mathbf{v}(t_2,\hat{\lambda}(t_k))]\,dt_2, \qquad (12.4.7)$$

which includes additionally accumulation inside the k-th period. Accuracy characteristic (12.4.5) remains the same (for $T = T_r$).

In the case of stationary interferences, by introducing their spectral matrix $S_{\text{III}}(\omega)$ it is easy to obtain

$$z_k=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\partial}{\partial\lambda^{(\alpha)}}\mathbf{V}(\omega;\hat{\lambda})\mathbf{S}_{\text{III}}^{-1}(\omega)\,[\mathbf{Y}(\omega)-\mathbf{V}(\omega;\hat{\lambda})]\,d\omega, \qquad (12.4.8)$$

where $\mathbf{V}(\omega)$, $\mathbf{Y}(\omega)$ – columns of spectra of signals $\mathbf{v}(t)$ and input realizations $\mathbf{y}(t)$ in the k-th period.

Formula (12.4.8) graphically illustrates rejector properties of the matrix of filters.

12.4.2. Gaussian Coherent Signals

As also in § 6.7, for generality we consider the case when each of the input voltages has the form

$$
{}^{(j)}y(t)=\sum_{k=1}^{p_j} {}^{(j)}u_{ak}(t;\lambda(t)){}^{(j)}E_k(t)\cos[\omega_0 t+{}^{(j)}\varphi_k(t)+
$$

$$
+{}^{(j)}\psi_k(t;\lambda(t))]+{}^{(j)}n(t)\ (j=1,\ldots,m), \qquad (12.4.9)
$$

i.e., consists of a whole set of p_j differently modulated components with various correlation properties, depending in this case on several parameters $\boldsymbol{\lambda}(t) = \{\lambda^{(1)}(t), \ldots, \lambda^{(l)}(t)\}$, and white noises ${}^{(j)}n(t)$. The whole set of functions (12.4.9) can be presented in the form

$$
\mathbf{y}(t)=\mathrm{Re}\{\mathbf{U}^+(t;\boldsymbol{\lambda}(t))\,\boldsymbol{\theta}(t)\,e^{i\omega_0 t}\}+\mathbf{n}(t), \qquad (12.4.10)
$$

where $\mathbf{U}(t)$ – complex nearly diagonal matrix of coefficients of modulation $u(t)$, whose j-th diagonal element is column

$$
\{{}^{(j)}u_1(t),\ldots,{}^{(j)}u_{p_j}(t)\},
$$

where ${}^{(j)}u_k(t) = {}^{(j)}u_{ak}(t)\exp\{i{}^{(j)}\psi_k(t)\}$; $\boldsymbol{\theta}(t)$ – complex column vector whose j-th subcolumn element is set

$$
{}^{(j)}\boldsymbol{\theta}(t)=\{{}^{(j)}E_1(t)e^{i{}^{(j)}\varphi_1(t)},\ldots,{}^{(j)}E_{p_j}(t)e^{i{}^{(j)}\varphi_{p_j}(t)}\},
$$

reflecting random modulations; $\mathbf{y}(t)$, $\mathbf{n}(t)$ – simple column vectors of order $(m \times 1)$, composed of input realizations ${}^{(i)}y(t)$ and noises ${}^{(i)}n(t)$. The correlation matrix of the whole set of mixtures (12.4.9) is equal to

$$
\mathbf{R}(t_1,t_2)=\overline{\mathbf{y}(t_1)\mathbf{y}^+(t_2)}=\mathrm{Re}\{\mathbf{U}^+(t_1,\boldsymbol{\lambda})\,\mathbf{r}(t_1-t_2)\times\mathbf{U}^*(t_2,\boldsymbol{\lambda})\,e^{i\omega_0(t_1-t_2)}\}+\mathbf{N}\delta(t_1-t_2), \quad (12.4.11)
$$

where $\mathbf{r}(t_1 - t_2) = \frac{1}{2}\overline{\theta'(t_1)\theta^+(t_2)}$ – complex matrix function of order $\left(\sum_{l=1}^{m} p_l \times \sum_{l=1}^{m} p_l\right)$ with submatrix elements ${}^{(ij)}\mathbf{r}(t_1 - t_2) = \|{}^{(ij)}r_{pq}(t_1 - t_2)\|$, where ${}^{(ij)}r_{pq}(t_1 - t_2)$ is the function of crosscorrelation of random processes ${}^{(i)}E_p(t_1)\ {}^{\cos}_{\sin}\ {}^{(i)}\varphi_p(t_1)$ and ${}^{(j)}E_q(t_2)\ {}^{\cos}_{\sin}\ {}^{(j)}\varphi_q(t_2)$, considered stationary. Matrix $\mathbf{N}$ in (12.4.11) is the matrix of spectral densities of white noises, as also in Paragraph 12.4.1, and is not necessarily diagonal.

Matrix $\mathbf{W}(t_1, t_2)$, the reciprocal of (12.4.11), which is necessary for construction of the likelihood functional, as also in § 6.7 is sought in the form

$$
\mathbf{W}(t_1,t_2)=-\mathrm{Re}\{\mathbf{N}^{-1}\mathbf{U}^+(t_1)\,\mathbf{w}(t_1,t_2)\,\mathbf{U}^*(t_2)\,\mathbf{N}^{-1}e^{i\omega_0(t_1-t_2)}\}+\mathbf{N}^{-1}\delta(t_1-t_2), \qquad (12.4.12)
$$

from which for auxiliary matrix function $\mathbf{w}(t_1, t_2)$, structurally similar to $\mathbf{r}(t_1 - t_2)$, we have integral equation

$$\frac{1}{2}\int \mathbf{r}(t_1-s)\,\mathbf{B}_0(s)\,\mathbf{w}(s,t_2)\,ds+\mathbf{w}(t_1,t_2)=\mathbf{r}(t_1-t_2), \qquad (12.4.13)$$

where $\mathbf{B}_0(s) = \mathbf{U}^*(s)\mathbf{N}^{-1}\mathbf{U}^+(s)$ – matrix, determined by the form of regular modulations and power properties of the signals.

Assuming modulation of all separate components of mixtures to be rapidly changing, it is possible to average $\mathbf{B}_0(s)$ over the time under the sign of the integral, introducing constant matrix

$$\overline{\mathbf{B}}_0=\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{U}^*(s)\mathbf{N}^{-1}\mathbf{U}^+(s)\,ds. \qquad (12.4.14)$$

Then the Fourier transform from $\mathbf{w}(t_1, t_2) = \mathbf{w}(t_1 - t_2)$ will be expressed in the form

$$\tilde{\mathbf{W}}(\omega)=\left[\mathbf{I}+\frac{1}{2}\mathbf{S}_c(\omega)\overline{\mathbf{B}}_0\right]^{-1}\mathbf{S}_c(\omega), \qquad (12.4.15)$$

where $\mathbf{S}_c(\omega)$ – spectral matrix of fluctuations of the whole set of useful signals [i.e., the Fourier transform from $\mathbf{r}(t)$]. Thus,

$$\begin{aligned} l(\mathbf{y},\boldsymbol{\lambda},t)={}&\mathrm{const}\sum_{i,j=1}^{m}\sum_{k=1}^{p_i}\sum_{l=1}^{p_j}\int_{t_0}^{t} {}^{(i)}x_k(\tau_1,\boldsymbol{\lambda}(\tau_1))\times\\ &\times {}^{(ij)}h_{kl}^{(1)}(t-\tau_1)\cos\omega_{\text{пр}}(t-\tau_1)\,d\tau_1\int_{t_0}^{t}{}^{(j)}x_l(\tau_2,\boldsymbol{\lambda}(\tau_2))\times\\ &\times {}^{(ij)}h_{kl}^{(2)}(t-\tau_2)\cos\omega_{\text{пр}}(t-\tau_2)\,d\tau_2, \end{aligned} \qquad (12.4.16)$$

where

$${}^{(i)}x_k(t,\boldsymbol{\lambda})={}^{(i)}y(t)\,{}^{(i)}u_{ak}(t,\boldsymbol{\lambda})\cos[(\omega_0+\omega_{\text{пр}})t+{}^{(i)}\psi_k(t,\boldsymbol{\lambda}(t))]$$

is the result of correlation processing of the i-th mixture for the purpose of singling out from it the k-th component of the signal and with simultaneous transfer of oscillations to intermediate frequency $\omega_{\text{пр}}$. Pulse response envelopes ${}^{ij}h_{kl}^{(1)}(t)$ and ${}^{ij}h_{kl}^{(2)}(t)$ are selected physically realizable and such that this equality is satisfied:

$$\int_{-\infty}^{+\infty}{}^{(ij)}h_{kl}^{(1)}(t_1)\,e^{-i\omega t_1}dt_1\int_{-\infty}^{+\infty}{}^{(ij)}h_{kl}^{(2)}(t_2)\,e^{i\omega t_2}dt_2={}^{ij}\tilde{w}_{kl}(\omega), \qquad (12.4.17)$$

where ${}^{(ij)}\tilde{w}_{kl}(\omega)$ – element of complex matrix $\tilde{\mathbf{w}}(\omega)$, determined by relationship (12.4.15).

Operation of the discriminator when generalized signal-to-noise ratios do not depend on $\boldsymbol{\lambda}(t)$ is expressed by a formula obtained by differentiation of (12.4.16) with respect to $\lambda^{(\alpha)}$

$$z^{(\alpha)}(t) = \text{const} \sum_{i,j=1}^{m} \sum_{k=1}^{p_i} \sum_{l=1}^{p_j} \int_{t_0}^{t} \frac{\partial}{\partial \lambda^{(\alpha)}} {}^{(i)}x_k [\tau_1, \hat{\lambda}(\tau_1)] [^{(ij)} h_{kl}^{(1)} (t-\tau_1) \times$$
$$\times \cos \omega_{\text{пр}} (t-\tau_1) \, d\tau_1 \int_{t_0}^{t} {}^{(j)}x_l [\tau_2; \hat{\lambda}(\tau_2)] [^{(ij)} h_{kl}^{(2)} (t-\tau_2) \times$$
$$\times \cos \omega_{\text{пр}} (t-\tau_2) \, d\tau_2. \qquad (12.4.18)$$

Examples of optimum processing of several input signals already were given in the preceding chapters; therefore in view of the excessive generality of case (12.4.18) the circuit of the discriminators is best given only in the study of concrete examples.

For matrix element $\overline{K(t)} = K$ in the most general case of a set of normal processes with correlation matrix $\mathbf{R}(t_1, t_2; \lambda)$ and its reciprocal $\mathbf{W}(t_1, t_2; \lambda)$ we have a formula, analogous to (5.7.65):

$$K^{(\alpha\beta)} = -\lim_{T\to\infty} \frac{1}{2T} \int_0^T \int_0^T \text{spur} \left\{ \frac{\partial}{\partial \lambda^{(\alpha)}} \mathbf{W}(t_1, t_2; \lambda) \times \frac{\partial}{\partial \lambda^{(\beta)}} \mathbf{R}^+(t_1, t_2; \lambda) \right\} dt_1 dt_2 =$$
$$= -\lim_{T\to\infty} \frac{1}{2T} \int_0^T \int_0^T \text{spur} \left\{ \frac{\partial}{\partial \lambda^{(\beta)}} \mathbf{W}(t_1, t_2; \lambda) \frac{\partial}{\partial \lambda^{(\alpha)}} \mathbf{R}^+(t_1, t_2; \lambda) \right\} dt_1 dt_2. \qquad (12.4.19)$$

In the considered case this gives

$$K^{(\alpha\beta)} = \lim_{T\to\infty} \text{Re} \frac{1}{2T} \int_0^T \int_0^T \text{spur} \, \{ \mathbf{B}_2^{(\alpha\beta)}(t_1) \, \mathbf{w}(t_1 - t_2) \overline{\mathbf{B}}_0 \mathbf{r}(t_1 - t_2) +$$
$$+ \mathbf{B}_1^{(\alpha)}(t_1) \, \mathbf{w}(t_1 - t_2) \, \mathbf{B}_1^{(\beta)}(t_2) \, \mathbf{r}(t_1 - t_2) \} \, dt_1 dt_2, \qquad (12.4.20)$$

where

$$\mathbf{B}_2^{(\alpha\beta)}(t) = \frac{\partial}{\partial \lambda^{(\alpha)}} \mathbf{U}^*(t, \lambda) \mathbf{N}^{-1} \frac{\partial}{\partial \lambda^{(\beta)}} \mathbf{U}^+(t, \lambda),$$
$$\mathbf{B}_1^{(\alpha)}(t) = \mathbf{U}(t, \lambda) \mathbf{N}^{-1} \frac{\partial}{\partial \lambda^{(\alpha)}} \mathbf{U}^{*+}(t, \lambda) \qquad (12.4.21)$$

are matrices structurally similar to matrix $\overline{\mathbf{B}}_0$.

If there exist constant limits

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{B}_2^{(\alpha\beta)}(t) \, dt = \overline{\mathbf{B}}_2^{(\alpha\beta)},$$
$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{B}_1^{(\alpha)}(t) \, dt = \overline{\mathbf{B}}_1^{(\alpha)}, \qquad (12.4.22)$$

taking into account (12.4.15) we can simplify relationship (12.4.20):

$$K^{(\alpha\beta)}=\frac{1}{4\pi}\int_{-\infty}^{+\infty}\mathrm{spur}\left\{\left[\mathbf{I}+\frac{1}{2}\mathbf{S}_c(\omega)\overline{\mathbf{B}}_0\right]^{-1}\mathbf{S}_c(\omega)\times[\overline{\mathbf{B}}_0\mathbf{S}_c^+(\omega)\overline{\mathbf{B}}_2^{(\alpha\beta)}+\overline{\mathbf{B}}_1^{(\beta)}\mathbf{S}_c^+(\omega)\overline{\mathbf{B}}_1^{(\alpha)}]\right\}d\omega. \quad (12.4.23)$$

For the simple example when $p_i = 1$, $\mathbf{N} = N_0\mathbf{I}$, and power of the i-th signal is P_i, so that ${}^{(ik)}S_c(\omega) = \frac{\sqrt{P_iP_k}}{\Delta f_c}S_0(\omega)$ $(S_0(0) = 1)$, we have

$$K^{(\alpha\beta)}=J_1^{(\alpha\beta)}(U)\,J_2(h_\Sigma, S_0), \quad (12.4.24)$$

where

$$\left.\begin{aligned}
&J_1^{(\alpha\beta)}(U)=[b_2^{(\alpha\beta)}-b_1^{(\alpha)}b_1^{(\beta)}];\\
&J_2=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{h_\Sigma^2\,S_0^2(\omega)\,d\omega}{1+h_\Sigma S_0(\omega)};\\
&b_2^{(\alpha\beta)}=\lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{k=1}^m a_k\left[\frac{\partial{}^{(k)}u_a(t)}{\partial\lambda^{(\alpha)}}\,\frac{\partial{}^{(k)}u_a(t)}{\partial\lambda^{(\beta)}}+\right.\\
&\left.+{}^{(k)}u_a^2(t)\,\frac{\partial{}^{(k)}\psi(t)}{\partial\lambda^{(\alpha)}}\,\frac{\partial{}^{(k)}\psi(t)}{\partial\lambda^{(\beta)}}\right]dt;\\
&b_1^{(\alpha)}=\lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{k=1}^m a_k\,{}^{(k)}u_a^2(t)\,\frac{\partial{}^{(k)}\psi(t)}{\partial\lambda^{(\alpha)}}\,dt;\\
&a_k=P_k/\sum_{i=1}^m P_i;\quad h_\Sigma=\sum_{i=1}^m P_i/2N_0\Delta f_c.
\end{aligned}\right\} \quad (12.4.25)$$

A sufficient (but not necessary) condition of diagonalness of matrix **K** here is the presence in each signal of only amplitude or only phase modulation, depending on no more than one parameter. But this diagonalness can also be observed on average in a number of other cases.

Absolutely in that same plan we can consider the case of weakly correlated sendings of a coherent pulse signal (see § 6.7). Subsequently we shall only discuss the limiting case of incoherent independently fluctuating Gaussian sendings.

12.4.3. Gaussian Uncorrelated Sendings

Let us consider again the case of many input mixtures ${}^{(i)}y(t)$ $(i = 1, \ldots, m)$, each of which can contain a whole series of useful components, possibly, with different forms of modulation. In distinction from Paragraph 12.4.2 we assume that fluctuations of useful components are independent from period to period, but inside the period are completely correlated. Interferences in all mixtures we shall consider white noises. Such a model is valid to equal measure for coherent and incoherent signals, since additional phase shift from incoherence already changes

nothing in the case of absence of correlation of fluctuations of signals in neighboring periods. With these assumptions it is possible to conduct synthesis of circuits, considering separate periods of repetition. Construction of the likelihood function in this case is conducted analogously to Paragraph 12.4.2. The column of the set of mixtures in a separate period will take form

$$\mathbf{y}(t) = \operatorname{Re}\{\mathbf{U}_0^+(t)\ \theta e^{i\omega_0 t}\} + \mathbf{n}(t), \tag{12.4.26}$$

where $\mathbf{U}_0(t)$ — nearly diagonal matrix of coefficients of regular modulation in one period;

θ — complex column-vector of their random amplitudes and phases: $\theta^{(k)} = \{E_1^{(k)} e^{i\varphi_1^{(k)}}, \ldots, E_{p_k}^{(k)} e^{i\varphi_{p_k}^{(k)}}\}$ (in distinction from Paragraph 12.4.2 it is constant inside the period);

$\mathbf{n}(t)$ — column vector of white noises.

The correlation matrix of mixtures and its reciprocal can be recorded

$$\begin{aligned}
\mathbf{R}(t_1, t_2) &= \operatorname{Re}\{\mathbf{U}_0^+(t_1)\, \mathbf{r}\mathbf{U}_0^*(t_2) e^{i\omega_0(t_1 - t_2)}\} + \mathbf{N}\delta(t_1 - t_2),\\
\mathbf{W}(t_1, t_2) &= -\operatorname{Re}\{\mathbf{N}^{-1}\mathbf{U}_0^+(t_1)\left[\mathbf{I} + \tfrac{1}{2}\mathbf{r}\overline{\mathbf{B}}_0\right]^{-1}\mathbf{r}\mathbf{U}_0^*(t_2) \times \\
&\times \mathbf{N}^{-1} e^{i\omega_0(t_1 - t_2)}\} + \mathbf{N}^{-1}\delta(t_1 - t_2),
\end{aligned} \tag{12.4.27}$$

where $\mathbf{r}$ — matrix $\left\{\sum_{i=1}^{m} p_i \times \sum_{i=1}^{m} p_i\right\}$ of correlation between all components of mixtures;

$$\overline{\mathbf{B}}_0 = \frac{1}{T_r}\int_0^{T_r} \mathbf{U}_0^*(s)\mathbf{N}^{-1}\mathbf{U}_0^+(s)\, ds. \tag{12.4.28}$$

In the correlation variant the operation of the discriminator according to (12.4.27) has form

$$z^{(\alpha)} = \text{const} \sum_{i,j=1}^{m} \sum_{k=1}^{p_i} \sum_{l=1}^{p_j} {}^{(ij)}w_{kl} \left[\int_0^{T_r} {}^{(i)}S_k(t, \lambda)\, dt \times \int_0^{T_r} \frac{\partial\, {}^{(j)}S_l(t, \lambda)}{\partial \lambda^{(\alpha)}} + \int_0^{T_r} {}^{(i)}C_k(t, \lambda)\, dt \int_0^{T_r} \frac{\partial\, {}^{(j)}C_l(t, \lambda)}{\partial \lambda^{(\alpha)}}\, dt, \right.$$

where

$$\begin{aligned}
{}^{(i)}S_k(t, \lambda) &= {}^{(i)}y(t)\, {}^{(i)}u_{ak}(t, \lambda) \sin[\omega_0 t + {}^{(i)}\psi_k(t, \lambda)],\\
{}^{(i)}C_k(t, \lambda) &= {}^{(i)}y(t)\, {}^{(i)}u_{ak}(t, \lambda) \cos[\omega_0 t + {}^{(i)}\psi_k(t, \lambda)]
\end{aligned}$$

are results of correlation processing.

An element of matrix $\mathbf{K}$ by the formula (12.4.19) can easily be reduced to a form analogous to (12.4.23):

$$K^{(\alpha\beta)} = \frac{1}{T_r}\operatorname{spur}\left\{\left[\mathbf{I} + \frac{1}{2}\mathbf{r}\overline{\mathbf{B}}_0\right]^{-1}\mathbf{r}\,[\overline{\mathbf{B}}_0\mathbf{r}\overline{\mathbf{B}}_2^{(\alpha\beta)} + \overline{\mathbf{B}}_1^{(\alpha)}\mathbf{r}\overline{\mathbf{B}}_1^{(\beta)}]\right\}, \tag{12.4.29}$$

where

$$\bar{\mathbf{B}}_1^{(\alpha)} = \frac{1}{T_r}\int_0^{T_r}\frac{\partial \mathbf{U}_0(s)}{\partial \lambda^{(\alpha)}}\mathbf{N}^{-1}\mathbf{U}_0^{*+}(s)ds, \quad \bar{\mathbf{B}}_2^{(\alpha\beta)} = \frac{1}{T_r}\int_0^{T_r}\frac{\partial \mathbf{U}_0(s)}{\partial \lambda^{(\alpha)}}\mathbf{N}^{-1}\frac{\partial \mathbf{U}_0^{*+}(s)}{\partial \lambda^{(\beta)}}ds \qquad (12.4.30)$$

are matrices, reflecting the coding of measured quantities in regular modulation of components of mixtures.

In that particular case when the mixtures contain only one useful component each, where these components are completely correlated, and white noises are independent, $K^{(\alpha\beta)}$ again is expressed by formula (12.4.24), where J_2 is replaced by

$$J_2 = \frac{2q_\Sigma^2}{(1+q_\Sigma)T_r}, \qquad (12.4.31)$$

and $q_\Sigma = \frac{\sum_{i=1}^{m} \mathcal{E}_i}{2N_0}$ is the ratio of total energy of the useful signal in all channels to the intensity of white noise in one of them.

12.4.4. Methods of Approximation for Construction of Discriminators and the Method of Comparison of Performance of Circuits

As also in the one-dimensional case, in engineering applications permissible and inevitable are certain deviations from optimum processing during realization of discriminators, hopefully close to optimum. In particular, a natural method of forming a discriminator is replacement of the exact expression for the derivative of $l(\mathbf{y}|\boldsymbol{\lambda})$ with respect to $\lambda^{(\alpha)}$ by a finite difference

$$z^{(\alpha)}(t) \approx \frac{1}{\Delta\lambda^{(\alpha)}}\left[l\left(t, \mathbf{y}, \hat{\boldsymbol{\lambda}} + \frac{\Delta\lambda^{(\alpha)}}{2}\right) - l\left(t, \mathbf{y}, \hat{\boldsymbol{\lambda}} - \frac{\Delta\lambda^{(\alpha)}}{2}\right)\right], \qquad (12.4.32)$$

where detuning is implied performed only with respect to the α-th parameter by a quantity smaller than the width of the peak of autocorrelation of the signal with respect to $\lambda^{(\alpha)}$. If it is technically difficult to have two channels at once, in a number of cases it is permissible to use one, alternately detuning it in different directions from the measured value of some parameter. In principle during realization of a multi-dimensional discriminator of l parameters it is possible to use just one channel if we select a switching function, which as it were "examines" points of the likelihood function removed small magnitudes of $\pm\Delta\lambda^{(\alpha)}$ ($\alpha = 1, \ldots, l$) from $\hat{\lambda}(t)$ consecutively for all parameters.

Nontrivial in the multi-dimensional case is the question of comparison of different circuits of discriminators, close to optimum or far from them, by their accuracy characteristics. As was shown in Paragraph 12.2.3, these characteristics are described by the matrix of equivalent spectral densities $\mathbf{K}^{-1}$ in the optimum

case and matrix $\mathbf{S}_{\text{ЭКВ}} = \mathbf{K}_{\text{Д}}^{-1}\mathbf{S}_0\mathbf{K}_{\text{Д}}^{-1+}$ in the general case. The geometric interpretation of these matrices can be, as it is known, an ellipsoid of dispersion in the parameter space with the same second m on all axes. The first rough method of comparison of geometric dimensio the ellipsoid is comparison of moments on the axes $\lambda^{(\alpha)}$ ($\alpha = 1, \ldots, l$) without taking into account mixed moments. This is sequential comparison of diagonal elements of two matrices. If we need more precise comparison it is possible to compare volumes of the ellipsoids, expressed through determinants of the matrices. If both these methods indicate approximate coincidence of $\mathbf{S}_{\text{ЭКВ}}$ with $\mathbf{K}^{-1}$, further attempts to improve processing, naturally, are unnecessary.

§ 12.5. Synthesis of Smoothing Circuits and Resultant Accuracy of Measurement

As also in the one-dimensional case, synthesis of smoothing circuits is analogous (for the two-loop variant of the meter) to synthesis of multi-dimensional Wiener filters. However the equivalent Wiener problem is very complicated, inasmuch as in it, it is necessary to consider the interconnection not only of measured parameters, but also of interferences superimposed on them. Such questions have been insufficiently studied in the literature. Therefore, the material below is arranged in separate parts so that it will be useful to those who are interested in the purely Wiener problem. Different classes of correlation matrices studied below are fully analogous to different correlation functions of the measured parameters considered for the one-dimensional case in § 6.8.

12.5.1. Parameters — Random Processes with Stationary Increments

We start our consideration with the case of stationary random processes, included as a subclass in random processes with stationary increments. Equation (12.3.19) for $\overline{\mathbf{K}(t)} = \mathbf{K}$ here has the form

$$\mathbf{c}(t, \tau) + \int_{t_0}^{t} \mathbf{c}(t, s)\mathbf{K}\mathbf{R}(s-\tau)\,ds = \mathbf{R}(t-\tau) \qquad (12.5.1)$$

and is an integral matrix equation (or system of integral equations) with a nucleus depending on the difference of arguments. Methods of solutions of such equations were studied, for instance, in [55]. Exact solution for an arbitrary time of observation in principle can be obtained by a method analogous to that described in § 6.6, i.e., by transition to the proper system of differential equations. However to solve these equations is complicated even in the case of two parameters. Therefore, it is useful to immediately pass to a study of limiting operators,

corresponding to a time of observation large as compared to the interval of correlation of parameters, when equation (12.5.1) passes into the matrix generalization of the Wiener-Hopf equation:

$$\mathbf{c}(t)+\int_0^\infty \mathbf{c}(\tau)\,\mathbf{KR}(t-\tau)\,d\tau=\mathbf{R}(t)\quad (t>0), \qquad (12.5.2)$$

inasmuch as $\mathbf{c}(t)$ is sought as a function of the difference of its own arguments. Direct application of Fourier transformation to (12.5.2) leads to a solution in the class of physically unrealizable filters, for which $\mathbf{c}(t) \neq 0$ when $t < 0$.*

To find a physically realizable solution in the literature there have been offered several methods. Below we give them names, not to any extent conventional.

a. The method of factoring a matrix [55, 56], having the most general significance, is based on presentation $\mathbf{I} + \mathbf{KS}(\omega)$, where $\mathbf{S}(\omega)=\int_{-\infty}^{+\infty}\mathbf{R}(t)e^{-i\omega t}dt$ is the spectral matrix of parameters, in the form of the product

$$\mathbf{I}+\mathbf{KS}(\omega)=\mathbf{\Psi}_+(\omega)\,\mathbf{\Psi}_-(\omega), \qquad (12.5.3)$$

where $\mathbf{\Psi}_-(\omega)$, $\mathbf{\Psi}_+(\omega)$ – matrix functions with analytic elements and determinants differing from zero correspondingly in the upper and lower half-planes of the complex variable ω.

The fundamental potential of factoring is proved with very wide assumptions about matrix $\mathbf{I} + \mathbf{KS}(\omega)$. The physically realizable solution of (12.5.2) has Fourier transform

$$\mathbf{C}(i\omega)=[\mathbf{S}(\omega)\,\mathbf{\Psi}_-^{-1}(\omega)]_+\,\mathbf{\Psi}_+^{-1}(\omega)=\frac{1}{2\pi}\int_0^\infty e^{-i\omega t}dt\int_{-\infty}^{+\infty}e^{iut}du\,\mathbf{S}(u)\,\mathbf{\Psi}_-^{-1}(u)\,\mathbf{\Psi}_+^{-1}(\omega), \qquad (12.5.4)$$

where $[\;]_+$ signifies the operation of taking that part of the expression in parentheses which has poles only in the upper half-plane of ω. Analytically, forming of $\mathbf{C}(i\omega)$ from the factored matrix is presented in (12.5.4) by two integrals.

Practically, however, factoring of (12.5.3) is a difficult problem. With rational spectral matrices, the most important for applications, a series of other methods turn out to be more convenient.

b. The method of factoring the determinant [57] is based on the expansion

$$\det[\mathbf{I}+\mathbf{KS}(\omega)]=\psi_+(\omega)\,\psi_-(\omega), \qquad (12.5.5)$$

where $\psi_+(\omega)\psi_-(\omega)$ – scalar factors with the same properties, as $\mathbf{\Psi}_+(\omega)$, $\mathbf{\Psi}_-(\omega)$ in

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*In principle operations of "physically unrealizable" filters can be executed by means of recording the input realization and subsequent processing, but we are interested in filtration with direct delivery of the result.

(12.5.3). The solution has the form

$$\mathbf{C}(i\omega)=[\mathbf{S}(\omega)[\mathbf{I}+\mathbf{KS}(\omega)]^{-1}\psi_+(\omega)]_+\frac{1}{\psi_+(\omega)}+\frac{1}{\psi_+(\omega)}\quad \mathbf{C}_0(i\omega), \tag{12.5.6}$$

where $\mathbf{C}_0(i\omega)$ – matrix with elements in the form $\sum_i \frac{A_i^{\alpha\beta}}{\omega-\gamma_i}$;

γ_i – the pole of matrix $[\mathbf{I} + \mathbf{KS}(\omega)]^{-1}\psi_+(\omega)\psi_-(\omega)$ in the upper half-plane of ω;

$A_i^{(\alpha\beta)}$ – undetermined factors, selected from condition

$$[\mathbf{C}(i\omega)[\mathbf{I}+\mathbf{KS}(\omega)]_+=[\mathbf{S}(\omega)]_+. \tag{12.5.7}$$

c. The method of undetermined coefficients [58] consists of seeking $C^{(\alpha\beta)}(i\omega)$ in the form of rational functions of $\sum_{k=1}^{m-1}\frac{B_k^{(\alpha\beta)}\omega^k}{\prod_{i=1}^{m}(\omega-\gamma_i)}$, where γ_i – all zeroes of the numerator of $\psi_+(\omega)$ from (12.5.5). Undetermined factors $B_k^{(\alpha\beta)}$ will be selected from the condition of turning into zero after reduction to common denominators of all numerators of elements of matrix $\mathbf{C}(i\omega)[\mathbf{I} + \mathbf{KS}(\omega)] - \mathbf{S}(\omega)$ at points $\omega = \gamma_i$ and $\omega = \delta_i$, where δ_i – pole of elements $S(\omega)$ in the upper half-plane.

Formulas (12.5.3)-(12.5.7) are applicable also to the general case of parameters with stationary increments of the k-th order if we introduce their spectral densities according to [59] and during factoring expand factors ω^{2k} in denominators of elements and the determinant of $\mathbf{I} + \mathbf{KS}(\omega)$ as $\omega^{2k} = (i\omega)^k(-i\omega)^k$. It sometimes is more convenient to consider processes with stationary increments as limits from certain stationary processes with the same orders of denominators of the spectral matrix and to pass to the limit after obtaining the solution of (12.5.4) for auxiliary processes.

As a first example we shall consider a joint meter of l parameters in the form of stationary processes of very simple form with complete correlation, where $S^{(\alpha\beta)}(\omega) = \sigma_\alpha\sigma_\beta[1 + (\omega T)^2]^{-1}$. Seeking our solution in the form $\mathbf{C}(i\omega) = C(i\omega)\|\sigma_\alpha\sigma_\beta\|$, it is easy to obtain

$$\mathbf{C}(i\omega)=\frac{2T\,\|\sigma_\alpha\sigma_\beta\|}{(1+a)(a+i\omega T)},\quad \Sigma_{\text{min}}=\frac{2\,\|\sigma_\alpha\sigma_\beta\|}{1+a},$$

$$\mathbf{G}(i\omega)=\frac{2T\,\|\sigma_\alpha\sigma_\beta\|}{(1+a)(1+i\omega T)},\quad a=\sqrt{1+2T\sum_{\alpha,\beta=1}^{l}K^{(\alpha\beta)}\sigma_\alpha\sigma_\beta}. \tag{12.5.8}$$

Smoothing circuits (Fig. 12.12) consist here of l inertialess input amplifiers, an adder, a common first-order link (RC-circuit) with time constant T, met in the

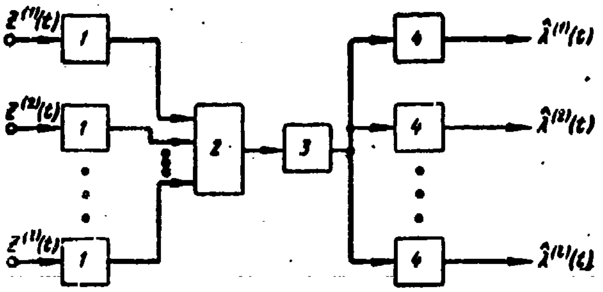

Fig. 12.12. Smoothing circuits for stationary parameters with total correlation: 1, 4) inertialess amplifiers; 2) adder; 3) inertial link of the first order.

spectral matrix, and l inertialess output amplifiers. As compared to separate measurement of the α-th parameter in the shown circuit we can obtain gain in variance of a factor

$$\left(1+2T\sum_{\alpha,\beta=1}^{l}K^{(\alpha\beta)}\sigma_\alpha\sigma_\beta\right)^{1/2}\times\left(1+2TK^{(\alpha\alpha)}\sigma_\alpha^2\right)^{-1/2},$$

considerable when $\sum_{\alpha,\beta=1}^{l}K^{(\alpha\beta)}\sigma_\alpha\sigma_\beta\gg$ $\gg K^{(\alpha\alpha)}\sigma_\alpha^2\gg 1$, i.e., when the α-th parameter has smaller variance, and its coding in the signal is more strongly suppressed by interferences than for other parameters.

A second example pertains to the analogous case of l parameters in the form of Wiener processes with total correlation, when $S^{(\alpha\beta)}(\omega) = \sqrt{B_\alpha B_\beta}/\omega^2$. The same method we obtain

$$\mathbf{C}(i\omega)=\frac{\left\|\sqrt{B_\alpha B_\beta}\right\|}{\sum_{\gamma,\delta=1}^{l}K^{(\gamma\delta)}\sqrt{B_\gamma B_\delta}+i\omega},\quad \mathbf{G}(i\omega)=\frac{\left\|\sqrt{B_\alpha B_\beta}\right\|}{i\omega},$$

$$\mathbf{\Sigma}_{\text{вых}}=\frac{\left\|\sqrt{B_\alpha B_\beta}\right\|}{\sum_{\gamma,\delta=1}^{l}K^{(\gamma\delta)}\sqrt{B_\gamma B_\delta}}. \tag{12.5.9}$$

Change of structure of the smoothing circuits as compared to Fig. 12.12 consists only in replacement of the common RC-circuit by an ideal integrator. Great gain in accuracy as compared to separate measurement of the α-th parameter is observed when

$$\sum_{\alpha,\beta=1}^{l}K^{(\alpha\beta)}\sqrt{B_\alpha B_\beta}\gg K^{(\alpha\alpha)}B_\alpha.$$

A third example pertains to simultaneous measurement of a parameter in the form of a Wiener process and the integral of it, when the correlation matrix has the form

$$\mathbf{S}(\omega)=\begin{Vmatrix} B/\omega^2 & iB/\omega^3 \\ -iB/\omega^3 & B/\omega^4 \end{Vmatrix}. \tag{12.5.10}$$

By the method of undetermined coefficients with diagonalness of matrix $\mathbf{K}$ we can obtain

$$C(i\omega)=\frac{B}{(g_1+g_2)\left[g_2+\sqrt{g_1+g_2}\,i\omega+(i\omega)^2\right]}\times$$
$$\times\left\|\begin{matrix} g_2+\sqrt{g_1+g_2}\,i\omega & i\omega \\ \sqrt{g_1+2g_2}+i\omega & \frac{g_1+g_2}{g_2}+\frac{\sqrt{g_1+2g_2}}{g_2}\,i\omega \end{matrix}\right\|, \qquad (12.5.11)$$

$$\Sigma_{вых}=\frac{B}{g_1+g_2}\left\|\begin{matrix} \sqrt{g_1+2g_2} & 1 \\ 1 & \sqrt{g_1+2g_2}/g_2 \end{matrix}\right\|, \qquad (12.5.12)$$

$$G(i\omega)\,K=\left\|\begin{matrix} \frac{g_1}{g_1+g_2}\,\frac{\sqrt{g_1+2g_2}}{i\omega} & \frac{g_2}{g_1+g_2}\,\frac{g_2}{i\omega} \\ \frac{g_1}{g_1+g_2}\,\frac{\sqrt{g_1+2g_2}+i\omega}{(i\omega)^2} & \frac{g_2}{g_1+g_2}\,\frac{g_2+\sqrt{g_1+2g_2}\,i\omega}{(i\omega)^2} \end{matrix}\right\|, \qquad (12.5.13)$$

where $g_1 = BK_1$, $g_2 = (BK_2)^{1/2}$.

According to (12.5.13) smoothing circuits of the single-loop variant of the meter have the form of Fig. 12.13. The estimate of the Wiener process will be formed by integration (with various amplification) of both output voltages of the discriminator and addition of the results; obtaining of the estimate of the integral of the first parameter anticipates transmission of output voltages of integrators through additional integrators with correcting RC-circuits with time constants $(g_1 + 2g_2)^{-1/2}$ and $(g_1 + 2g_2)^{1/2}g_2^{-1}$ and again addition of results. Reduction of smoothing circuits for cases of suppression by noises of the coding of one of the parameters ($K_1 = 0$, $K_2 \neq 0$ or $K_1 \neq 0$, $K_2 = 0$) is presented in Fig. 12.14. However only when $K_1 = 0$, $K_2 \neq 0$ do all errors of measurement remain finite; in the opposite case ($K_1 \neq 0$, $K_2 = 0$) finite only is error of measurement of the Wiener process, while error of determination of the integral of it grows infinitely in time. This corresponds to a phenomenon known in technology, consisting in the fact that integration of speed for determination of a coordinate leads to gradual uncompensated increase of error.

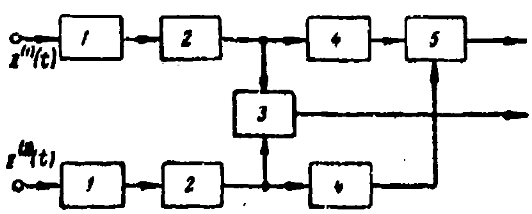

Fig. 12.13. Smoothing circuits for meter of position and speed: 1) inertialess amplifiers; 2) integrators; 3, 5) adders; 4) integrators with correction.

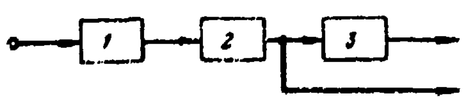

Fig. 12.14. Simplified smoothing circuits: 1) inertialess amplifier; 2) integrator; 3) integrator with correction.

In the general case ($K_1 \neq 0$, $K_2 \neq 0$) the gain as compared to the case of separate closing of smoothing circuits is given by formulas

$$\frac{(\sigma_1^2)_{\text{разд}}}{(\sigma_1^2)_{\text{совм}}}=\frac{1+\frac{g_2}{g_1}}{\sqrt{1+2\frac{g_2}{g_1}}},\quad \frac{(\sigma_2^2)_{\text{разд}}}{(\sigma_2^2)_{\text{совм}}}=\frac{1+\frac{g_1}{g_2}}{\sqrt{1+\frac{g_1}{2g_2}}}. \qquad (12.5.14)$$

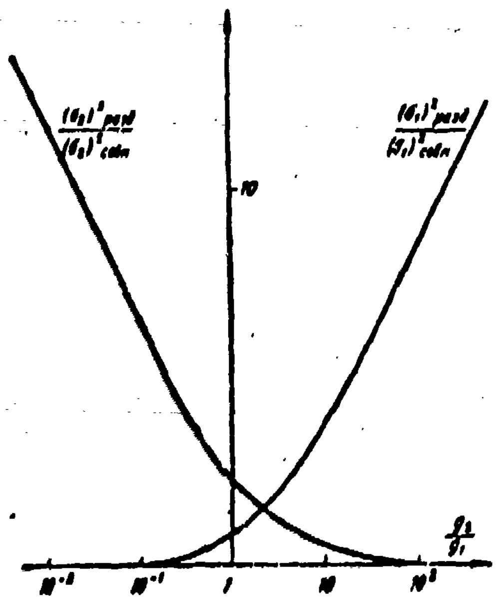

Fig. 12.15. Gain in variance during joint measurement.

Depending upon ratio g_2/g_1 it is observed for the first or second parameters, and when $\max\left\{\frac{g_1}{g_2},\frac{g_2}{g_1}\right\}\gg 1$ it is very marked (Fig. 12.15).

We shall also consider parameters in the form of correlated Wiener processes with spectral matrix

$$\mathbf{S}(\omega)=\begin{Vmatrix}\frac{B_1}{\omega^2} & \varkappa\frac{\sqrt{B_1B_2}}{\omega^2}\\ \varkappa\frac{\sqrt{B_1B_2}}{\omega^2} & \frac{B_2}{\omega^2}\end{Vmatrix} \qquad (12.5.15)$$

with arbitrary interconnection of codings

$$\mathbf{K}=\begin{Vmatrix}K_1 & \mu\sqrt{K_1K_2}\\ \mu\sqrt{K_1K_2} & K_2\end{Vmatrix}.$$

Coefficients of matrices $\mathbf{C}(i\omega)$, $\Sigma_{\text{BЫX}}$ and $\mathbf{G}(i\omega)$ we find by the method of undetermined coefficients:

$$\mathbf{C}(i\omega)=\frac{1}{(a+i\omega)(b+i\omega)}\times$$

$$\times\begin{Vmatrix}B_1\left[\sqrt{\frac{g_2}{g_1}}\beta+\frac{\left(1+\sqrt{\frac{g_2}{g_1}}\beta\right)i\omega}{a+b}\right] & \sqrt{B_1B_2}\left[-\mu\beta+\frac{\varkappa-\mu\beta}{a+b}i\omega\right]\\ \sqrt{B_1B_2}\left[-\mu\beta+\frac{\varkappa-\mu\beta}{a+b}i\omega\right] & B_2\left[\sqrt{\frac{g_1}{g_2}}\beta+\frac{\left(1+\sqrt{\frac{g_1}{g_2}}\beta\right)i\omega}{a+b}\right]\end{Vmatrix}, \qquad (12.5.16)$$

$$\Sigma_{\text{вых}}=\frac{1}{a+b}\begin{Vmatrix}B_1\left(1+\sqrt{\frac{g_2}{g_1}}\beta\right) & \sqrt{B_1B_2}(\varkappa-\mu\beta)\\ \sqrt{B_1B_2}(\varkappa-\mu\beta) & B_2\left(1+\sqrt{\frac{g_1}{g_2}}\beta\right)\end{Vmatrix}, \qquad (12.5.17)$$

$$\mathbf{G}(i\omega)=\frac{1}{(a+b)i\omega}\begin{Vmatrix}B_1\left(1+\sqrt{\frac{g_2}{g_1}}\beta\right) & \sqrt{B_1B_2}(\varkappa-\mu\beta)\\ \sqrt{B_1B_2}(\varkappa-\mu\beta) & B_2\left(1+\sqrt{\frac{g_1}{g_2}}\beta\right)\end{Vmatrix}. \qquad (12.5.18)$$

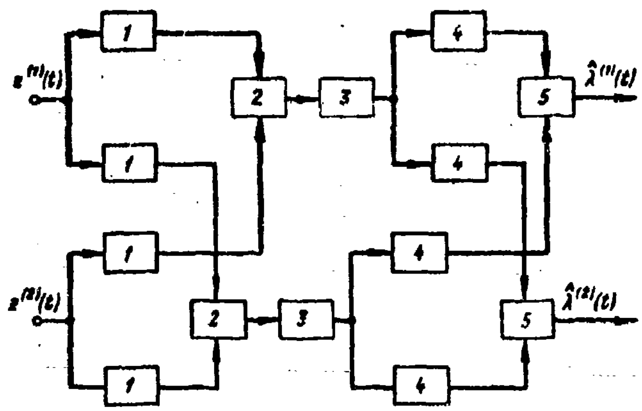

Fig. 12.16. Smoothing circuits for two correlated Wiener processes: 1, 4) inertialess amplifiers; 2, 5) adders; 3) integrators.

where $\beta = \sqrt{\frac{1-\varkappa^2}{1-\mu^2}}$; $g_i = K_i B_i$ ($i = 1, 2$), and $-a^2$, $-b^2$ are roots of equation

$$x^2 + (g_1 + g_2 + 2\mu\varkappa\sqrt{g_1 g_2})\,x + g_1 g_2 (1-\mu^2)(1-\varkappa^2) = 0.$$

Smoothing circuits of the single-loop variant according to (12.5.18) consist of two integrals [sic], to each of which there are fed through inertialess amplifiers output voltages of the discriminator $z^{(1)}(t)$ and $z^{(2)}(t)$ (Fig. 12.16).

According to (12.5.17) the gain provided by allowance for interconnection of parameters and their codings with respect to mean square error of measurement of the first parameter comprises

$$\frac{(\sigma_1)_{\text{разд}}}{(\sigma_1)_{\text{совм}}} = \left[1 + \frac{\nu^2}{4\mu^2} + \nu\varkappa + \frac{\nu}{\mu}(1-\mu^2)(1-\varkappa^2)\right]^{1/4} \times \left[1 + \frac{\nu}{2\mu}\sqrt{\frac{1-\varkappa^2}{1-\mu^2}}\right]^{-1/2},$$

where it is considered that $g_2/g_1 = \nu^2/4\mu^2$ ($\nu^2 = B_2/B_1$).

If the influence of the interconnection of coding here is nonessential when $\mu < 0.1$, strong correlation of parameters ($\varkappa > 0.5$) ensures noticeable gains (Fig. 12.17), especially with great strength of the second parameter ($\nu > 1$). This also proves the advantage of joint measurement.

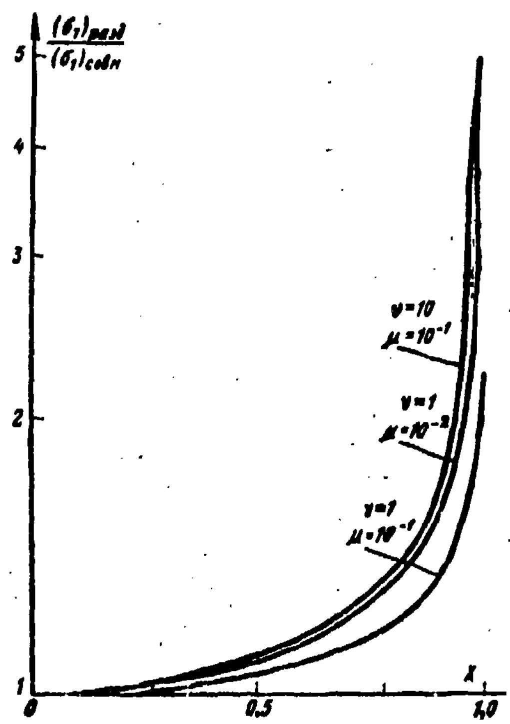

Fig. 12.17. Gain in mean square error during joint measurement.

12.5.2. Parameters – Linear Combinations of Known Functions with Random Factors

In a number of cases, especially during measurement of coordinates of bodies travelling by determined laws, presentation of measured parameters in the form of quasiregular (degenerate) random processes is permissible:

$$\lambda^{(\alpha)}(t)=\sum_{k=1}^{q_\alpha}\mu_k^{(\alpha)}f_k^{(\alpha)}(t)+\overline{\lambda^{(\alpha)}(t)}, \tag{12.5.19}$$

where $f_k^{(\alpha)}(t)$, $\overline{\lambda^{(\alpha)}(t)}$ – known functions;

$\mu_k^{(\alpha)}$ – random normally distributed quantities ($\overline{\mu^{(\alpha)}} = 0$, $\overline{\mu_k^{(\alpha)}\mu_j^{(\beta)}} = M_{kj}^{(\alpha\beta)}$, $\alpha, \beta = 1, \ldots, l$; $k = 1, \ldots, q_\alpha$; $j = 1, \ldots, q_\beta$).

The correlation matrix for such parameters in general can be expressed in the form

$$\mathbf{R}(t, \tau)=\mathbf{F}^{+}(t)\mathbf{M}\mathbf{F}(\tau) \tag{12.5.20}$$

where $\mathbf{F}(t)$ – complex nearly diagonal matrix with elements in the form of columns $\mathbf{F}^{(\alpha\beta)}(t) = \mathbf{F}^{(\alpha)}\delta_{\alpha\beta}$, $\mathbf{F}^{(\alpha)}(t) = \{f_1^{(\alpha)}(t), \ldots, f_{q_\alpha}^{(\alpha)}(t)\}$,

$\mathbf{M}$ – complex matrix $\left(\sum_{\alpha=1}^{l}q_\alpha\times\sum_{\alpha=1}^{l}q_\alpha\right)$ with submatrices $\mathbf{M}^{(\alpha\beta)} = \|M_{kj}^{(\alpha\beta)}\|$.

Then from (12.3.19), (12.3.20) and (12.5.20) we can obtain

$$\begin{aligned}\mathbf{c}(t, \tau)&=\mathbf{F}^{+}(t)[\mathbf{M}^{-1}+\mathbf{U}(t)]^{-1}\mathbf{F}(\tau),\\ \mathbf{g}(t, \tau)&=\mathbf{F}^{+}(t)[\mathbf{M}^{-1}+\mathbf{U}(\tau)]^{-1}\mathbf{F}(\tau),\\ \mathbf{\Sigma}_{\text{вых}}(t)&=\mathbf{F}^{+}(t)[\mathbf{M}^{-1}+\mathbf{U}(t)]^{-1}\mathbf{F}(t),\end{aligned} \tag{12.5.21}$$

where $\mathbf{U}(t) = \int_{t_0}^{t}\mathbf{F}(s)\mathbf{K}\mathbf{F}^{+}(s)ds$ – block matrix $\left(\sum_{\alpha=1}^{l}q_\alpha\times\sum_{\alpha=1}^{l}q_\alpha\right)$.

If between coefficients $\mu_k^{(\alpha)}$, pertaining to different parameters, there exists complete correlational coupling, matrix $\mathbf{M}$ turns out to be singular, $\mathbf{M}^{-1}$ does not exist, and to find $\mathbf{c}(t, \tau)$, $\mathbf{\Sigma}_{\text{вых}}(t)$ and $\mathbf{g}(t, \tau)$ another method is more convenient. It is based on presentation of the column vector of parameters in the form

$$\boldsymbol{\lambda}(t)=\mathbf{\Phi}(t)\boldsymbol{\mu}+\overline{\boldsymbol{\lambda}}(t), \tag{12.5.22}$$

where $\mathbf{\Phi}(t)$ – matrix ($l \times q$);

q – total number of coefficients $\boldsymbol{\mu}$, through which all parameters are expressed.

This leads to the correlation matrix of parameters

$$\mathbf{R}(t, \tau)=\mathbf{\Phi}(t)\mathbf{M}_0\mathbf{\Phi}^{+}(\tau),\ \mathbf{M}_0=\overline{(\boldsymbol{\mu}\boldsymbol{\mu}^{+})}, \tag{12.5.23}$$

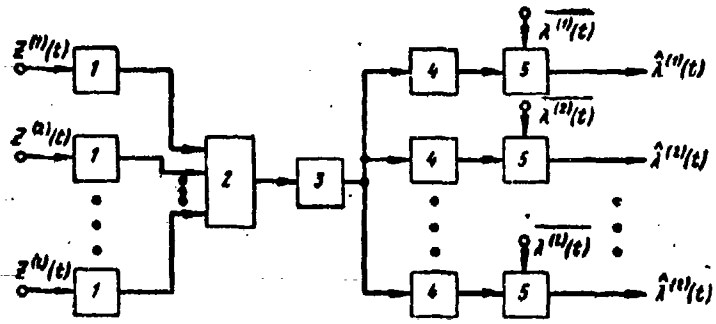

Fig. 12.18. Smoothing circuits for quasiregular processes depending on one coefficient: 1, 4) variable-gain amplifiers; 2) adder; 3) integrators; 5) adders.

and the solution of the problem is sought by a method analogous to the presented one, and gives instead of (12.5.21):

$$\begin{aligned}
\mathbf{c}(t, \tau) &= \Phi(t)\,[\mathbf{M}_0^{-1} + \mathbf{V}(t)]^{-1}\Phi^{+}(\tau),\\
\mathbf{g}(t, \tau) &= \Phi(t)\,[\mathbf{M}_0^{-1} + \mathbf{V}(\tau)]^{-1}\Phi^{+}(\tau),\\
\Sigma'_{\text{вых}}(t) &= \Phi(t)\,[\mathbf{M}_0^{-1} + \mathbf{V}(t)]^{-1}\Phi^{+}(t),\\
\mathbf{V}(t) &= \int_{t_0}^{t}\Phi^{+}(s)\,\mathbf{K}\Phi(s)\,ds.
\end{aligned} \tag{12.5.24}$$

Let us consider the particular case when all $q_\alpha = 1$, and coefficients $\mu^{(\alpha)}$ are equal to one another ($\mu^{(\alpha)} = \mu$, $\overline{\mu^2} = \sigma^2$). Using (12.5.24), it is easy to obtain

$$\begin{aligned}
g^{(\alpha\beta)}(t, \tau) &= \frac{\sigma^2 f^{(\alpha)}(t)\, f^{(\beta)}(\tau)}{1 + \sigma^2 \sum\limits_{\gamma,\delta=1}^{l} K^{(\gamma\delta)} \int\limits_{t_0}^{\tau} f^{(\gamma)}(s)\, f^{(\delta)}(s)\, ds},\\
\Sigma^{(\alpha\beta)}(t) &= \frac{\sigma^2 f^{(\alpha)}(t)\, f^{(\beta)}(t)}{1 + \sigma^2 \sum\limits_{\gamma,\delta=1}^{l} K^{(\gamma\delta)} \int\limits_{t_0}^{t} f^{(\gamma)}(s)\, f^{(\delta)}(s)\, ds}.
\end{aligned} \tag{12.5.25}$$

Smoothing circuits (Fig. 12.18) consist of l multipliers of output voltages of the discriminator by function $\sigma^2 f^{(\alpha)}(\tau)\left[1 + \sum\limits_{\gamma,\delta=1}^{l} K^{(\gamma\delta)} \sigma^2 \int\limits_{t_0}^{t} f^{(\gamma)}(s)\, f^{(\delta)}(s)\, ds\right]^{-1}$, an adder and an integrator. At the output of the integrator there will be formed the estimate of the unknown coefficient μ. After multiplication of this estimate by known laws of variation of parameters $f^{(\alpha)}(t)$ and addition with mean values there are formed estimates $\hat{\lambda}^{(\alpha)}(t)$. According to (12.5.25) errors of measurement of all parameters approach zero with increase of the time of observation $t - t_0$. In the general case

smoothing circuits contain $\sum_{\alpha=1}^{l} q_\alpha$ input and as many output multipliers. After multiplications of input voltages by known functions there is produced summation in groups equal to the nu r of coefficients $\mu_1^{(\alpha)}$, from the formed voltages the integrators form estimates of these coefficients, and from the estimates by formulas (12.5.19) and (12.5.22) there are formed the measured parameters.

§ 12.6. Examples of Multi-Dimensional Meters

12.6.1. Measurement of Range and Speed by Coherent Radar

Let us consider joint measurement of range d(t) and speed d'(t) by a coherent radar, working on carrier frequency $f_0 = \omega_0/2\pi$ (wavelength $\lambda = c/f_0$) with square width of the spectrum of modulation

$$\Delta f_{\kappa B} = \frac{\Delta\omega_{\kappa B}}{2\pi}, \quad \Delta\omega^2_{\kappa B} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \omega^2 |U(i\omega)|^2 d\omega. \qquad (12.6.1)$$

Signal-to-noise ratio h and the width of the spectrum of fluctuations Δf_c are determined according to Chapters VII, IX and X. Piloted targets maneuver, where radial speed can be described by a Wiener process with parameter B, expressed in m^2/sec^3. Range, naturally, is the integral of the Wiener process (second integral of white noise) with the same parameter B. The influence of angular coordinates of the target on measurement of d(t) and d'(t) we shall disregard.

According to results of Chapters VII and IX during approximation of the spectrum of fluctuations of the signal by function

$$S_0(\omega) = [1 + (\omega/2\Delta f_c)^2]^{-1} \qquad (12.6.2)$$

we have the following formulas for characteristics of optimum discriminators of a speed meter and a range finder:

$$K_\omega = \frac{1}{2\Delta f_c}\frac{h^2}{\sqrt{1+h}\,(1+\sqrt{1+h})^2},$$
$$K_\tau = 2\Delta\omega^2_{\kappa B}\Delta f_c \frac{h^2}{\sqrt{1+h}\,(1+\sqrt{1+h})}, \qquad (12.6.3)$$

where subscripts ω and τ indicate that parameters of the signal are considered the (circular) carrier frequency ω and time delay τ. Approximation (12.6.2) gives a minimum value of K_ω for large h (with fixed band Δf_c); with respect to K_τ the form of the approximation of $S_0(\omega)$ makes practically no difference.

As it was shown above, in multi-dimensional meters it is necessary to characterize the quality of a joint discriminator also by the mixed derivative of

the logarithm of the likelihood function with respect to d and d'. This derivative for a coherent signal, as one can prove by formula (12.4.20), is close to zero if many periods of repetition are correlated and the signal-to-noise ratio is not too great. Therefore, matrix **K** is completely determined by its diagonal elements (12.6.3), and the circuit of the optimum joint discriminator of range and speed consists of circuits of discriminators of parameters τ and ω simultaneously tuned to $\hat{\tau}(t)$ and $\hat{\omega}(t)$ (Fig. 12.19).

Henceforth it is convenient to consider as parameters d and d' directly. Inasmuch as

$$\tau=\frac{2d}{c},\quad \omega=2\omega_0\frac{d'}{c}, \tag{12.6.4}$$

elements of (diagonal) matrix **K** in new parameters will have the form

$$K_{d'}=K_{\omega}\left(\frac{2\omega_0}{c}\right)^2=\left(\frac{2\omega_0}{c}\right)^2\frac{1}{2\Delta f_c}\frac{h^2}{\sqrt{1+h}\,(1+\sqrt{1+h})^2},$$
$$K_d=K_{\tau}\left(\frac{2}{c}\right)^2=\left(\frac{2}{c}\right)^2 2\Delta\omega^2_{\text{к}\text{э}}\Delta f_c\frac{h^2}{\sqrt{1+h}\,(1+\sqrt{1+h})}. \tag{12.6.5}$$

Let us turn to smoothing circuits and resultant accuracy of joint measurement. Inasmuch as the matrix of spectral densities of d(t) and d'(t) has the form (12.5.10), and matrix **K** is diagonal, it is possible to directly use the solution of Paragraph 12.5.1, substituting in (12.5.11)-(12.5.13) $g_{d'} = g_1$ and $g_d = g_2$ – coefficients of the dimensionality of the square of frequency. Due to the correlatedness of parameters smoothing circuits of the joint meter (see Fig. 12.13) are tightly entwined. The measured value of speed of a piloted object will be formed by integration (with various weights) of both output voltages of the joint discriminator and addition of results. This processing is hardly obvious, but it leads to minimum mean square error of measurement of speed.

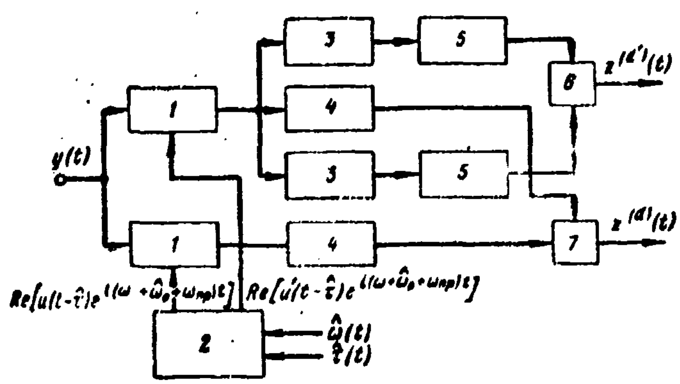

Fig. 12.19. Joint discriminator of range and speed for a coherent signal: 1) multipliers (mixers and gated amplifiers); 2) unit of formation of reference signals; 3) i-f amplifiers with detuning; 4) i-f amplifiers; 5) square-law detectors; 6) subtractor; 7) phase detector.

The measured value of range will be formed by separate transmission of output voltages of the two mentioned integrators through additional integrators, equipped

with different correcting RC-circuits, and addition of results.

The influence of one or another of the output voltages of the joint discriminator is determined by the relationship between coefficients g_d and $g_{d'}$. If $g_{d'} > g_d$, the basic role is played by the speed discriminator, and, conversely, when $g_d > g_{d'}$ the basic role is played by the range discriminator.

In particular cases when $g_d \ll g_{d'}$, or $g_{d'} \ll g_d$, smoothing circuits could have been simplified (Fig. 12.14). However, with inaccurate measurements of the range finder such simplification is technically and theoretically unjustified, inasmuch as resultant error for range grows without limit in time. A multidimensional meter, losing selection for range, also loses it for speed, as a result of which all measurement ceases. In other words, inaccurate measurement of range is better than failure of this measurement.

Otherwise, with very rough primary measurements of speed (wide spectrum of fluctuations, large wavelength, and so forth) tracking by both coordinates can be performed from readings of the range finder with finite quantities of errors of selection, if the signal-to-noise ratio h is sufficiently great.

Variances of errors for speed and range in steady-state operating conditions according to (12.5.12) are equal to

$$\sigma_{d'}^2 = \sqrt{\frac{B}{K_{d'}}}\,\frac{\sqrt{1+2/\varkappa}}{1+1/\varkappa},$$
$$\sigma_d^2 = \sqrt{2}\left(\frac{B}{K_d^3}\right)^{1/4}\frac{\sqrt{1+\varkappa/2}}{1+\varkappa}, \qquad (12.6.6)$$

where $\varkappa = \sqrt{g_{d'}/g_d}$.

Comparison of (12.6.6) with results of Chapter VI shows that as compared to separate closing of loops of range and speed tracking there is observed gain in mean square error of speed and range by factors of $(1 + 2/\varkappa)^{1/4}(1 + 1/\varkappa)^{1/2}$ and $(1 + \varkappa/2)^{1/4}(1 + \varkappa)^{-1/2}$, respectively. To find conditions of increase of accuracy for d or d' we investigate quantity $\varkappa$ as a function of tactical and technical parameters of the radar. Using (12.6.5) it is easy to prove that

$$\varkappa = \frac{\pi}{\sqrt{2}}\,\frac{\Delta f_{\text{доп}}}{\Delta f_c}\,\frac{f_c}{\Delta f_{\text{кв}}}\,\varphi(h), \qquad (12.6.7)$$

where

$$\Delta f_{\text{доп}} = \sqrt{\frac{4B}{\lambda^2 \Delta f_c}}$$

is root mean square magnitude of drift of Doppler frequency due to maneuver of the target during the interval of correlation of fluctuations of the signal;

$$\varphi(h) = \frac{h}{(1+h)^{1/4}(1+\sqrt{1+h})^{5/2}}$$

is a function of the signal-to-noise ratio h, asymptotically changing with growth and decrease of h, as

$$\varphi(h) \underset{h \to 0}{\to} 2^{-5/2}h,$$
$$\varphi(h) \underset{h \to \infty}{\to} h^{-1/2},$$

and reaching a maximum equal to 0.146 when h ≈ 6.2 (Fig. 12.20). The influence of measurements of speed is seen to the maximum degree for mean values of the signal-to-noise (h = 0.1 to 10^3), which usually are operating values. This influence is stronger, the faster the target is piloted, the more it maneuvers, the shorter the effective wavelength λ and the narrower the width of the spectra of fluctuations Δf_c and of regular modulation Δf_{KB}. Concrete values of $\varkappa/\varphi(h)$ we shall estimate for two examples.

Thus, when λ = 3 cm ($f_0 = 10^{10}$ cps), Δf_c = 30 cps, B = 30 m^2/sec^3, $\Delta f_{KB} = 10^6$ cps we have $\varkappa/\varphi(h) = 4.9 \cdot 10^4$.

In the other case, when λ = 10 cm, Δf_c = 10 cps, B = 10 m^2/sec^3, $\Delta f_{KB} = 6 \cdot 10^6$ cps, $\varkappa/\varphi(h) \simeq 2.2 \cdot 10^3$.

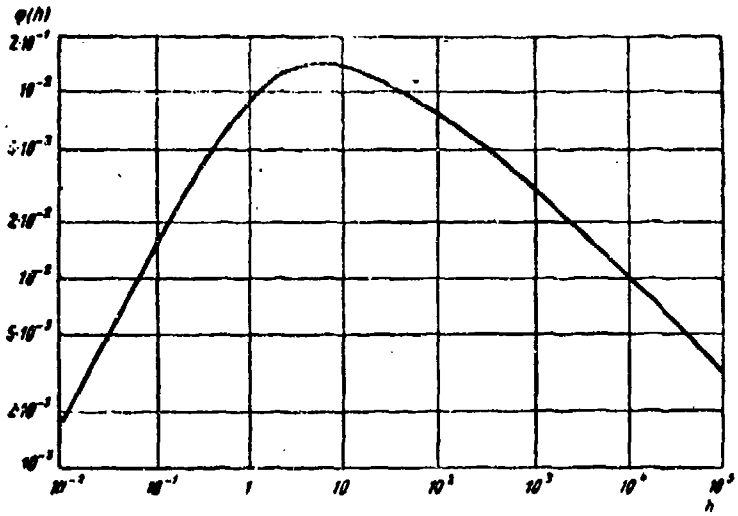

Fig. 12.20. Dependence of φ(h) on the signal-to-noise ratio.

For radars of various types and in various conditions of application $\varkappa/\varphi(h)$ varies from 10^3 and 10^5. Therefore, with h = (1-50), when $\varphi(h) \geq 0.1$ coefficient

$\varkappa$ is not less than 10^2-10^4, which gives a gain in mean square error σ_d of measurement of range of $(2\varkappa)^{1/4} = (4$-$12)$ times. In these conditions instead of (12.6.6) one should use formulas

$$\sigma_{d'}^2 = \sqrt{\frac{B}{K_{d'}}} = \frac{\lambda}{2}\sqrt{\frac{B\Delta f_c}{\pi}\frac{(1+h)^{1/4}(1+\sqrt{1+h})^{3/2}}{h}},$$

$$\sigma_d^2 = \frac{1}{\sqrt{K_d K_{d'}}} = \frac{\lambda c}{(4\pi)^2}\frac{1}{\Delta f_{\text{кв}}}\frac{(1+h)^{1/2}(1+\sqrt{1+h})^2}{h^2}. \tag{12.6.8}$$

When $h \gg 1$, as long as condition $\varkappa \gg 1$ is preserved, these approximate formulas are valid

$$\sigma_{d'}^2 = \frac{\lambda}{2}\sqrt{\frac{B\Delta f_c}{\pi}},$$

$$\sigma_d^2 = \frac{\lambda c}{(4\pi)^2}\frac{1}{\Delta f_{\text{кв}}}\frac{1}{\sqrt{h}}. \tag{12.6.9}$$

Paradoxically, error for range does not depend in the formulated conditions on average maneuverability, determined by parameter B. The fact is that with growth of maneuverability (i.e., of the magnitude of bursts of speed) the speed discriminator, basically determining accuracy of measurement, works ever more efficiently. We also notice that change of the width of the spectrum of fluctuations Δf_c leads to growth of noises at the output of the speed discriminator and decrease of them at the output of the range discriminator, so that in a joint meter there occurs compensation of two factors, and $\sigma_Д$ turns out to depend on Δf_c only implicitly, through magnitude h. Considering for example $\lambda = 3$ cm, $\Delta f_{КВ} = 10^6$ cps, $h = 100$, we have a minute noise component of error for range

$$\sigma_Д \approx 0{,}08\ м,$$

which is hundredths of the interval of range resolution. If, however, we seek the same quality of measurement of range with separate construction of the range finder and speed meter, it is necessary to expand by many times the spectrum of regular modulation.

Thus, during joint measurement of range and speed of a maneuvering target by a coherent signal it turns out to be possible to sharply increase accuracy of measurement of range due to the high accuracy of primary measurements of speed.

12.6.3 Measurement of Range and Speed by Incoherent Radar*

Let us consider now measurement of the same parameters of motion $d'(t)$ and $d(t)$ by an incoherent single-channel radar. We keep the former designations for

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*The series of results of paragraph 12.6.2 was obtained by V. K. Smirnov.

wavelength, carrier frequency and mean square width of the spectrum of modulation. The pulse signal of the radar we shall consider fluctuating independently of the period to a period of duration T_r. In distinction from Paragraph 12.6.1 we shall consider a nonmaneuvering target, flying at the radar with constant, but unknown speed and undetermined position at the initial moment. Influence of measurement of angular coordinates, as also in Paragraph 12.6.1, we shall disregard.

In examining properties of an optimum joint discriminator in accordance with the character of the signal it is permissible to consider one period of repetition. By formula (12.4.29) in conditions of a symmetric form of the sending we have matrix **K** (with parameters ω, τ) in the form

$$\mathbf{K}_{\omega\tau}=\frac{2q^2}{T_r(1+q)}\begin{Vmatrix}\tau^2_{\text{кв}} & k\tau_{\text{кв}}\omega_{\text{кв}}\\ k\tau_{\text{кв}}\omega_{\text{кв}} & \omega^2_{\text{кв}}\end{Vmatrix}, \qquad (12.6.10)$$

where $u(t) = u_a(t)e^{i\psi(t)}$ — regular modulation;

q — signal-to-noise ratio;

$$\tau_{\text{кв}}=\sqrt{\frac{1}{T_r}\int\limits_{-\infty}^{+\infty}|u(t)|^2t^2dt}$$

— mean square duration of the sending;

$$k=-\operatorname{Re}\frac{\frac{1}{T_r}\int\limits_{-\infty}^{+\infty}u'(t)u^*(t)\,it\,dt}{\tau_{\text{кв}}\omega_{\text{кв}}} \qquad (12.6.11)$$

— dimensionless coefficient of interconnection of codings of the two parameters of modulation.

It also is possible to express matrix **K** very graphically through the derivative of the autocorrelation function of the pulse sending (see Chapter I),

$$C(\tau,\omega)=\frac{1}{T_r}\int\limits_{-\infty}^{+\infty}u(t-\tau)u^*(t)e^{-i\omega t}dt \quad (C(0,0)=1), \qquad (12.6.12)$$

taken for zero arguments

$$\mathbf{K}_{\omega\tau}=-\frac{2q^2}{T_r(1+q)}\operatorname{Re}\begin{Vmatrix}\frac{\partial^2C(0,0)}{\partial\omega^2} & \frac{\partial^2C(0,0)}{\partial\tau\partial\omega}\\ \frac{\partial^2C(0,0)}{\partial\omega\partial\tau} & \frac{\partial^2C(0,0)}{\partial\tau^2}\end{Vmatrix}. \qquad (12.6.13)$$

In other words, with an accuracy of a coefficient monotonically depending on the signal-to-noise ratio q matrix $\mathbf{K}_{\omega\tau}$ consists of quantities characterizing

sharpness of the peak of the autocorrelation function in various sections. If lines of constant level $C(\tau, \omega)$ for small arguments are ellipses whose axes are oriented along axes ω, τ, the mixed derivative turns into zero, which bears witness to the absence of interconnection of codings of range and speed in the signal. Such a situation occurs, for instance, for simple pulse modulation, intrapulse triangular frequency modulation and phase-code manipulation, when integral $\int_{-\infty}^{+\infty} u'(t)\, u^*(t)\, it\,dt$ turns out to be a pure imaginary quantity or zero. In the general case the interconnection of two parameters exists, which is easy to prove by the example of linear intrapulse frequency modulation (FM)

$$u(t) = u_a(t)\, e^{\frac{iVt^2}{2}}, \tag{12.6.14}$$

where V — rate of rise (decrease) of the frequency of the filling.

Here coefficient k is proportional to the steepness of FM

$$k = V \frac{\tau_{\text{кв}}}{\omega_{\text{кв}}}, \tag{12.6.15}$$

We shall study resultant errors for d' and d for fixed duration of the sending $\tau_{\text{КВ}}$ and width of its spectrum $\omega_{\text{КВ}}$. Transition in matrix $K_{\omega\tau}$ to parameters d' and d, in accordance with (12.6.4) and (12.6.10), gives

$$\mathbf{K}_{d'd} = \begin{Vmatrix} K_{d'} & k\sqrt{K_{d'} K_d} \\ k\sqrt{K_{d'} K_d} & K_d \end{Vmatrix}, \tag{12.6.16}$$

where $K_{d'} = \frac{2q^2}{1+q} \frac{\tau^2_{\text{кв}}}{T_r} \left(\frac{2\omega_0}{c}\right)^2$; $K_d = \frac{2q^2}{1+q} \frac{\omega^2_{\text{кэ}}}{T_r} \left(\frac{2}{c}\right)^2$, and the value of k is kept equal to (12.6.11).

Let us turn to synthesis of smoothing circuits. Here it is convenient to use the method described by relationships (12.5.22)-(12.5.24). Actually,

$$\begin{gathered} d(t) = d_0 + d'_0 (t - t_0) + \overline{d(t)}, \\ d'(t) = d'_0 + \overline{d'(t)}. \end{gathered} \tag{12.6.17}$$

where $\overline{d'(t)}$, $\overline{d(t)}$ are a priori mean values of range and radial speed, and d'_0, d_0 are unknown deviations of the initial (i.e., moment t_0) speed and position with zero mean values and the matrix of moments

$$\mathbf{M}_0 = \begin{Vmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_0^2 \end{Vmatrix}.$$

It is possible to present $\lambda(t) = \{d'(t), d(t)\}$ in the form (12.5.22), where

$$\Phi(t)=\left\|\begin{matrix}1 & 0\\ t-t_0 & 1\end{matrix}\right\|, \quad \mu=\left\|\begin{matrix}d'_0\\ d_0\end{matrix}\right\|,$$

and, directly using (12.5.24), to record the solution

$$c(t,\tau)=\frac{1}{\det(t)}\left\|\begin{matrix} K_d(t-t_0)+\frac{1}{\sigma_0^2} & \begin{matrix}K_d(t-t_0)(\tau-t_0)-\frac{K_d(t-t_0)^2}{2}-\\ -k\sqrt{K_dK_{d'}}(t-t_0)+\frac{\tau-t_0}{\sigma_0^2}\end{matrix}\\ \begin{matrix}\frac{K_d(t-t_0)^2}{2}-k\sqrt{K_dK_{d'}}(t-t_0)+\\ +\frac{t-t_0}{\sigma_0^2}\end{matrix} & \begin{matrix}\frac{K_d(t-t_0)^2(\tau-t_0)}{2}-\frac{K_d(t-t_0)^3}{6}-\\ -k\sqrt{K_dK_{d'}}(t-t_0)(\tau-t_0)+\\ +K_{d'}(t-t_0)+\frac{(t-t_0)(\tau-t_0)}{\sigma_0^2}+\frac{1}{\sigma_1^2}\end{matrix}\end{matrix}\right\| \tag{12.6.18}$$

$$g(t,\tau)=\frac{1}{\det(\tau)}\left\|\begin{matrix} K_d(\tau-t_0)+\frac{1}{\sigma_0^2} & \begin{matrix}\frac{K_d(\tau-t_0)^2}{2}-k\sqrt{K_dK_{d'}}(\tau-t_0)+\\ +\frac{\tau-t_0}{\sigma_0^2}\end{matrix}\\ \begin{matrix}K_d(\tau-t_0)(t-t_0)-\frac{K_d(\tau-t_0)^2}{2}-\\ -k\sqrt{K_dK_{d'}}(\tau-t_0)+\frac{t-t_0}{\sigma_0^2}\end{matrix} & \begin{matrix}\frac{K_d(t-t_0)(\tau-t_0)^2}{2}-\frac{K_d(\tau-t_0)^3}{6}-\\ -k\sqrt{K_dK_{d'}}(t-t_0)(\tau-t_0)+\\ +K_{d'}(\tau-t_0)+\frac{(t-t_0)(\tau-t_0)}{\sigma_0^2}+\frac{1}{\sigma_1^2}.\end{matrix}\end{matrix}\right\| \tag{12.6.19}$$

where

$$\det(t)=\frac{K_d^2(t-t_0)^4}{12}+K_{d'}K_d(1-k^2)(t-t_0)^2+\\ +\frac{K_d(t-t_0)}{\sigma_1^2}+\frac{1}{\sigma_0^2}\left[K_1(t-t_0)+k\sqrt{K_dK_{d'}}(t-t_0)^2+\right.\\ \left.+\frac{K_d(t-t_0)^3}{3}\right]+\frac{1}{\sigma_0^2\sigma_1^2}.$$

The algorithm of joint processing of data from the output of the discriminator in the single-loop variant of meter is presented in Fig. 12.21, where the meaning of operations is intelligible in light of the descriptions of § 12.5.

From the matrix of resultant errors $\Sigma_{\text{вых}}(t)$ we extract elements on the main diagonal, again obtained after transition to continuous time:

$$\sigma_d^2(t)=\frac{4}{K_d(t-t_0)}\times\frac{1-3kb(t)+3b^2(t)+3b_0^2(t)+3b_1^2(t)}{1+12b^2(t)(1-k^2)+4(1+3kb(t)+3b^2(t))b_0^2(t)+12b_1^2(t)+12b_0^2(t)b_1^2(t)},$$

$$\sigma^2_{d'}(t) = \frac{12}{K_d\,(t-t_0)^3} \times$$

$$\times \frac{1+b_1^2(t)}{1+12b^2(t)\,(1-k^2)+4(1+3kb\,(t)+3b^2(t))b_0^2(t)+12b_1^2(t)+12b_0^2(t)b_1^2\,(t)}, \qquad (12.6.20)$$

where

$$b\,(t)=\sqrt{\frac{K_{d'}}{K_d}}\frac{1}{(t-t_0)}; \quad b_0(t)=\frac{1}{\sigma_0\sqrt{K_d\,(t-t_0)}};$$

$$b_1(t)=\frac{1}{\sigma_1(t-t_0)\sqrt{K_d\,(t-t_0)}} \qquad (12.6.21)$$

are coefficients depending on the time of observation and approaching zero with increase of it. The first of them b(t) is actually proportional to the ratio of unit (i.e., in a period) error of the range discriminator $\sqrt{T_r/K_d}$ to the advance of error in range during the time of observation $t - t_0$ due to unit error for speed $\sqrt{T_r/K_{d'}}$ with independent codings of $d'(t)$ and $d(t)$ in the signal:

$$b\,(t)=\frac{\tau_{\text{кв}}}{t-t_0}\cdot\frac{f_0}{f_{\text{кв}}}. \qquad (12.6.22)$$

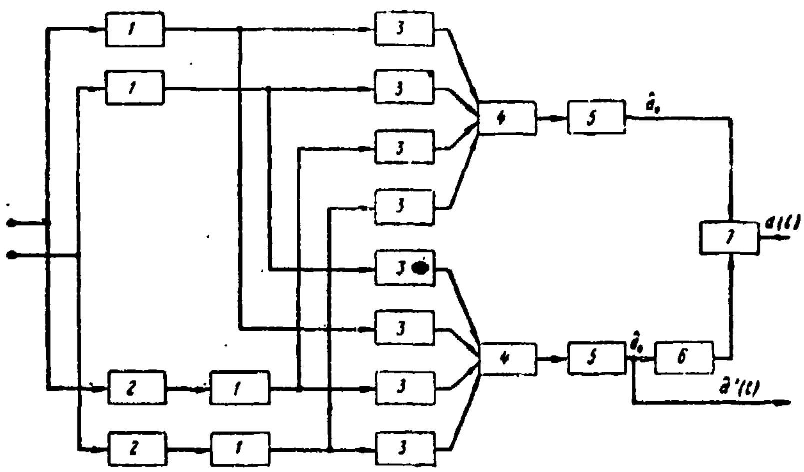

Fig. 12.21. Smoothing circuits for a meter of range and radial speed: 1) integrators; 2, 5, 6) variable-gain amplifiers; 3) inertialess amplifiers; 4, 7) adders.

The second and third coefficients $b_0(t)$ and $b_1(t)$ in (12.6.21) are determined by accuracy of a priori data and may be negligible, in any case, with a sufficiently large time of observation. We shall investigate namely the last case when, approximately, we have

$$\sigma_d^2(t) \approx \frac{4}{K_d(t-t_0)} \frac{1-3kb(t)+3b^2(t)}{1+12b^2(t)(1-k^2)},$$
$$\sigma_{d'}^2(t) \approx \frac{12}{K_d(t-t_0)^3} \frac{1}{1+12b^2(t)(1-k^2)}. \qquad (12.6.23)$$

As relationships (12.6.23) show, for resultant errors not indifferent are the absolute value and sign of the coefficient k of interconnection of codings of parameters in the signal. Let us discuss the dependence of range error $\sigma_d(t)$ on k. For the two extreme and the intermediate cases (k = -1; 0; 1) we have:

$$\sigma_d^2(t)|_{k=-1} = \frac{4}{K_d(t-t_0)}[1+3b^2(t)+3b(t)],$$
$$\sigma_d^2(t)|_{k=0} = \frac{4}{K_d(t-t_0)} \frac{1+3b^2(t)}{1+12b^2(t)}, \qquad (12.6.24)$$
$$\sigma_d^2(t)|_{k=1} = \frac{4}{K_d(t-t_0)}[1+3b^2(t)-3b(t)],$$

i.e., with increase of k from -1 to +1 error decreases. The family of dependences of ratio $\alpha = \frac{\sigma_d^2}{\sigma_d^2|_{k=0}}$ on k for several values of b is shown in Fig. 12.22.

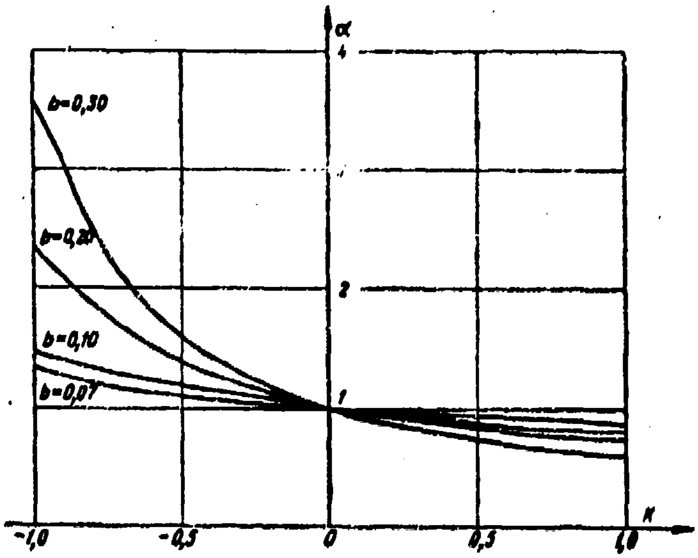

Fig. 12.22. The dependence of range error on the interconnection of coding of range and speed.

For typical values of width of the spectrum of modulation and duration of the sending (of the order of 1 Mc and 1 millisecond), wavelength 0.5-1 m and small times of observation (of the order of 10 sec) we have:

$$\frac{\sigma_d^2(t)|_{k=-1}}{\sigma_d^2(t)|_{k=1}} = 1,5 - 1,8.$$

However, with increase of the time of observation in any case error of range asymptotically seeks $4/K_d(t - t_0)$, i.e., is determined only by primary measurements of the range finder. In Fig. 12.23 is the family of curves $\sigma_d^2(t)\ \frac{K_d(t - t_0)}{4}$. Here k is a fixed parameter, $K_{d'}/K_d = 1\ \text{sec}^2$.

These results show that the dependence of error of measurement of range on coefficient k appears only with a comparatively small time of smoothing.

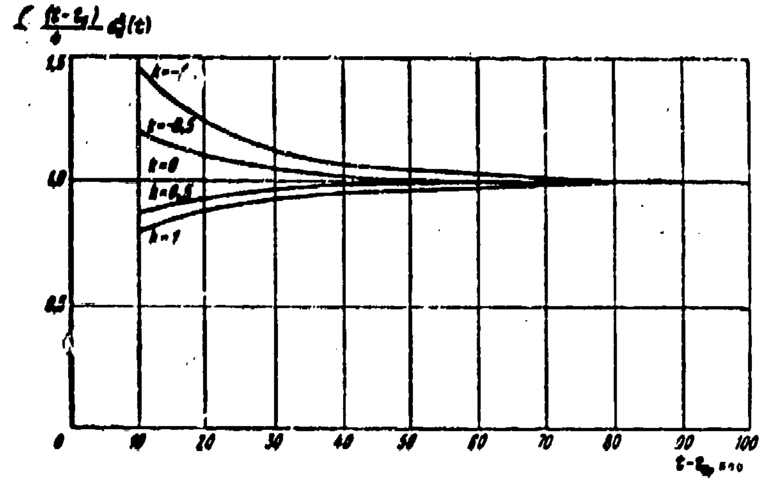

Fig. 12.23. Asymptotic behavior of range error.

On the other hand, with increase of the time of observation accuracy of measuring speed is increased. Indeed, if we use only readings of the speed meter, we have

$$[\sigma_{d'}^2(t)]_{\text{раздр}} \to \frac{1}{K_{d'}(t - t_0)},$$

and with joint measurement we have, according to (12.6.23),

$$[\sigma_{d'}^2(t)]_{\text{совм}} \to \frac{12}{K_d(t - t_0)^3},$$

as a result of which

$$\frac{[\sigma_{d'}(t)]_{\text{разд}}}{[\sigma_{d'}(t)]_{\text{совм}}} \approx \frac{1}{\sqrt{12\, b(t)}}. \qquad (12.[illegible])$$

Thus, when $K_{d'}/K_d = 1$ sec² observation for [illegible] sec gives a gain in mean square error of approximately a factor of [illegible]. This gain is explained by the fact that in a range finder for determination of speed there is the possibility of effectively

using the time base of the extended interval of measurement. The speed discriminator after expiration of the period of rough measurement of frequency due to noises ceases to feel the small remainder of the unmeasured difference of the Doppler frequency and practically does not participate in the process of further definitizing of data.

A similar effect of gradual decrease of the influence of measurements of speed is also observed with coherent radiation, for which the above mentioned results preserve their force, if in (12.6.5) as b(t) we substitute

$$b(t) = \frac{1}{2(\Delta f_c t)_-}\left(\frac{f_0}{\Delta f_{KB}}\right)\frac{1}{1+\sqrt{1+h}}\,(t_0=0). \qquad (12.6.26)$$

With growth of h and t function b(t) decreases, although for average times of observation ($\Delta f_c\ t$ = 2-30 and h ≈ 100) it may reach 15-25. This, according to the middle formula of (12.6.23) gives a gain in mean square error of range of a factor of approximately 2, inasmuch as

$$\sigma_d^2(t) \approx \frac{1}{K_d t}. \qquad (12.6.27)$$

Formula (12.6.27) is intelligible inasmuch as the range finder must definitize only the initial position of the target, and its speed in the first moments turns out to be known with sufficient accuracy.

Returning to the case of an incoherent pulse signal, we shall discuss in somewhat greater detail the case of deep linear frequency modulation ($|k| \approx 1$). Here matrix $\mathbf{K}_{d'd}$ becomes singular, and elements of matrices $\mathbf{K}^{-1}$ increase infinitely. The latter, at first sight, bears witness to huge unit errors (in recalculation of parameters to the discriminator input). In fact the case of deep FM does not give any practical inconveniences. It is possible not always to recalculate the output voltage of the discriminator of range and speed to the input, i.e., to reduce them to quantities of the dimensionality of d' and d in such a manner that in one quantity there is not contained range, and in other there is not contained speed. We encounter this case with deep FM. With increase of V outputs quantities of the discriminator z_i^d and $z_i^{d'}$ become all the more similar. The fact is that here speed is coded in frequency shift, and range is coded in frequency shift and delay of amplitude modulation of the sending:

$$u(t,\tau,\omega_Д) = u_a(t-\tau)\,e^{-\frac{iV(t-\tau)^2}{2}+i\omega_Д t} \equiv$$
$$\equiv u_a(t-\tau)\,e^{\frac{iVt^2}{2}-i(V\tau+\omega_Д)t}\,e^{\frac{iV\tau^2}{2}}.$$

With increase of the gain factor V the channel of the range discriminator, separating mismatch in amplitude modulation, takes an ever smaller weight, so that already with a width of the FM spectrum exceeding the width of the spectrum of the sending envelope by 5-10 times the additional channel can be disregarded. Then z_1^d and $z_1^{d'}$ turn out to be identical accurate to a proportionality factor ω_0/V, and for further processing it is sufficient to take only one output quantity, as a result of which the smoothing circuits are simplified somewhat.

Transition to a smaller number of output voltages of the discriminator is also useful in all cases when matrix **K** turns out to be singular, inasmuch as singularity appears due to the presence of linear interconnection between output voltages of the discriminator. The number of output quantities not accounted for is equal to the number of linear couplings superimposed on the measured quantities. Here there remain in force all the above mentioned formulas; it is necessary only to manifest caution during inversion of matrix **K**. Inasmuch as this inversion is nevertheless necessary during comparison of optimum circuits with real ones, it is necessary to pass to a smaller number of unknown parameters. In particular, in the considered example as $\lambda(t)$ it is possible to consider

$$\lambda(t)=d(t)+\frac{\omega_0}{V}d'(t)$$

or

$$\lambda(t)=d'(t)+\frac{V}{\omega_0}d(t),$$

which formally reduces the problem to a one-dimensional one.

From the practical side investigation of accuracy of measurement of range and speed in incoherent radars showed the necessity of allowance for interconnection of these parameters in trajectory and in coding in the common signal. During motion by a definite (linear) law the role of primary measurements of speed with respect to resultant accuracy decreases in time. Accuracy for range and speed is basically determined by the range finder, where in speed there is observed great gain as compared to the case of its independent measurement on the basis of the Doppler effect. For small times of observation (10-20 sec) measurements of speed increase accuracy, where the concrete gain depends on the form of modulation. Modulation with a coefficient of interconnection of errors $k \to +1$ (for instance, linear FM with increase of frequency) has the best qualities; modulation with $k \to -1$ (linear FM with decrease of frequency) has the worst qualities. All modulations with $k = 0$ (simple pulse modulation, triangular FM and phase-code intrapulse modulation) g[illegible].

intermediate results. Linear FM with variable slope gives approximately the same result.

§ 12.7. Conclusion

During the analysis of tracking multi-dimensional meters known in practice, as in the one-dimensional case, it is convenient to divide interconnected discriminators and smoothing circuits. Discrimination and fluctuation characteristics of multi-dimensional discriminators, which for l measured quantities are determined in the number l and $l(l + 1)/2$, respectively, turn out to depend in general on all l parameters. With unconnected codings of parameters and absence of coupling in smoothing circuits during the analysis of accuracy fully permissible is reduction of the multi-dimensional problem to l one-dimensional problems, which is the basis for results of Chapters VII-XI. In the general case with known characteristics of the discriminator and an assigned scheme of smoothing circuits resultant errors are found by solving integral matrix equations. In the case of smoothing circuits with constant parameters and stationary measured parameters calculation of errors is comparatively simple and includes algebraic operations on matrices of equivalent spectral densities of the multi-dimensional discriminator $\mathbf{S}_{\text{эк}}$ of spectral densities of parameters $\mathbf{S}_{\lambda}(\omega)$, the matrix of gain factors of the discriminator $\mathbf{K}_{Д}$ and the transmission matrix of smoothing circuits.

As also in one-dimensional theory, the most suitable apparatus of synthesis of optimum multi-dimensional meters is the theory of statistical solutions. There is performed synthesis of an optimum meter of any set of arbitrarily connected parameters with Gaussian and Markovian statistics with Gaussian approximation of the multi-dimensional likelihood function. There are given meter circuits equivalent in quality: single-loop and two-loop tracking circuits and a nontracking circuit.

Tracking variants, practically the most convenient, contain as basic elements multi-dimensional nonlinear units — discriminators and accuracy units — and a matrix (or two matrices) of smoothing circuits which for Gaussian statistics of parameters are linear. Output quantities of the discriminator characterize current mismatch for all parameters, and output quantities of the accuracy unit characterize current accuracy of separate measurements. With interconnected codings of parameters in signals there simultaneously appears dependence of separate output quantities of the discriminator directly on several mismatches, and interconnection of errors of measurements of the shown parameters. The latter leads to the necessity of creating a special accuracy unit characterizing the current measure of

interconnection of errors. In a number of cases, however, the unit of accuracy and control of smoothing circuits from the realization of the signal may be rejected, which leads to circuit simplifications.

Without concretization of the nature of the parameters it is possible to note a series of laws governing synthesis of discriminators for different statistical properties of input signals and synthesis of smoothing circuits for different correlation properties of the parameters.

The given examples of synthesis of joint meters show large gains in accuracy of measurement of one or another of the measured interconnected quantities. In particular, during joint measurement of speed and range of a body with determined character of motion there is noted considerable gain in speed, growing in time; during measurement of the speed and range of a maneuvering target with a coherent signal there is the possibility of obtaining great gain in range.

However the theory developed above cannot in any measure be called exhaustive. In particular, analysis of multi-dimensional meters is conducted on the assumption of the absence of a number of important components of errors. Absolutely ignored are questions of breakoff of tracking in multi-dimensional circuits, although in principle the device of investigation of nonlinear random phenomena in similar systems can be offered (Fokker-Planck equations).

If we pass to questions of synthesis, then it is immediately necessary to stipulate that the theory given in Chapter XII with full success can be used during synthesis of meters of coordinates of single signals, and also signals against a background of interferences of arbitrary (including, similar) structure with known parameters.

If, however, we are interested in the problem of multipurpose radar (resolution and identification of similar signals in the course of measurement), the above-mentioned device, strictly speaking, can be used only partially. As before indisputable is the necessity of application of the theory of statistical solutions (filtration); however, during synthesis it is necessary to consider the multipeak structure of the likelihood function. The latter appears in view of the similarity of signals, so that transposition of like parameters of them leads to a situation, "likely" almost to the same degree as the true one. Different peaks correspond to all possible transpositions of parameters. As the signals draw close, when there appears the problem of resolution, separate peaks merge, so that a Gaussian approximation of the likelihood function turns out to be inapplicable in principle.

From the point of view of broadening the theory developed above it is necessary to continue consideration of functionals of parameters (including, non-linear), to construct other different forms of smoothing circuits for cases of interest for practice, and so forth.

In conclusion we indicate that results of analysis and synthesis of multi-dimensional radar meters, as also results of Chapter VI, can be used with success in diverse fields of technology: optics, infrared and ultrasonic technology, communications, broadcasting, television; everywhere where they use signals carrying information.

CHAPTER XIII

RESOLVING POWER

§ 13.1. Introductory Remarks

In connection with the expansion of the area of use of radar and the increasing complexity of problems solved by it in recent years there has arisen interest in problems connected with simultaneous separate observation (resolution) of many targets. Inasmuch as development of a theory for this problem is far from complete, there naturally exists quite a lot of different approaches to the problem of resolution of targets, in part concerning quantitative appraisal of resolving power of various instruments, and also in part concerning formulation of the problem of synthesis of systems optimum with respect to resolving power.

As it is known, the concept of resolving power was first used by Rayleigh in the theory of optical instruments. There, by resolving power there was understood the ability of an instrument to a sufficient degree to separately reproduce at its exit images of observed sources of light. As the quantitative characteristic of resolving power of an instrument Rayleigh proposed using the minimum distance $\Delta_{МИН}$ between identical point sources starting from which the total response of the instrument, considered as a function of a given coordinate, has two maxima. Obviously, $\Delta_{МИН}$ coincides with the width of the instrument response, properly defined.

The characteristic of resolving power of optical instruments introduced by Rayleigh found application in other areas, too, including radar. Initially this characteristic was used for appraisal of the performance of systems, principles of whose construction were selected, basically, empirically. After F. Woodward and J. Davis developed the theory of optimum reception of radar signals in noise, there were conducted more general investigations of the resolving power of such a receiver.

Inasmuch as in a receiver which is optimum with respect to noises there is produced multiplication of the received signal by the expected and integration, the intensity of the response to the useful signal is proportional to the square of the modulus of the autocorrelation function of modulation $|C(\tau, \Omega)|^2$ (function of uncertainty, considered in Chapter I, Vol. I): In accordance with this Rayleigh characteristics of resolving power of such a receiver with respect to delay and Doppler frequency coincide with the width of the principal maximum of the function of uncertainty with respect to τ and Ω, respectively.

Investigation of properties of the function of uncertainty (see Chapter I, Vol. I) showed that the width of the maximum on the τ-axis is inversely proportional to the width of the spectrum of modulation, and on the Ω-axis it is inversely proportional to the duration of the signal. From this there arose the common idea that range resolving power is determined by the width of the spectrum, and speed resolving power is determined by the duration of the sounding signal.

The Rayleigh characteristic of resolving power is introduced very conditionally and has meaning only in reference to resolved signals identical in strength. During detection of a weak signal against the background of a strong signal the behavior of the response of the instrument (of the function of uncertainty in the case of correlation reception) for all values of mistuning between signals, and not only within the limits of the principal maximum of the response, becomes of interest. Inasmuch as the response may not decrease monotonically with increase of detuning, and may have spurious maxima, for the case of targets of arbitrary, differing strength it is not possible to introduce in general the idea of an interval of resolution. Resolving power is characterized by the magnitude of response as a function of detuning.

In connection with the problem of resolution of targets of different strength the function of uncertainty over the whole plane τ, Ω was subjected to thorough investigation [4, 36], and there were formulated requirements on modulation from the point of view of resolving power. These requirements, naturally, reduced to decrease of the values of $|C(\tau, \Omega)|^2$ outside the principal maximum ($\tau = 0$, $\Omega = 0$) and narrowing of this maximum. Increase of resolving power by selection of modulation during correlation reception is limited by integral properties of the function of uncertainty (see Chapter I). Nonetheless in this direction there were obtained a series of results very useful for practice (for instance, the creation

of phase-code manipulation).

Likewise it is possible to consider the problem of resolution of angular coordinates. Increase of angle resolving power is provided by decrease of the level of side lobes of directional patterns on reception and transmission and increase of the dimensions of antennas. During consideration of these questions it is possible, by virtue of the known theorem of mutuality [67], to limit oneself to consideration of the receiving pattern. Here there becomes clear the analogy between problems of resolution for angles, range and speed.

The expected signal in the problem of processing the field in the aperture constitutes a plane wave $\mathbf{E}_0 e^{i\omega t - i\mathbf{kr}}$, where $\mathbf{k}$ — wave vector; $\mathbf{r}$ — radius vector of the point of observation. As a result of multiplication of the field in the aperture $y(\mathbf{r}, t)$ by the expected signal and integration over the aperture σ, we obtain

$$Z(\mathbf{k}, t) = \int_{\sigma} y(\mathbf{r}, t)\, e^{-i\mathbf{kr}}\, d\mathbf{r}. \qquad (13.1.1)$$

If the leading edge of the expected wave is parallel to the aperture ($\mathbf{kr} = 0$ in the aperture), transformation of (13.1.1) corresponds to cophasal summation of the aperture field. Obviously, this operation ensures obtaining of maximum energy of the useful signal for given dimensions of the aperture and, consequently, is optimum with respect to natural noises of the receiver. For dimensions of the aperture which are large as compared to wavelength operation (13.1.1) is optimum also with respect to a background of the type of white noise (components of the background arriving from different directions are identical on average in intensity and are statistically independent) [[illegible]]. Correctness of this last affirmation is clear from the space-time analogy.

An analog of the function of uncertainty is, obviously, the dependence of the square of the modulus of response of a system with cophasal summation on a plane wave on the direction of arrival of the wave, i.e., function

$$|C(\mathbf{k})|^2 = \left|\frac{1}{\sigma}\int_{\sigma} e^{i\mathbf{kr}}\, d\mathbf{r}\right|^2, \qquad (13.1.2)$$

coinciding with the directional pattern of the antenna with respect to power with cophasal summation of the field in the aperture.

In practice decrease of side lobes is provided by selection of a method of processing the field, differing from cophasal summation. Here, of course, there is

loss in power of the received signal and, consequently, in the signal-to-noise ratio. Making an analogy between space and time processing of a signal, it is possible to say that this method of increasing resolving power corresponds to selection of processing of the time signal in the receiver, and to selection of the form of the time signal there corresponds, although not exactly, to selection of the shape and dimensions of the aperture.

In theoretical consideration of the question of possible means and limits of increase of resolving power it is natural to use both selection of the method of processing, and also selection of the form of the signal. Complete solution of this problem should include synthesis of the system of processing the received signal, optimum in resolving power, and finding the form of signal which under the imposed limitations during optimum processing provides the best performance indices of the system. Of great interest also is investigation of the criticality of optimum processing and of the form of the signal. Two approaches to solution of the problem at hand are possible, differing in the criteria of optimality utilized:

1. The criterion of optimality can be formulated directly for the response of the instrument, not connecting it directly with functions of the radar (detection, measurement of coordinates and so forth). Resolving power here is understood in the classical sense as the ability of an instrument to separate signals from observed targets. Subsequently we shall call systems synthesized with this approach systems of optimum separation of signals.

2. An optimum system can, as this is done in the theory of radar, be sought by methods of the theory of statistical solutions, proceeding from the quality of execution by the radar of various functions in the presence of many targets [43, 44, 4]. The problem of separation of signals in this case is not posed beforehand. The radar which in the very best manner produces detection or measurement of coordinates (or executes some other function) in the presence of several closely located targets is considered optimum in its resolving power. Resolving power in this approach is understood as the ability to execute some function in the presence of many targets. This concept thus turns out to be very broad.

Each of the shown approaches has its advantages and deficiencies. The first approach is preferable when there is no a priori information about the number and location of resolved targets. The advantage is also simplicity of formulation of the problem and of the obtained results. A deficiency of such an approach is the

difficulty of correct selection of a criterion of optimality having meaning for a broad class of multitarget problems.

The second approach is more rigorous; however, consistent use of it requires assignment of a priori information about the number and position of targets. Such an approach may be preferable, e.g., in the case of simultaneous measurement of coordinates of all detected targets. It has also great value for finding the limits of increase of quality of multitarget systems.

Below we shall give examples of use of both approaches and show cases when they lead to close results.

§ 13.2. Synthesis of a System of Optimum Separation of Signals

13.2.1. Solving the Problem in a General Form

In accordance with the classification introduced above during synthesis of a system of optimum separation of signals the criterion of optimality is formulated directly for the response of the system, which should be small for signals interfering with respect to a given channel and large for the useful (detected) signal. The optimum is obtained as a result of a compromise between these two requirements. As examples of criteria of optimality of systems of separation of signals it is possible to indicate the criterion of a maximum ratio of the output power of the useful signal to the total power of interference and the interfering signals; the criterion of minimax of the response for interfering signals with an assigned value of response to the separated signal; the criterion of a maximum of the signal-to-interference ratio with complete suppression of interfering signals, the presence of which is assumed possible, etc.

We shall first synthesize an optimum system in accordance with the last of the enumerated criteria. A basic advantage of this criterion is independence of the synthesized operations on the received signal from the relationship of strengths of the useful and interfering signals, which in real conditions is usually unknown. Subsequently we shall consider the results of synthesis based on the crite- [illegible] of the signal [illegible] ([illegible] + [illegible]) [illegible] and we shall show that for signal-to-interference ratios large as compared [illegible] results of synthesis based on application of [illegible] criteria are identical [illegible] coincide with re[illegible] obtained [illegible] methods of the theory of [illegible] [illegible] difference between [illegible]

very large overlaps of the resolved signals, where for receiving any reliable solutions for separate targets from the observed set there is required such an increase of power of the useful signal as compared to the case of absence of interfering signals that for many typical radar problems this is equivalent to the impossibility of resolution with the given signal parameters (width of the spectrum of modulation, dimensions of the aperture, time of observation).

During solution of the problem of synthesis we shall consider the system of separation of signals linear (which is very natural, inasmuch as we want to suppress the sum of interfering signals with unknown coefficients), and we shall limit our consideration to such times of separation of signals in which fluctuating changes of these signals are negligible.

This limitation is not essential, inasmuch as separation of signals by parameters of modulation depending on range and angular coordinates is best realized in each period of modulation, which is usually small as compared to the time of correlation of fluctuations.

Let us turn to solution of the problem at hand with the imposed limitations. Mathematically this problem can be formulated in the following way. Let us assume that in multi-dimensional* domain S_0 there is assigned a system of random signals of form $\mathrm{Re}\, A_\lambda \varphi(s, \lambda)$, where λ — parameter taking values in domain Λ_0, A_λ — independent random complex coefficients not depending on s. For each of the signals we introduce a correlation function

$$R_\lambda(s_1, s_2) = E_\lambda \,\mathrm{Re}\, \varphi(s_1, \lambda)\varphi^*(s_2, \lambda). \qquad (13.2.1)$$

Functions $\varphi(s, \lambda)$ are assumed complex and such that

$$\int_{S_0} \varphi(s,\lambda_1)\,\varphi(s,\lambda_2)\,ds = 0$$

for all λ_1, λ_2. Let us separate from the introduced signals useful signal $\varphi(s, \lambda_0)$ and signals $\varphi(s, \lambda_1)$, ..., $\varphi(s, \lambda_n)$, ..., to be suppressed (interfering signals). Let us assume that in that same domain S_0 there is assigned interference with correlation function $R(s_1, s_2)$. It is necessary to find such a reference signal $\psi(s, \lambda_0)$, that the signal-to-interference ratio

*Consideration of the multi-dimensional case is not connected with any difficulties and at the same time permits us to cover the problem of angle resolution.

$$q_0 = \frac{E_0}{2} \frac{\left|\int\limits_{S_0} \psi(s,\lambda_0)\,\varphi^*(s,\lambda_0)\,ds\right|^2}{\iint\limits_{S_0} \psi(s_1,\lambda_0) R(s_1,s_2) \varphi^*(s_2,\lambda_0)\, ds_1\, ds_2} \qquad (13.2.2)$$

is ma[illegible]mum on the condition

$$\int\limits_{S_0} \psi(s,\lambda_0)\,\varphi^*(s,\lambda)\,ds = 0 \quad \text{when} \quad \lambda \in \Lambda_1, \qquad (13.2.3)$$

where Λ_1 – domain of variation of parameters of interfering targets, signals of which it is necessary to suppress.

O[illegible]viously, without disturbing generality it is possible to consider

$$\int\limits_{S_0} \psi(s,\lambda_0)\,\varphi^*(s,\lambda_0)\,ds = 1. \qquad (13.2.4)$$

Fulfillment of this condition is ensured by selection of a constant factor for $\psi(s, \lambda_0)$. Here the problem is reduced to minimization of the q[illegible]atic functional

$$\iint\limits_{S_0} \psi(s_1,\lambda_0)\,R(s_1,s_2)\,\psi^*(s_2,\lambda_0)\,ds_1 ds_2 \qquad (13.2.5)$$

under conditions (13.2[illegible] and (13.2.4). In order to solve this problem without passing to consideratio[illegible] real and imaginary parts of $\psi(s, \lambda_0)$, one should add to (13.2.3) and (13.2.4) complex conjugate conditions.

Let us consider the case when Λ_1 consists of discrete points $\lambda_1, \ldots, \lambda_n$. In accordance with the known Lagrange method the problem of minimization of the functional with the considered additional conditions is equivalent to the problem of finding the absolute minimum of the functional

$$\iint\limits_{S_0} \psi(s_1,\lambda_0) R(s_1,s_2)\,\psi^*(s_2,\lambda_0)\,ds_1 ds_2 -$$
$$-\sum_{k=0}^{n}\left[w'_{0k}\int\limits_{S_0}\psi(s,\lambda_0)\,\varphi^*(s,\lambda_k)\,ds + w_{0k}\int\limits_{S_0}\psi^*(s,\lambda_0)\,\varphi(s,\lambda_k),ds\right], \qquad (13.2.6)$$

where w_{0k} and w'_{0k} are determined from conditions (13.2.3) and (13.2.4) and their conjugates, respectively.

Equating to zero the first variations of (13.2.6) with respect to $\psi(s, \lambda_0)$ and $\psi^*(s, \lambda_0)$, we find $w'_{0k} = w^*_{0k}$ and

$$\int\limits_{S_0} \psi(s_1,\lambda_0)\,R(s_1,s_2)\,ds_1 = \sum_{k=0}^{n} w_{0k}\,\varphi(s_2,\lambda_k). \qquad (13.2.7)$$

Multiplying both parts of (13.2.7) by function $W(s_2, s)$, determined by equation

$$\int_{S_0} W(s_1, s') R(s', s_2)\, ds' = \delta(s_1 - s_2), \tag{13.2.8}$$

and integrating over s_2, we obtain

$$\psi(s, \lambda_0) = \sum_{k=0}^{n} w_{0k} \int_{S_0} W(s, s_1)\, \varphi(s_1, \lambda_k)\, ds_1. \tag{13.2.9}$$

Substituting this expression in (13.2.3) and (13.2.4), it is easy to see that coefficients w_{0k} are elements of the first line (with zero index) of a matrix, the reciprocal of matrix $\|C(\lambda_j, \lambda_k)\|$ ($j, k = 0, 1, \ldots, n$), where

$$C(\lambda_j, \lambda_k) = \iint_{S_0} \varphi(s_1, \lambda_j)\, W(s_1, s_2)\, \varphi^*(s_2, \lambda_k)\, ds_1 ds_2. \tag{13.2.10}$$

Obviously $C(\lambda_j, \lambda_k)$ constitutes the generalized analog of the autocorrelation function of the sounding signal. Substituting (13.2.9) in (13.2.2), we obtain the signal-to-interference ratio with optimum separation of signals:

$$q_0 = \frac{E_0}{2w_{00}}. \tag{13.2.11}$$

If the useful and interfering signals differ so strongly that $C(\lambda_0, \lambda_k) = 0$ when $k \neq 0$, $w_{0k} = C^{-1}(\lambda_0, \lambda_0)\delta_{0k}$ and $q_0 = q_0' = \frac{E_0}{2}C(\lambda_0, \lambda_0)$. Ratio

$$\Gamma = \frac{q_0'}{q_0} = w_{00} C(\lambda_0, \lambda_0) \tag{13.2.12}$$

characterizes the loss in the signal-to-interference ratio which is the cost of separation. It is possible in principle to separate signals as close as one wishes, if only there is a sufficient reserve of power of the useful signal.

It is easy to prove that loss Γ increases with increase of the number of signals which it is necessary to suppress. Let us assume that to the considered set there is added one signal. In accordance with the property of elements of inverse matrices, already used in Chapter IV Vol. I [see (4.9.20)], we have*

$$w_{jk}^{(n+1)} - w_{jk}^{(n)} = \frac{w_{j\,n+1}^{(n+1)} w_{n+1\,k}^{(n+1)}}{w_{n+1\,n+1}^{(n+1)}}, \tag{13.2.13}$$

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*Formula (13.2.13) is derived in the following way. Multiplying both parts of

where the upper index signifies the order of the matrix.

From (13.2.13) we obtain

$$w_{00}^{(n+2)} - w_{00}^{(n+1)} = \frac{w_{0\,n+2}^{(n+2)} w_{n+2\,0}^{(n+2)}}{w_{n+2\,n+2}^{(n+2)}} = \frac{|w_{0\,n+2}^{(n+2)}|^2}{w_{n+2\,n+2}^{(n+2)}} \geq 0, \tag{13.2.14}$$

since $\|w_{jk}\|$, as $\|C(\lambda_j, \lambda_k)\|$, is a Hermitian matrix. From (13.2.14) it follows that $\Gamma(n+1) \geq \Gamma(n)$, since $w_{n+2n+2}^{(n+2)} > 0$ by virtue of the positive definitiveness of functions $C(\lambda, \lambda')$.

Above we did not touch on the question of selection of parameters of the suppressed signals. In those cases when parameters of the available targets are known exactly, or there is assigned an a priori distribution for these parameters, we naturally equate $\lambda_1, \ldots, \lambda_n$ to parameters of the interfering targets or use the principle of least risk in some form. In the absence of a priori data, encountered most frequently, we naturally use the minimax principle, giving to parameters of the suppressed signals those values at which these signals present the greatest danger for execution by the radar of its functions.

In particular it is possible to combine suppressed signals with coordinates of spurious maxima $\lambda_{m1}, \lambda_{m2}, \ldots$ of the function of uncertainty $|C(\lambda_0, \lambda_1)|^2$. Here, due to the fact that in the reference signal $\psi(s, \lambda_0)$ there are present signals with parameters $\lambda_{m1}, \lambda_{m2}, \ldots$, there appear additional spurious maxima near each point λ_{m1}. These maxima of the response will have a magnitude of the order of the square of the initial maxima. If the resulting attenuation of spurious maxima is recognized as insufficient, it is possible to repeat the whole procedure, combining

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[FOOTNOTE CONT'D FROM PRECEDING PAGE]

equation

$$\sum_l w_{jl}^{(n)} C_{l\nu} = \delta_{j\nu}$$

by $w_{\nu k}^{(n+1)}$ and summing for ν from 1 to n, we obtain

$$w_{jk}^{(n+1)} - w_{jk}^{(n)} (1 - \delta_{k\,n+1}) = -w_{n+1\,k}^{(n+1)} \sum_{l=1}^{n} w_{jl} C_{l\,n+1}.$$

Setting k = n + 1, we find

$$w_{j\,n+1}^{(n+1)} = -w_{n+1\,n+1}^{(n+1)} \sum_{l=1}^{n} w_{jl}^{(n)} C_{l\,n+1}.$$

From the last two formulas when j, k ≤ n there immediately follows (13.2.13). It is considered that conditions of existence of the corresponding inverse matrices are realized.

$\lambda_1, \ldots, \lambda_n$ both with the principal and with additional maxima, etc.

In certain cases the described procedure leads to complete elimination of spurious maxima of the response. Thus are matters, in particular, for phase-code manipulation [see Paragraph 13.2.2].

For a number of problems more justified is consideration of the continuum of parameters of suppressed signals, i.e., the case when domain Λ_1, introduced in connection with formula (13.2.3), is not assumed to consist of discrete points. This case is obtained from the considered one by passage to the limit with distances between points approaching zero, and the number of points approaching infinity. As a result relationship (13.2.9) is replaced by

$$\begin{aligned}\psi(s, \lambda_0) = w_0 \int_{S_0} W(s, s_1)\varphi(s_1, \lambda_0)\, ds_1 + \\ + \int_{\Lambda_1} w(\lambda_0, \lambda_1)\, d\lambda_1 \int_{S_0} W(s, s_1)\varphi(s_1, \lambda_1)\, ds_1,\end{aligned} \tag{13.2.15}$$

where w_0 and $w(\lambda_0, \lambda_1)$ are determined by equation

$$w_0 C(\lambda_0, \lambda) + \int_{\Lambda_1} w(\lambda_0, \lambda_1) C(\lambda_1, \lambda)\, d\lambda_1 = \begin{cases} 0 & \text{when } \lambda \in \Lambda_1, \\ 1 & \text{when } \lambda = \lambda_0. \end{cases} \tag{13.2.16}$$

It is possible to simplify these formulas somewhat. Determining w_0 from (13.2.16) for $\lambda = \lambda_0$, we obtain

$$w_0 = \frac{1 - \int_{\Lambda_1} w(\lambda_0, \lambda_1) C(\lambda_1, \lambda_0)\, d\lambda_1}{C(\lambda_0, \lambda_0)}. \tag{13.2.17}$$

Substituting this expression in (13.2.16) when $\lambda \in \Lambda_1$, we have

$$\begin{aligned}\int_{\Lambda_1} w(\lambda_0, \lambda_1)[C(\lambda_1, \lambda_0) C(\lambda_0, \lambda) - C(\lambda_0, \lambda_0) C(\lambda_1, \lambda)]\, d\lambda_1 = \\ = C(\lambda_0, \lambda).\end{aligned} \tag{13.2.18}$$

Loss in the signal-to-interference ratio, as before, is determined by formula (13.2.12), where instead of w_{00} one should substitute w_0, determined from (13.2.17). As a result we have

$$\Gamma = 1 - \int_{\Lambda_1} w(\lambda_0, \lambda_1) C(\lambda_1, \lambda_0)\, d\lambda_1. \tag{13.2.19}$$

Thus, solution of the problem of separation of signals for the considered case reduces to solution of a Fredholm integral equation of the first kind (13.2.18).

In certain cases loss in the signal-to-interference ratio, connected with total suppression of the selected interfering signals, can be impermissibly large and even infinite. In these cases it is expedient to use partial suppression of signals. Corresponding operations are most simply found from the condition of a maximum of the ratio of the power of the useful signal to the sum of the powers of interference and the selected interfering signals with assigned strengths of interfering signals at the input of the system.

We will give here only the result of solution of the problem in this formulation, inasmuch as the method of solution does not differ from the considered one. The reference signal in this case, as before, is recorded in the form (13.2.9), but matrix $\|w_{jk}\|$ is replaced by $\|v_{jk}\|$, the reciprocal of $\|C(\lambda_j, \lambda_k) + \frac{2}{E_k}\delta_{jk}\|$, where E_k – energy of the k-th signal.

The formula for the signal-to-interference ratio taking into account all interfering signals has the form

$$q_0 = \frac{E_0}{2v_{00}} - 1. \tag{13.2.20}$$

In § 13.3 it will be shown that optimum processing of a signal when using such a criterion coincides with optimum processing in the problem of detection of a separated signal against a background of interference and interfering signals.

It is easy to see that for large values of the signal-to-interference ratio for all the considered signals at the input of the system operations of partial and total suppression of signals, and also output signal-to-interference ratios, coincide. As a confirming example we shall consider the very simple case of one interfering target. In this case from formula (13.2.11) and (13.12.20) we obtain:

with total suppression of the interfering signal

$$q_0 = q'_0(1-\gamma^2); \tag{13.2.21}$$

with partial suppression of the interfering signal

$$q_0 = q'_0\left(1 - \frac{q'_1}{1+q'_1}\gamma^2\right), \tag{13.2.22}$$

where q'_0 and q'_1 -- signal-to-interference ratios for the useful and interfering signals without suppression; $\gamma = \left|\frac{C(\lambda_1, \lambda_0)}{C(\lambda_0, \lambda_0)}\right|$.

In Fig. 13.1 there is given the dependence of the ratio Γ of quantities (13.2.22) and (13.2.21) on γ^2 for different q_1', characterizing loss in the signal-to-interference ratio due to use of total separation instead of partial. Assigned a permissible value of loss Γ we can find the γ at which loss starts to exceed the permissible:

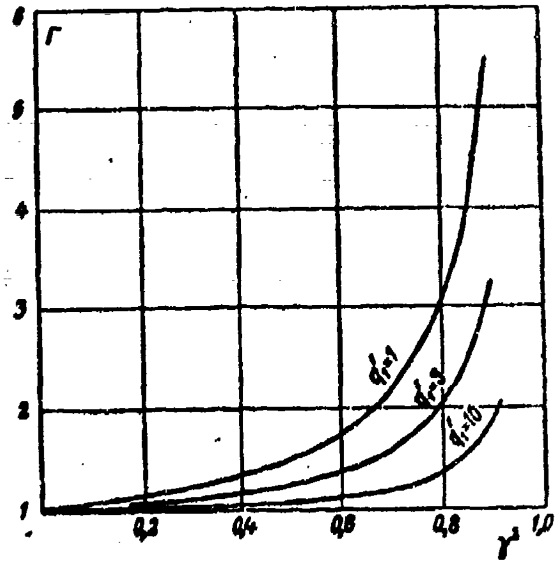

Fig. 13.1. Dependence of loss in the signal-to-interference ratio with replacement of partial separation of signals by total.

$$\gamma^2 = 1 - \frac{1}{(\Gamma - 1)(1 + q_1') + 1}.$$

Introducing the density of strength of the interfering signals and passing to the limit as $n \to \infty$ and with the interval between signals approaching zero, we can obtain the proper formulas for the continuum of interfering targets. Here the problem coincides with that which was considered in § 4.11 Vol. I in connection with selection of a signal against a background of passive interferences.

13.2.2. Separation of Signals by Range

As a particular case we shall consider the problem of separation of signals by range in the presence of interference in the form of white noise. Here, if the signal is periodic S_0 coincides with the period of modulation:

$$\varphi(t, \tau) = u(t - \tau) e^{i\omega_0 t}, \quad R(t_1, t_2) = N_0 \delta(t_1 - t_2),$$
$$W(t_1, t_2) = \frac{1}{N_0} \delta(t_1 - t_2),$$

$$C(\tau_j, \tau_k) = \frac{1}{N_0} \int_0^{T_r} u(t - \tau_j) u^*(t - \tau_k) dt = \frac{T_r}{N_0} C_0(\tau_k - \tau_j), \tag{13.2.23}$$

$$\psi(t, \tau_0) = \frac{1}{N_0} \sum_{k=0}^{n} \omega_{0k} u(t - \tau_k) e^{i\omega_0 t}. \tag{13.2.24}$$

Thus, the reference signal in a system of optimum separation is a linear combination of all $n + 1$ considered signals, orthogonal to n interfering signals. Multiplication by reference signal $\psi(t, \tau_0)$ and integration can be carried out

in the usual circuit of heterodyne processing, shown in Fig. 13.2. Under certain conditions the same result can be obtained in principle in a circuit with filtration.

Let us consider first the case when there is applied a filter designed to process the whole period of modulation. This is inevitable in the case of continuous radiation and can be considered one of the variants of processing periodic pulse modulation. We consider $\psi(t, \tau_0)$ a function of difference $\tau_0 - t$

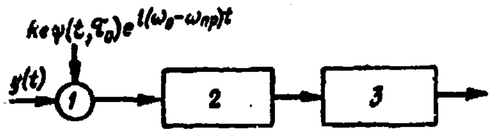

Fig. 13.2. Block diagram of heterodyne processing: 1 — mixer; 2 — i-f filter; 3 — detector.

$$\psi(t, \tau_0) = h_1(\tau_0 - t)$$

and consider $h_1(\tau_0 - t)$ equal to zero outside interval $\tau_0 - T_r$, τ_0. Then $h_1(t)$ it can be considered the pulse response of a physically realizable filter. It is easy to see that passage of a signal through such a filter ensures separation of signals optimum in the considered meaning, where the separated signal is obtained at the output at a moment equal to the value of delay of the separated signal. Really, if to the filter input there proceeds signal $u(t - \tau')e^{i\omega_0 t}$, the output signal at time t is recorded in the form

$$Q_\Phi(t) = \int_{t-T_r}^{t} h_1(t-x)\, u^*(x-\tau')\, e^{-i\omega_0 x}\, dx =$$

$$= \frac{1}{N_0}\int_{t-T_r}^{t} \sum_{k=0}^{n} w_{0k} u(\tau_0 - \tau_k - t + x)\, u^*(x-\tau')\, dx = \frac{T_r}{N_0}\sum_{k=0}^{n} w_{0k} C_0(\tau_0 - \tau_k + \tau' - t);$$

and at time $t = \tau'$, in the form

$$Q_\Phi(\tau') = \frac{T_r}{N_0}\sum_{k=0}^{n} w_{0k} C_0(\tau_0 - \tau_k) = 1.$$

At the same moment all signals for which $\tau' - \tau = \tau_0 - \tau_\nu$ ($\nu = 1, 2, \ldots, n$), are completely suppressed:

$$Q_{\Phi M}(\tau') = \frac{T_r}{N_0}\sum_{k=0}^{n} w_{0k} C_0(\tau_\nu - \tau_k) = 0.$$

Thus, we obtain a system whose output signal in each moment of time t coincides with the output signal, delayed this time, of an optimum circuit with total suppression of signals, separated $\tau_0 - \tau_\nu$ ($\nu = 1, \ldots, n$). In a circuit with filtration points $\lambda_0, \lambda_1, \ldots, \lambda_n$ shift with change of t, keeping their relative position.

Somewhat more complicated is the matter with use of filtration for processing separate pulses. For physical realizability of the filter it is necessary that $\psi(t_1, \tau_0)$ turn into zero for t larger than some value t_0. Inasmuch as $u(t)$ has limited duration, it is possible to set $u(t) = 0$ when $t > 0$. Then by virtue of (13.2.24) $\tau_0 = \max_k \tau_k$, and it is possible to consider $\psi(t, \tau_0) = h(\max \tau_k - t)$ the pulse response of a physically realizable filter. If $\max_k \tau_k \neq \tau_0$ the result of optimum processing of a signal with $\tau = \tau_0$ is obtained with delay with respect to the moment of termination of this signal by time $\max_k \tau_k - \tau_0$.

In principle it is possible to eliminate the delay in delivery of results of processing separate pulses if we beforehand require equality $\psi(t, \tau_0) = 0$ when $t > \tau_0$. Here it is possible to preserve the whole described approach if we consider $u(t - \tau_k) = 0$ when $t > \tau_0$ for all the considered signals. Such "truncation" of signals should, of course, be taken into account during calculation of elements of matrices $C(\tau_j, \tau_k)$ and w_{jk}. When using filtration with "truncation" of signals at the output of the filter at every moment of time there is separated a signal with $\tau = t$, and all signals with $\tau = t - \tau_0 + \tau_k$, existing at the same moment of time, are suppressed. From qualitative considerations it is quite clear that "truncation" of signals can lead to increase of corresponding values of $C(\tau_j, \tau_k)$ and that this increase should cause worsening of the signal-to-noise ratio as compared to the "untruncated" case. However there is not excluded cases where there exist signals for which "truncation" does not lead to essential losses. Further investigations in this direction are of considerable interest.

Let us consider two examples of signals with phase-code manipulation (see Chapter I, Vol. I). In this case, combining delays of the suppressed signals with spurious maxima of the function of uncertainty, it is possible to achieve complete elimination of the spurious maxima. It is simple to show that the response of a linear instrument with a phase-manipulated reference signal to a phase-manipulated signal with the same code interval Δ, but delayed time τ, is determined by a formula analogous to (1.2.22):

$$Q(\tau) = Q(\mu\Delta) - \left(\frac{\tau}{\Delta} - \mu\right)\{Q(\mu\Delta) - Q[(\mu+1)\Delta]\}, \qquad (13.2.25)$$

where μ — the integral part of ratio $\frac{\tau}{\Delta}$.

Seeking absences of response for τ, which are multiples of Δ, we also provide suppression of signals with intermediate values of τ.

Example 1. Let us consider as an example a continuous signal, manipulated in phase by the code of Hoffman [?]. As it is known, for such a code $C_0(\mu\Delta) = -\frac{1}{n}$ when $\mu \neq 0$. Let us find a reference signal (pulse response of the filter) at which response of the instrument to signals $\tau - \tau_0 = \mu\Delta (\mu \neq 0)$ is equal to zero. Here it is sufficient to consider values $|\mu| \leq \frac{n-1}{2}$, where n is the number of elements of the code, since for larger $|\mu|$ the whole picture is repeated periodically. It is easy to show that

$$w_{jk} = \frac{N_0}{T_r} \frac{n}{n+1} (1 + \delta_{jk}),$$

$$\psi(t, \tau_0) = \frac{N_0 e^{i\omega_0 t}}{T_r} \frac{n}{n+1} \Big\{ 2u(t - \tau_0) + \tag{13.2.26}$$

$$+ \sum_{1}^{(n-1)/2} [u(t - \tau_0 - \mu\Delta) + u(t - \tau_0 + \mu\Delta)] \Big\}. \tag{13.2.27}$$

In different code intervals $u(t)$ takes value $1/\sqrt{n}$ or $-1/\sqrt{n}$ (phase is equal to 0 or π). In the reference signal $\psi(t, \tau_0)$ at certain code intervals there appear zero values (signal gaps). Thus, the reference signal turns out to be modulated in amplitude. For instance, when $n = 2^3 - 1 = 7$ the sequence of values of $u(t - \tau_0)$ has the form 1, -1, -1, -1, +1, +1, -1, and for $\psi(t, \tau_0)$ 0, -1, -1, -1, 0, 0, -1 [for brevity of recording we rejected in the expressions for $u(t)$ and $\psi(t, \tau_0)$ the normalizing factors].

By the formula (13.2.12) it is easy to calculate the loss in the signal-to-noise ratio due to separation. Substituting (13.2.23) and (13.2.26) in (13.2.12), we obtain

$$\Gamma = \frac{2n}{n+1}.$$

For large n loss is approximately double.

Example 2. Let us assume we have a pulse signal, manipulated in phase by the code of Barker with n = 3. Such a code has the form 1, 1, -1.

The corresponding function of $C_0(\mu\Delta)$ is ...; 0; $-\frac{1}{3}$; 0; 1; 0; $-\frac{1}{3}$; 0; ... We shall consider for simplicity the problem of separation on an unlimited interval and use for inversion of matrix $C(\tau_\mu, \tau_\nu)$ discrete Fourier transformation. Then

$$w_{jj+\nu}=\frac{N_0}{\pi}\int_0^{\pi}\frac{\cos\nu\lambda\,d\lambda}{1+\frac{2}{3}\cos 2\lambda}=$$

$$=\begin{cases}0, & \nu=2k+1\\ (-1)^k\frac{3\pi}{\sqrt{5}}N_0\left(\frac{3-\sqrt{5}}{2}\right)^{|k|}, & \nu=2k.\end{cases}$$

Coefficients $w_{jj+\nu}$ rapidly decrease with growth of ν. Therefore in practice it is possible to limit the duration of the reference signal. Quantity w_{jj} in accordance with (13.2.12) determines loss in the signal-to-interference ratio during separation of signals: $\Gamma = \frac{3\pi}{\sqrt{5}} \approx 4.2$. The considered method can be used for finding processing of codes with a larger number of elements. Difficulties arising here are of a purely calculating character.

13.2.3. Separation of Signals in the Direction of Arrival

The described approach can be used for synthesis of processing of the field in the antenna aperture, providing increase of angle resolving power. Here domain S_0 is the surface of the aperture of the antenna, and signal $\varphi(s, \lambda)$ is a plane wave $e^{-i\mathbf{kr}}$, where $\mathbf{r}$ – radius vector of a point in the aperture, which we consider flat; $\mathbf{k}$ – wave vector (parameter of the signal). The dependence of the signal on time may be ignored if we are not concerned with joint separation with respect to angles and range.*

Interference in the problem of separation by angles can be considered to consist of two components: background radiation and natural noises of the receiver. We shall consider these components separately.

Background radiation we shall consider superposition of plane waves arriving from different directions with random and independent amplitudes and phases. Here the correlation function for background in the aperture has the form [51]

$$R_\Phi(\mathbf{r}_1-\mathbf{r}_2)=N_\Phi\frac{1}{\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}e^{i\frac{\omega}{c}(\mathbf{r}_1-\mathbf{r}_2)\,\rho}\sin\theta\cos\theta\,d\varphi\,d\theta=$$

$$=N_\Phi\frac{2J_1\left(\frac{\omega}{c}|\mathbf{r}_1-\mathbf{r}_2|\right)}{\frac{\omega}{c}|\mathbf{r}_1-\mathbf{r}_2|};$$

---

*The problem of simultaneous separation with respect to angles and range was considered by Yu. I. Shirman [illegible] for the case of an aperture in the form of an infinitely small slot.

where $J_1(x)$ — Bessel function of the first kind;

N_Φ — spectral current density of power of the background through the aperture at the considered frequency;

$\frac{\omega}{c}\rho = k_\sigma$ — projection of the wave vector on the surface of the aperture.

The formula for $R_\Phi(\mathbf{r}_1, \mathbf{r}_2)$ is derived in the following way. Let us consider an elementary plane wave striking the aperture at an angle of ϑ to the normal and angle φ with respect to an arbitrarily selected direction $\mathbf{x}$ in the aperture plane. We consider that the phase and amplitude of the wave are random, where all values of phase in interval $(0, 2\pi)$ are equiprobable. Then, obviously, the correlation function of the field of this plane wave for any two points of the aperture is

$$\overline{A}\,\mathrm{Re}\, e^{i(\mathbf{r}_1-\mathbf{r}_2)\mathbf{k}} = \overline{A}\,\mathrm{Re}\, e^{i\frac{\omega}{c}(\mathbf{r}_1-\mathbf{r}_2)\boldsymbol{\rho}},$$

where $\mathbf{k}$ — wave vector,

$\overline{A}$ — average density of the flux of power, corresponding to the given plane wave.

Inasmuch as on the receiver of radiation there usually acts part of the power flux passing through the aperture, we must multiply $\overline{A}$ by $\cos\vartheta$. If waves arriving from different directions are not correlated, the correlation function for their sum is obtained by summation of correlation functions of the components. Introducing the angular density of the strength of $\frac{1}{\pi}N_\Phi$ in accordance with formula

$$\overline{A} = \frac{N_\Phi}{\pi}\sin\vartheta\, d\varphi\, d\vartheta,$$

we obtain as a result of integration the above-mentioned expression for $R_\Phi(\mathbf{r}_1, \mathbf{r}_2)$.

To correlation functions $R_\Phi(\mathbf{r}_1, \mathbf{r}_2)$ there corresponds spectral density

$$S_\Phi(\mathbf{p}) = \begin{cases} \frac{1}{\pi}N_\Phi\lambda^2, & |\mathbf{p}| < \frac{\omega}{c}, \\ 0 & |\mathbf{p}| \geq \frac{\omega}{c}. \end{cases}$$

Plane waves in the aperture of an antenna constitute space harmonics of form $e^{-i\mathbf{pr}}$, where $\mathbf{p} = \mathbf{k}_\sigma$, and, obviously, $|\mathbf{k}_\sigma| < \frac{\omega}{c}$. Thus, for all values of $\mathbf{p}$ interesting background radiation has uniform spectral density and it can be considered equivalent to space white noise with correlation function $N_\Phi\lambda^2\delta(\mathbf{r}_1 - \mathbf{r}_2)$. More strict analysis of [61, 63] shows that this equivalence occurs only for dimensions of the aperture, large as compared to wavelength, with disregard of boundary effects.

In examining natural noises one should distinguish noises added to the signal in the course of space processing (output power of these noises depends on the character of processing) (noises of the first kind), and noises added to the signal after space processing (noises of the second kind). Noises of the first kind play a large role in systems with a phased antenna array, in which every element of the array is connected to a noisy receiver. Space processing of a signal in this case consists in weighted summation of output voltages of these receivers. With a large number of closely located elements summation can be replaced by integration, considering noises δ-correlated. Allowance for the influence of noises of the first kind and of background and synthesis of a system for separation of signals are produced in full conformity with the general method described in 13.2.1.

Noises of the second kind, added to a signal after space processing, play a basic role in those cases when space processing is produced without preliminary amplification of signals received from separate elements of the antenna. The power of these noises does not depend on the nature of space processing; therefore such a case requires special consideration.

Taking into account the presence of a δ-correlated background with spectral density $\frac{1}{\pi}N_{\Phi}\lambda^2$, natural noises of the first kind, also assumed δ-correlated, and noises of the second kind, we write the signal-to-interference ratio, which is required to be maximized, in the form

$$q_0 = \frac{E_0}{2}\,\frac{\left|\frac{1}{\sigma}\int\limits_{\sigma}\psi(\mathbf{r},\mathbf{k}_0)\,e^{i\mathbf{k}_0\mathbf{r}}d\mathbf{r}\right|^2}{N_2 + N_0\frac{1}{\sigma}\int\limits_{\sigma}|\psi(\mathbf{r},\mathbf{k}_0)|^2 d\mathbf{r}}. \qquad (13.2.28)$$

In this formula σ – area of the aperture, N_2 – spectral density of noises of the second kind, $N_0 = N_1 + \frac{1}{\pi}N_{\Phi}\lambda^2$ – sum of spectral densities of noises of the first kind (N_1) and the background [we consider factors with $\delta(\mathbf{r}_1 - \mathbf{r}_2)$ in expressions for correlation functions]. In (13.2.28) by means of introduction of factor $\frac{1}{\sqrt{\sigma}}$ with $\psi(\mathbf{r}, \mathbf{k}_0)$, it is considered that in the absence of amplification the energy received from the aperture cannot be greater than the incident energy (there occurs summation of the field with respect to power, and not to voltage).

With maximization of expression (13.2.28) on function ψ there should be imposed

an additional condition, limiting amplification of the signal to addition of noises of the second kind. If this condition is absent, the result is trivial: the constant factor in $|\psi(\mathbf{r}, \mathbf{k}_0)|$ should be so large that component N_2 in the denominator of (13.2.28) can be ignored.

If we write the shown condition in the form

$$\frac{1}{\sigma}\int_{\sigma}|\psi(\mathbf{r}, \mathbf{k}_0)|^2 d\mathbf{r} = 1,$$

i.e., assume that amplification at separate points of the aperture is possible, but on the average over the aperture it is absent, then, as it is easy to see, the expression for $\psi(\mathbf{r}, \mathbf{k}_0)$ with accuracy of a constant factor coincides with that obtained for $N_2 = 0$:

$$\psi(\mathbf{r}, \mathbf{k}_0) = A\sum_{j=0}^{n} w_{0j} e^{-i\mathbf{k}_j\mathbf{r}}, \qquad (13.2.29)$$

where $\mathbf{k}_j (j = 1, \ldots, n)$ — wave vectors of suppressed signals: [maximization of (13.2.28) is performed with the additional conditions (13.2.3)].

Coefficient A is determined from condition

$$\frac{|A|^2}{\sigma}\int_{\sigma}\sum_{j,l} w_{0j}w^*_{0l} e^{-i(\mathbf{k}_j-\mathbf{k}_l)\mathbf{r}} d\mathbf{r} =$$

$$= \frac{|A|^2}{\sigma}\sum_{j,l} w_{0j}w^*_{0l} C(\mathbf{k}_j, \mathbf{k}_l) = |A|^2 w_{00} = 1,$$

so that

$$q_0 = E_0/[2w_{00}(N_0 + N_2)].$$

In these formulas w_{jk} — elements of a matrix, the reciprocal of $\|C(\mathbf{k}_j, \mathbf{k}_l)\|$, and

$$C(\mathbf{k}_j, \mathbf{k}_l) = \frac{1}{\sigma}\int_{\sigma} e^{-i(\mathbf{k}_j-\mathbf{k}_l)\mathbf{r}} d\mathbf{r}.$$

If amplification and attenuation of the signal are absent at every point of the aperture, $|\psi(\mathbf{r}, \mathbf{k}_0)| \equiv 1$, and separation is carried out only by phase processing. Function $\psi(\mathbf{r}, \mathbf{k}_0)$ in this case can be recorded in the form

$$\psi(\mathbf{r}, \mathbf{k}_0) = e^{i\theta(\mathbf{r}, \mathbf{k}_0)}, \qquad (13.2.30)$$

where $\theta(\mathbf{r}, \mathbf{k}_0)$ – real function.

Substituting (13.2.30) in (13.2.8) and (13.2.3), we obtain

$$q_0 = \frac{E_0}{2(N_2 + N_0)} \left\{ \left(\frac{1}{\sigma} \int_\sigma \cos[\theta(\mathbf{r}, \mathbf{k}_0) + \mathbf{k}_0 \mathbf{r}]\, d\mathbf{r} \right)^2 + \right.$$
$$\left. + \left(\frac{1}{\sigma} \int_\sigma \sin[\theta(\mathbf{r}, \mathbf{k}_0) + \mathbf{k}_0 \mathbf{r}]\, d\mathbf{r} \right)^2 \right\}. \qquad (13.2.31)$$

The problem reduces to maximization of (13.2.31) under condition

$$\frac{1}{\sigma} \int_\sigma \cos[\theta(\mathbf{r}, \mathbf{k}_0) + \mathbf{k}_j \mathbf{r}]\, d\mathbf{r} = \frac{1}{\sigma} \int_\sigma \sin[\theta(\mathbf{r}, \mathbf{k}_0) + \mathbf{k}_j \mathbf{r}]\, d\mathbf{r} = 0. \qquad (13.2.32)$$

Using the method of undetermined Lagrange factors, we obtain

$$\psi(\mathbf{r}, \mathbf{k}_0) = e^{i\theta(\mathbf{r}, \mathbf{k}_0)} = \frac{\sum_{j=0}^{n} a_j e^{-i\mathbf{k}_j \mathbf{r}}}{\left| \sum_{j=0}^{n} a_j e^{-i\mathbf{k}_j \mathbf{r}} \right|}, \qquad (13.2.33)$$

where coefficients a_j are determined from conditions

$$a_0 = \frac{1}{\sigma} \int_\sigma \frac{\sum_j a_j e^{-i(\mathbf{k}_j - \mathbf{k}_0)\mathbf{r}}}{\left| \sum_j a_j e^{-i\mathbf{k}_j \mathbf{r}} \right|}\, d\mathbf{r},$$

$$\frac{1}{\sigma} \int_\sigma \frac{\sum_j a_j e^{-i(\mathbf{k}_j - \mathbf{k}_l)\mathbf{r}}}{\left| \sum_j a_j e^{-i\mathbf{k}_j \mathbf{r}} \right|}\, d\mathbf{r} = 0, \quad l = 1, \ldots, n. \qquad (13.2.34)$$

Thus, the optimum weighting function $\psi(\mathbf{r}, \mathbf{k}_0)$ in the considered case is a linear combination of all the considered signals, multiplied by a certain function of the radius vector, ensuring constancy of $|\psi(\mathbf{r}, \mathbf{k}_0)|$. The signal-to-interference ratio with such a reference signal, as it is easy to prove by substitution of (13.2.33) and (13.2.34) in (13.2.28), has the form

$$q_0 = \frac{E_0}{2(N_2 + N_0)} |a_0|^2 = q'_0 |a_0|^2, \qquad (13.2.35)$$

where q'_0 – signal-to-interference ratio in a system without suppression of interfering signals.

The obtained formulas show that calculation of optimum processing of a signal in this case is considerably more complicated than in that considered earlier. We find the gain in the signal-to-noise ratio to which replacement of processing found without taking into account absence of amplification of power in the antenna by treatment of the form just now considered leads.

In the first case there is produced multiplication of the received signal by a reference signal of form (13.2.29), normalized, however, in such a way that $|\psi(\mathbf{r}, \mathbf{k}_0)| = 1$. Energy of the signal and the spectral density of noise at the output of such a system will be recorded in the form

$$E_{01} = E_0 \left| \frac{1}{\sigma} \int_\sigma \frac{\psi(\mathbf{r}, \mathbf{k}_0) e^{i\mathbf{k}_0\mathbf{r}} d\mathbf{r}}{\max|\psi(\mathbf{r}, \mathbf{k}_0)|} \right|^2,$$

$$N_{01} = N_2 + N_0 \frac{1}{\sigma} \frac{\int_\sigma |\psi(\mathbf{r}, \mathbf{k}_0)|^2 d\mathbf{r}}{\max|\psi(\mathbf{r}, \mathbf{k}_0)|^2},$$

where E_0 and N_0 — energy of the signal and spectral density of the background radiation in the absence of separation.

Substituting these expressions in (13.2.28), for the signal-to-interference ratio we obtain

$$q_0 = \frac{E_{01}}{2N_{01}} = \frac{E_0}{2} \frac{1}{N_2 \max\left|\sum_j w_{0j} e^{-i\mathbf{k}_j\mathbf{r}}\right|^2 + N_0 w_{00}}. \qquad (13.2.36)$$

The sought gain is characterized by the ratio of (13.2.35) and (13.2.36):

$$\Gamma_1 = \frac{|a_0|^2 \left(\max\left|\sum_j w_{0j} e^{-i\mathbf{k}_j\mathbf{r}}\right|^2 + \frac{N_0 w_{00}}{N_2}\right)}{1 + \frac{N_0}{N_2}}. \qquad (13.2.37)$$

Considering the problem of separation of signals in the direction of their arrival, we considered that the field in the antenna aperture is directly subjected to processing. In antennas of the image type accessible for processing usually is the diffracted field in the focus of the mirror, equivalent, as we know [69], to the diffracted field at great distances from the aperture in the absence of a focusing mirror. For large dimensions of the aperture this diffracted field can be calculated by a simplified Kirchhoff formula

$$y_1(\mathbf{k}') \sim \frac{1}{\sigma} \int_\sigma y(\mathbf{r}) e^{i\mathbf{k}'\mathbf{r}} d\mathbf{r}, \qquad (13.2.38)$$

where $\mathbf{k}' = \frac{\omega}{c}\rho$, and ρ — a unit vector characterizing the direction for which the diffracted wave is considered.

In accordance with this formula the plane wave $e^{-i\mathbf{kr}}$ will be transformed with accuracy of a factor into function $C(\mathbf{k} - \mathbf{k}')$, which in connection with this can be called the response of the aperture to the plane wave.

Let us establish conformity between processing of the field in the aperture of an antenna and the diffracted field. As can be seen from (13.2.38) as a result of multiplication of the field in the aperture by a plane wave and integration there is obtained a diffracted wave for a certain direction. Thus, multiplication by reference signal (13.2.29), which is a linear combination of plane waves, and integration over the aperture can be replaced, in principle, by formation of the same linear combination composed of values of the diffracted field for the corresponding directions. However, by virtue of the inevitable presence of noises of the second kind, such an operation is unprofitable in terms of power. The fact is that $|y_1(\mathbf{k})|^2$ is proportional to the density of the power flux of the diffracted field. In order to obtain a signal proportional to $y_1(\mathbf{k})$ it is necessary to take, in general, an infinitesimal elementary solid angle near the given direction $\mathbf{k}$. The energy of such a signal, and consequently also the ratio of this energy to the spectral density N_2 of noise of second kind, will be infinitesimal.

It is possible to construct another operation, equivalent to the considered one from the point of view of separation of signals and allowing us to completely use the energy reaching the aperture. Let us consider expression

$$Z(\mathbf{k}) = \int\limits_{|\mathbf{k}'|<\frac{\omega}{c}} y_1(\mathbf{k}') C(\mathbf{k}' - \mathbf{k})\, d\mathbf{k}' \sim \frac{1}{\sigma} \int\limits_{\sigma} y(\mathbf{r})\, d\mathbf{r} e^{i\mathbf{kr}} \times$$
$$\times \int\limits_{|\mathbf{k}'|<\frac{\omega}{c}} C(\mathbf{k}' - \mathbf{k}) e^{i(\mathbf{k}'-\mathbf{k})\mathbf{r}} d\mathbf{k}'. \qquad (13.2.39)$$

For dimensions of the aperture, large as compared with the wavelength, $C(\mathbf{k})$ rapidly decreases with growth of $|\mathbf{k}|$. Therefore, it is possible to replace the limits of integration in the integral over $\mathbf{k}'$ by infinite limits and to consider this integral the two-dimensional reverse Fourier transform. Inasmuch as the response of the aperture $C(\mathbf{k})$ is the Fourier transform of a function equal to $\frac{1}{\sigma}$ in the aperture and to zero outside the aperture, the considered integral for all points of the aperture has a constant value. Thus,

$$Z(\mathbf{k}) \sim \frac{1}{\sigma}\int_{\sigma} y(\mathbf{r})\, e^{i\mathbf{kr}} d\mathbf{r}$$

and multiplication of the field in the aperture by plane wave $e^{-i\mathbf{kr}}$ and integration over r is equivalent to multiplication of the diffracted field $y_1(\mathbf{k}')$ by response of the aperture $C(\mathbf{k}' + \mathbf{k})$ and integration in all directions (over the whole focal plane in the case of an image antenna).

To reference signal $\sum_j a_j e^{-i\mathbf{k}_j \mathbf{r}}$ in the aperture there corresponds, obviously, reference signal $\sum_j a_j C(\mathbf{k}' + \mathbf{k}_j)$ for the diffracted field or $\sum_j a_j C(\frac{\mathbf{r}_\Phi}{d_\Phi} + \mathbf{k}_j)$ for the focal plane (d_Φ – focal length; $\mathbf{r}_\Phi$ – radius vector in the focal plane). The obtained results determine the structure of the irradiator of an image (or lens) antenna, ensuring the required separation of signals with a maximum signal-to-interference ratio.

Example. Let us consider a linear aperture (an aperture with infinitesimal width), for which

$$C(\mathbf{k}_1 - \mathbf{k}_2) = \frac{\sin\frac{\pi d}{\lambda}(\sin\theta_1 - \sin\theta_2)}{\frac{\pi d}{\lambda}(\sin\theta_1 - \sin\theta_2)} \approx \frac{\sin\frac{\pi d}{\lambda}(\theta_1 - \theta_2)}{\frac{\pi d}{\lambda}(\theta_1 - \theta_2)} = C(\theta_1 - \theta_2),$$

where θ – angle between the wave vector and the line of the aperture:

d – length of the aperture;

λ – wavelength.

We demand suppressions of first spurious maxima, occurring for $\theta_m = 4.5\frac{\lambda}{\pi d}$, where

$$C(\theta_m) = \gamma_1 = -0{,}217, \quad C(2\theta_m) = \gamma_2 = 0{,}046.$$

With n = 2 and a symmetric location of interfering signals matrices $\|C(\lambda_j, \lambda_k)\|$ and w_{jk} have form

$$\|C(\lambda_j, \lambda_k)\| = \begin{Vmatrix} 1 & \gamma_1 & \gamma_2 \\ \gamma^*_1 & 1 & \gamma_1 \\ \gamma^*_2 & \gamma^*_1 & 1 \end{Vmatrix},$$

$$\|w_{jk}\| = \frac{1}{\Delta}\begin{Vmatrix} 1 - |\gamma_1|^2 & \gamma^*_1\gamma_2 - \gamma_1 & \gamma_1^2 - \gamma_2 \\ \gamma_1\gamma^*_2 - \gamma^*_1 & 1 - |\gamma_2|^2 & \gamma^*_1\gamma_2 - \gamma \\ \gamma_1^{2*} - \gamma^*_2 & \gamma_1\gamma^*_2 - \gamma^*_1 & 1 - |\gamma_1|^2 \end{Vmatrix},$$

where $\Delta = 1 - 2|\gamma_1|^2 - |\gamma_2|^2 + 2\,\mathrm{Re}\,\gamma_1^2\gamma_2^*$.

From this it is easy to find loss in the signal-to-interference ratio and optimum processing for the case when there is allowed amplification during processing of the field from separate sections of the aperture. Substituting w_{00} and $C(\lambda_0, \lambda_0)$ (elements on the intersection of diagonals in the above matrices) in (13.2.12), we find $\Gamma = w_{00} = (1 - |\gamma_2|^2)/\Delta \approx 1.1$. Loss comprises in all 10%. Optimum processing of the field in the aperture for the case when the selected target is on the axis of the antenna ($\theta_0 = 0$) is recorded in the form

$$\psi(x, \mathbf{k}_0) = \frac{1}{\sqrt{w_{00}}} \sum_{j=0}^{n} w_{0j} e^{i\frac{2\pi x}{\lambda}\theta_j} =$$

$$= \frac{1}{\Delta\sqrt{w_{00}}}\left[(\gamma_1\gamma_2^* - \gamma_1^*) e^{-i\frac{2\pi x}{\lambda}\theta_m} + (1 - |\gamma_2|^2) + \right.$$

$$\left. + (\gamma_1^*\gamma_2 - \gamma_1) e^{i\frac{2\pi x}{\lambda}\theta_m}\right] \approx 1 + 0{,}415\cos\frac{2\pi\theta_m}{\lambda}x,$$

where x – coordinate of the point in the aperture.

The field at every point of the aperture should be multiplied by $\psi(x, \mathbf{k}_0)$, which corresponds to a certain amplification or attenuation, inasmuch as phase shift is equal to zero, and is integrated over the aperture.

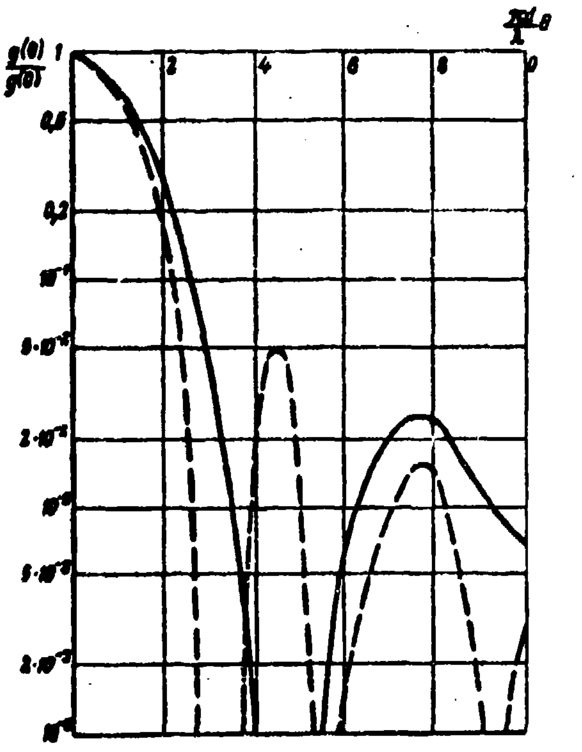

Fig. 13.3. Directional pattern of a linear antenna: ——— with suppression of the first side lobe; – – – – without suppression of the lobe.

Let us consider the directional pattern for power, obtained during suppression of first side lobes by the shown method:

$$g(\theta) = \left|\frac{1}{d}\int_{-d/2}^{d/2} \psi(x, \mathbf{k}_0) e^{i\frac{2\pi x}{\lambda}\theta} dx\right|^2 \approx$$

$$\approx |C(\theta) + 0{,}207\,[C(\theta - \theta_m) + C(\theta + \theta_m)]|^2.$$

Relationship $g(\theta)$ is shown in Fig. 13.3. In the same place for comparison there is plotted the directional pattern with an uncompensated side lobe. From comparison of patterns it is clear that suppression of the first side lobe is accompanied by certain expansion of the principal lobe and

increase of the second side lobe by 2 db. As already noted, by the shown processing it is possible to synthesize a system with suppression of any number of side lobes.

§ 13.3. Statistical Synthesis of Optimum Systems of Resolution

We shall now consider the second (of those enumerated in the introduction) approach to the problem of optimum resolution of targets, in which optimum from the point of view of resolving power is considered that system which best executes some function of the radar in the presence of many rather closely located targets.

Let us find general properties of optimum processing of the received signal in multitarget systems without specifying the solutions, taken for the set of targets. This can be done by using the concept of minimum sufficient statistics and a sufficient receiver (see § 3.7, Vol. I).

If there is no a priori information about the number and position of the targets, minimum sufficient statistics for any solutions taken for a set of targets is the set of values of likelihood ratios for all possible situations $\Lambda_1(y; \lambda_1)$, $\Lambda_2(y; \lambda_1, \lambda_2)$, ..., $\Lambda_n(y; \lambda_1, ..., \lambda_n)$, ..., where n (number of targets) has values from 1 to ∞, and parameters of targets λ_1, ..., λ_n pass through all possible a priori values. In accordance with this, sufficient is a receiver at whose output there will be formed the shown set. Having output signals of a sufficient receiver, it is possible to optimally find any solutions about the set of targets. The form of further transformations of these signals and what part of them is used depend on the presence of a priori information, the character of the sought solution and the rules utilized for solution.

Each of the likelihood ratios Λ_n is the ratio of the probability of reception of realization y in the presence of n signals with parameters λ_1, ..., λ_n to the probability of reception of the same realization without the signals. Instead of the likelihood ratios it is possible to consider the likelihood function; however in this case there arise certain formal difficulties connected with transition from discrete realizations to continuous.

Let us consider operations connected with forming of the likelihood ratio for rapid and slow fluctuations of the reflected signal, considering signals and interference statistically independent Gaussian random processes. Here, in order to have the possibility of comparing results of synthesis with results of optimum separation of signals, we will generalize expressions for likelihood ratios obtained in Chapters I and IV Vol. I to the multi-dimensional case.

Consideration of the multi-dimensional case is produced exactly as for the one-dimensional. We assume that in multi-dimensional domain S_0 there are assigned interference with correlation function $R(s_1, s_2)$ and a signal with correlation function $R_c(s_1, s_2)$, where both the signal and interference are Gaussian random processes. We consider the set of points $s_1, \ldots, s_N$. The likelihood ratio for values $y_j = y(s_j)$ is equal to

$$\Lambda(y_1, \ldots, y_N) = \frac{\exp\left\{\frac{1}{2}\sum_{j,\,k=1}^{N} (W_{jk} - W_{cn\,jk})\, y_j y_k\right\}}{\left|\delta_{jk} + \sum_{l=1}^{N} W_{jl} R_{c\,lk}\right|^{\frac{1}{2}}}, \tag{13.3.1}$$

where $R_{jk} = R(s_j, s_k)$; $R_{c\,jk} = R_c(s_j, s_k)$, and W_{jk} and $W_{cn\,jk}$ – elements of matrices, the reciprocals of correlation matrices of the interference and the signal with interference, respectively.

Passing to processing as $N \to \infty$ and bringing to zero the distances between points, it is possible to replace the sum by integrals and, using multi-dimensional analogs of formulas (1.4.1), (1.4.2), (1.4.12), (1.4.13), (4.2.4) and (4.2.5), write the likelihood ratio for the realization in the form

$$\Lambda(y) = \exp\left\{\frac{1}{2}\iint_{S_0} V(s_1, s_2)\, y(s_1)\, y(s_2)\, ds_1 ds_2 - \right.$$
$$\left. - \frac{1}{2}\int_0^1 d\alpha \int_{S_0} B(s, s, \alpha)\, ds\right\}, \tag{13.3.2}$$

where $V(s_1, s_2)$ and $B(s_1, s_2, \alpha)$ are determined by equations

$$\int_{S_0} V(s_1, s)\,[R(s, s_2) + R_c(s, s_2)]\, ds = W_1(s_1, s_2), \tag{13.3.3}$$

$$B(s_1, s_2; \alpha) + \alpha \int_{S_0} B(s_1, s; \alpha)\, W_1(s, s_2)\, ds = W_1(s_1, s_2), \tag{13.3.4}$$

$$W_1(s_1, s_2) = \int_{S_0} W(s_1, s)\, R_c(s, s_2)\, ds, \tag{13.3.5}$$

$$\int_{S_0} W(s_1, s)\, R(s, s_2)\, ds = \delta(s_1 - s_2). \tag{13.3.6}$$

Variables s in these formulas are multi-dimensional vectors.

We concretize the obtained relationships for the case when the signal is the

sum of n statistically independent signals with correlation functions of form (13.2.1). The radius of correlation of fluctuations of signals is considered large as compared to dimensions of domain S_0 (case of slow fluctuations). When

$$R_0(s_1, s_2) = \sum_l E_j \operatorname{Re} \varphi(s_1, \lambda_j) \varphi^*(s_2, \lambda_j),$$

solutions of equations (13.3.3) and (13.3.4) are presented, as it is simple to prove, in the form

$$V(s_1, s_2) = 2 \operatorname{Re} \sum_{j,k} v_{jk} X^*_j(s_1) X_k(s_2), \tag{13.3.7}$$

$$B(s_1, s_2; \alpha) = 2 \operatorname{Re} \sum_{j,k} b_{jk}(\alpha) \varphi^*(s_1, \lambda_j) X_k(s_2), \tag{13.3.8}$$

where

$$X_j(s_1) = \int_{S_0} W(s_1, s) \varphi(s, \lambda_j) \, ds, \tag{13.3.9}$$

and matrices $\|v_{jk}\|$ and $\|b_{jk}(\alpha)\|$ are reciprocals of matrices

$$\left\| \frac{2}{E_k} \delta_{jk} + C(\lambda_j, \lambda_k) \right\|, \quad \left\| \frac{2}{E_k} + \alpha C(\lambda_j, \lambda_k) \right\|$$

respectively, where $C(\lambda_j, \lambda_k)$, as before, is determined by formula (13.2.10).

Substituting (13.3.7) and (13.3.8) in (13.3.2), we obtain

$$\Lambda_n(y; \lambda_1, \ldots, \lambda_n) = \exp \Big\{ \sum_{j,k} v_{jk} Q^*_j Q_k - \\ - \int_0^1 \sum_{j,k} b_{jk}(\alpha) C(\lambda_k, \lambda_j) \, d\alpha \Big\}, \tag{13.3.10}$$

where

$$Q_j = \int_{S_0} y(s) X_j(s) \, ds. \tag{13.3.11}$$

In § 1.4 Vol. I it was shown that for an arbitrary matrix we have relationship

$$\frac{\partial \ln |R_{jk}|}{\partial \lambda} = \sum_{j,k} \frac{\partial R_{jk}}{\partial \lambda} W_{kj},$$

where λ – parameter.

Using this formula and considering properties of matrix $\|b_{jk}\|$, it is possible to rewrite (13.3.10) in the form

$$\Lambda_n (y; \lambda_1, \ldots, \lambda_n) = \exp \Big\{ \sum_{j,\,k} v_{jk} Q^*{}_j Q_k -$$
$$- \ln \Big| \delta_{jk} + \frac{E_k}{2} C (\lambda_j, \lambda_k) \Big| \Big\}. \qquad (13.3.12)$$

Let us consider operations connected with forming the logarithm of the likelihood ratio. The first component in $\Lambda_n(y; \lambda_1, \ldots, \lambda_n)$ can, taking into account (13.3.9) and (13.3.11), be transformed in the following way:

$$L_n (y) = \sum_{j,\,k} v_{jk} Q^*{}_j Q_k =$$
$$= \sum_j Q^*{}_j \int_{S_0} y (s) \Big(\sum_k v_{jk} \int_{S_0} W (s, s_1)\, \varphi (s_1, \lambda_k)\, ds_1 \Big) ds. \qquad (13.3.[illegible])$$

The function of s in parentheses under the integral coincides [see Paragraph 13.2.1] with the optimum reference signal $\psi(s, \lambda_j)$, ensuring a maximum ratio of the power of the considered signal to the sum of powers of interference and all other signals, considered as disturbing. As noted in § 13.2, for large signal-to-interference ratios for all targets this reference signal coincides with that which ensures total suppression of interfering signals.

Thus, transformations connected with obtaining the likelihood ratio include operations providing partial, and with high energies, practically total separation of signals, the hypothesis of whose presence is being proved.* Furthermore, these transformations include multiplication by reference signals $X_j^*(s)$, optimum for separation of a signal from interference, and integration. Results of both forms of correlation processing with identical indices are multiplied and totaled. The corresponding block-diagram is shown in Fig. 13.4.

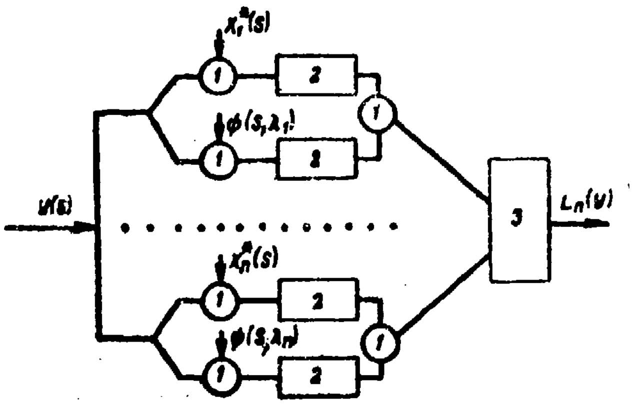

Fig. 13.4. Block diagram of optimum processing a signal with slow fluctuations: 1 – multiplier; 2 – integrator over domain S_0; 3 – adder.

*This is the difference from the case considered in §13.2, where there was ensured suppression of signals, the presence of which was not necessarily assumed, but was considered possible.

Let us consider analogously the case of rapid fluctuations, where we consider that fluctuation are fast with respect to one variable — time. In order to separate this coordinate, in our examination we everywhere shall write instead of one variable s two variables s and t. The correlation function of the signal we shall write in this case in the form

$$R_c(s_1, t_1, s_2, t_2) = \operatorname{Re} \sum_j E_j \varphi(s_1, t_1; \lambda_j) \varphi^*(s_2, t_2; \lambda_j) \rho_j(t_1 - t_2).$$

We assume also that the spectrum of fluctuations of interference is considerably wider than the spectrum of fluctuations of the signal, and we replace interference by white noise, i.e., we assume

$$R(s_1, t_1, s_2, t_2) = R_1(s_1, s_2) \delta(t_1 - t_2),$$

and that $\varphi(s, t; \lambda_j)$ is a periodic function of time for all j, where we shall consider the magnitude of the period T_r, common for all λ_j, small as compared to the time of correlation of fluctuations of the signal. Solutions of equations (13.3.3) and (13.3.4), as before, we shall seek in the form of (13.3.7) and (13.3.8), but v_{jk} and b_{jk} we shall consider to depend on t_1, t_2. All these assumptions and the subsequent derivation are completely analogous to those used in Chapter IV Vol. I in examining signals depending only on time.

Substituting (13.3.7) in (13.3.3) and leveling term by term components in the right and left parts of the equation, with the given assumptions we obtain

$$v_{jk}(t_1, t_2) + \frac{E_k}{2} \sum_l \int_0^T v_{jl}(t_1, t) C(\lambda_l, \lambda_k) \rho_k(t - t_2)\, dt = \frac{E_k}{2} \rho_k(t_1 - t_2), \tag{13.3.14}$$

where

$$C(\lambda_l, \lambda_k) = \frac{1}{T_r} \int_0^{T_r} dt \iint_{S_0} \varphi(s_1, t, \lambda_l) W(s_1, s_2) \varphi^*(s_2, t, \lambda_k)\, ds_1 ds_2. \tag{13.3.15}$$

Inasmuch as it is considered that $\Delta f_{c\,j} T \gg 1$ ($\Delta f_{c\,j}$ — effective width of the spectrum of fluctuations) for all j, it is possible, disregarding fringe effects, to solve equation (13.3.14) by Fourier transformation. The matrix of spectra $V_{jk}(\omega)$ is the reciprocal of matrix $\left\| \frac{2\Delta f_{c\,k}}{E_k S_k(\omega)} + C(\lambda_j, \lambda_k) \right\|$, where $S_k(\omega)$ — spectral

density, corresponding to $\rho_k(t)$, normalized so that max $S_k(\omega) = 1$. Likewise we find $B_{jk}(\alpha, \omega)$.

Substituting results of reverse Fourier transformation from $V_{jk}(\omega)$ and $B_{jk}(\alpha, \omega)$ in (13.3.7), (13.3.8), and (13.3.2), we obtain

$$\Lambda(y, \lambda_1, \ldots, \lambda_n) = \exp\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\sum_{j,k}' V_{jk}(\omega) Q^*_j(\omega) Q_k(\omega) - \right.\right. \\ \left.\left. - T \ln\left|\delta_{jk} + \frac{E_k S_k(\omega)}{2\Delta f_{ck}} C(\lambda_j, \lambda_k)\right|\right] d\omega\right\}, \tag{13.3.16}$$

where

$$Q_j(\omega) = \int_0^T dt \int_{S_0} y(s, t) X_j(s, t) e^{-i\omega t} ds. \tag{13.3.17}$$

As can be seen from the obtained formulas, in the case of rapid fluctuations separate spectral components of the signal obtained as a result of correlation processing must be subjected to operations of the same form as for slow fluctuations. Optimum processing is more complicated here than in the case of one target, because $V_{jk}(\omega)$ cannot, in general, be presented in the form of the product of certain frequency responses of filters: $H^*_j(i\omega)$ and $H_k(i\omega)$.

With a large signal-to-interference ratio we have equality $\|V_{jk}(\omega)\| \approx \|w_{jk}\|$ in frequency band Δf, where the ratio of spectral densities of the signal and interference is great.

This frequency band is broadened with increase of the signal-to-interference ratio. By analogy with results obtained for one signal (Chapters IV, VII, IX, X), it is possible to expect that expansion of band Δf with growth of the signal-to-interference ratio, after this band exceeds the width of the spectrum of fluctuations of the signal, and the form of the frequency responses do not substantially affect the performance characteristics of the system. Therefore we replace $V_{jk}(\omega)$ by $w_{jk}|H(i\omega)|^2$, where $H(i\omega)$ — frequency response of a filter with a passband, not considerably exceeding the width of the spectrum of fluctuations max $\Delta f_{c\,j}$. Here it is possible, by transforming the expression for $L_n(y)$ just as we did in examing one signal, to present a block diagram for processing in the form shown in Fig. 13.5.

Above we considered optimum transformations of a signal in a sufficient receiver, connected with forming the likelihood ratio for any assumed set of targets. If on the number and position of targets there are imposed limitations, optimum processing is substantially simplified. The simplest example of this kind

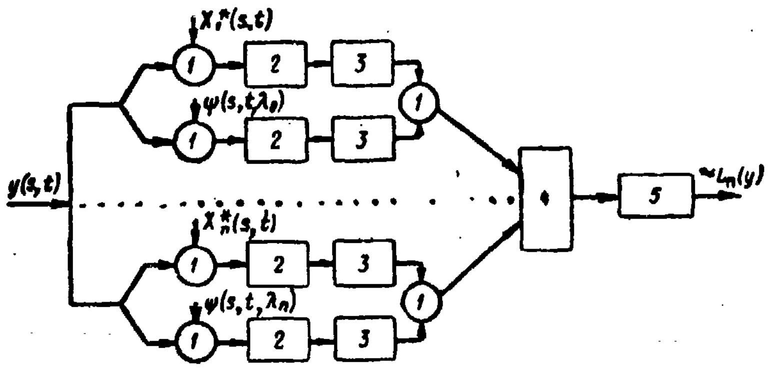

Fig. 13.5. Block diagram of quasi-optimum processing of a signal with fast fluctuations: 1 — multiplier; 2 — integrator over domain S_0; 3 — filter with frequency response $H(i\omega)$; 4 — adder; 5 — integrator during the time of observation.

is the case when parameters of n - 1 targets are known, and a solution is found only for the n-th target. The likelihood ratio for this problem equals ratio

$$\Lambda_n(y, \lambda_1, \ldots, \lambda_{n-1}, \lambda_n)/\Lambda_{n-1}(y, \lambda_1, \ldots, \lambda_{n-1}) = \Lambda(y).$$

Using the property (13.2.13) of inverse matrices, transformations of the signal connected with forming $\ln \Lambda(y)$ can be written in the form (with slow fluctuations)

$$L(y) = \frac{1}{v_{nn}}\left|\sum_{j=1}^{n} v_{nj} Q_j\right|^2. \tag{13.3.18}$$

In this case transformations of the signal ensure singling out of the useful signal with partial (and for large signal-to-interference ratios — total) suppression of interfering signals. Solution of the problem of detection of the n-th target against a background of n - 1 targets and interferences was conducted in [43] for finding potential possibilities of increasing resolving power.

Besides the mentioned work [43] there are a number of works in which there is posed and solved the problem of statistical synthesis of optimum resolution systems. In [44] by optimum resolution there is understood detection of a set of targets, producible by means of comparison of results of estimation for maximum likelihood of amplitudes of signals from these targets with a threshold. We consider the case of regular signals and white noise. Operations connected with estimating amplitudes coincide with those obtained in the problem of complete separation of signals.

In [64] there is offered another formulation of the problem, leading, however, under certain conditions to the same results. Considered best in resolving power

is the system providing the best estimate of the reflecting surface (level of reflected signal) as a function of the coordinates. As the loss function there is used*

$$I(\hat{\varphi}, \varphi) = \int_{S_0} |\hat{\varphi}(s) - \varphi(s)|^2 ds, \tag{13.3.19}$$

where $\varphi(s)$ — signal, corresponding to the true relief of reflecting surface; $\hat{\varphi}(s)$ — signal, corresponding to the estimated relief.

Minimization of mean risk is provided by selection of $\hat{\varphi}(s)$ at which $\overline{I(\hat{\varphi}, \varphi)}$ is minimum, where the dash here signifies averaging with the help of the a posteriori distribution of $p(\varphi|y)$. It is assumed that the signal $\varphi(s)$ is the superposition of signals from n point targets

$$\varphi(s) = \mathrm{Re} \sum_j A_j \varphi(s, \lambda_j),$$

where A_j — amplitude factors (signal $\varphi(s)$ is considered regular).

Here the problem reduces to finding $\hat{A}_j$, $\hat{\lambda}_j (j = 1, \ldots, n)$. Substituting (13.3.19) with $A_j = \hat{A}_j$, $\lambda_j = \hat{\lambda}_j$ in the formula for $\overline{I(\hat{\varphi}, \varphi)}$, differentiating with respect to A_j and equating the derivative to zero, we have

$$\hat{A}^*_j = 2 \sum_{l=1}^{n} B_{jl}^{-1} \Phi_l, \tag{13.3.20}$$

where B_{jl}^{-1} — element of a matrix, the inverse of $\|B_{jl}\|$:

$$B_{jl} = \int_{S_0} \varphi(s, \lambda_j) \varphi^*(s, \lambda_l) ds, \tag{13.3.21}$$

$$\Phi_j = \int_{S_0} \varphi(s, \lambda_j) \overline{\varphi(s)} ds. \tag{13.3.22}$$

Substituting (13.3.20) in the original formula for $\overline{I(\hat{\varphi}, \varphi)}$, we find that $\hat{\lambda}_j (j = 1, \ldots, n)$ must be selected from condition

$$\max_{\hat{\lambda}_j} \sum_{j, k} B_{jk}^{-1} \Phi^*_j \Phi_k. \tag{13.3.23}$$

If there is no a priori information about the position of the targets, $\overline{\varphi(s)} = y(s)$. Here the considered method of estimating the relief of the reflecting

*Here we do not hold to the designations adopted in the mentioned article.

surface coincides with the classical method of least squares. It is easy to see that transformation of (13.3.20) ensures total suppression of all signals, with the exception of that one whose amplitude is being estimated. It is not difficult to prove that this processing coincides with that considered in § 13.2, when $R(s_1, s_2) = N_0\delta(s_1 - s_2)$, i.e., when interference constitutes "white noise" with respect to all measurements.

Thus, the conducted consideration of transformations of the signal in multi-target systems which are optimum from different points of view showed that these transformations always include total or partial separation of signals from targets whose presence is assumed. In certain variants of formulation of the problem cross-processing of signals is exhausted by these operations of separation and further transformations of the separated signals are such as if each of the signals was unique (see Chapter IV Vol. 1). However in many cases this is not so, and for them it is interesting to compare characteristics of performance of an optimum system and a system in which separated signals are processed independently.

The fact is that practical use of optimum systems, in the which signal is processed in accordance with formulas (13.3.10) and (13.3.16), is possible, apparently, only with rather severe limitations (assigned in the form of a priori distribution) on the number of targets and their position. Thus are matters, e.g., during simultaneous tracking of several targets, the number and initial positions of which are sufficiently accurately known, or during detection of a group of a known number of targets, located at assigned points. In the absence of a priori information the optimum system is extraordinarily complicated, and it is advisable to use in this case a system with total suppression of assigned interfering signals and subsequent processing of the same form as for one signal.

In accordance with what has been said it is useful to take the following order of further consideration. In § 13.4 and § 13.5 we shall consider the problem of optimum detection and measurement of coordinates of an assigned set of targets. In § 13.6 we shall consider and compare with optimum ones simplified systems of detection and measurement, using the principle of total separation of signals.

§ 13.4. Optimum Detection of an Assigned Set of Targets

If possible positions of targets are fixed and it is required to determine which of these targets exist in fact, the optimum procedure of solution consists (see Chapter III, Vol I) of comparing likelihood ratios corresponding to all possible

situations with thresholds and among themselves. If losses connected with all possible errors in determination of the number and position of targets are identical, we should take such a solution in which the likelihood ratio has a maximum magnitude, exceeding the noise threshold. For instance, if the number of targets is not greater than two, it is necessary to compare with the threshold and among themselves

$$\Lambda_1(y, \lambda_1); \quad \Lambda_1(y, \lambda_2); \quad \Lambda_2(y, \lambda_1, \lambda_2).$$

In order to completely characterize the procedure of finding a solution in this case it is necessary to calculate the probabilities of all possible errors, the number of which, obviously, equals $\frac{n(n-1)}{2}$, where n is the number of possible solutions. For calculation of these probabilities it is necessary to find the joint distribution of probabilities for likelihood ratios. All these calculations turn out to be very cumbersome. Therefore we here will limit our consideration to the case of two targets. Let us calculate first the characteristics of detection of one of the targets, when the presence of the second target is exactly established. It is clear that this characteristic has the greatest bearing on the problem of resolution of closely located targets in conditions of detection. Absence of resolution will lead to detection of one target instead of two; therefore the quality of resolution can be estimated primarily by this characteristic.

Furthermore, we shall be interested in characteristics of detection of both targets when they are either present or absent simultaneously. Here we shall consider fluctuation of the signals slow.

In the first case the solution is found on the basis of comparision of quantity $\left|\sum_{j=1}^{2} v_{2j} Q_j\right|^2$ and, inasmuch as Q_j is distributed by normal law, the equation of characteristics of detection has the form (see Chapter IV Vol. I)

$$D = F^{\frac{1}{1+q_0}},$$

where q_0 – the signal-to-interference ratio at the output of the system forming $\sum_j v_{2j} Q_j$.

This ratio already was considered in § 13.2 in connection with the problem of separation of signals. Remember that the optimum system of detection in this case is a system of partial separation of signals, supplemented with a device comparing the square of the modulus of the output signal with the threshold.

Let us consider the case of a pair of simultaneously appearing targets. The solution is found on the basis of comparing with a threshold the likelihood ratio $\Lambda_2(y;\ \lambda_1,\ \lambda_2)$ or quantity

$$L_2(y)=\sum_{j,k=1}^{2} v_{jk}Q^*_jQ_k,$$

formed by the block diagram of Fig. 13.2. Writing the distribution for quantities Q_1, Q_2 in the form (see [19], and also formula (4.11.2) in Volume I)

$$p(Q_1,\ Q_2)=\frac{1}{\pi^2|R_{jk}|}\exp\left\{-\sum_{j,k=1}^{2} W_{jk}f_jf^*_k\right\},$$

where $\|W_{jk}\|$ – matrix, the inverse of R_{jk}, and

$$R_{jk}=\overline{Q_jQ^*_k}=C(\lambda_j,\ \lambda_k)+\sum_{l=1}^{2}\frac{E_l}{2}C(\lambda_j,\ \lambda_l)C(\lambda_l,\ \lambda_k), \tag{13.4.1}$$

it is easy to calculate the characteristic function of quantity $L_2(y)$:

$$\psi(\eta)=\left|\delta_{jk}-i\eta\sum_l v_{jl}R_{lk}\right|^{-1}. \tag{13.4.2}$$

Expanding the characteristic function to simple fractions, it is easy to find the corresponding density of distribution. Assuming for simplicity that the strengths of the signals are identical, we obtain for the probabilities of correct detection and false alarm the following expressions:

$$D(c)=\frac{1+\gamma}{2\gamma}e^{-\frac{c}{1+\gamma}}-\frac{1-\gamma}{2\gamma}e^{-\frac{c}{1-\gamma}}, \tag{13.4.3}$$

$$F(c)=\frac{1+\gamma+q'_0(1-\gamma^2)}{2\gamma}\exp\left[-\frac{(1+q'_0)^2-q'^2_0\gamma^2}{1+\gamma+q'_0(1-\gamma^2)}c\right]-$$
$$-\frac{1-\gamma+q'_0(1-\gamma^2)}{2\gamma}\exp\left[-\frac{(1+q_0)^2-q_0^2\gamma^2}{1-\gamma+q'_0(1-\gamma^2)}c\right], \tag{13.4.4}$$

where c – is the magnitude of the threshold, divided by q'_0; with which we compare L_2, $\gamma=|C(\lambda_1,\ \lambda_2)|/\sqrt{C(\lambda_1,\ \lambda_1)C(\lambda_2,\ \lambda_2)}$;

q'_0 – the signal-to-interference ratio in the absence of separation.

In order to simplify calculations connected with determination of threshold ratio q'_0, corresponding to the selected probabilities F and D, it is possible to use the fact that, as one may see from (13.4.3) and (13.4.4),

$$F(c,\ \gamma,\ q'_0)=D(c_1,\ \gamma_1), \tag{13.4.5}$$

where

$$\gamma_1 = \frac{\gamma}{1 + q'_0(1-\gamma^2)}; \quad c_1 = c\frac{(1+q'_0)^2 - q'^2_0\gamma^2}{1 + q'_0(1-\gamma^2)}. \tag{13.4.6}$$

If $F \ll 1$ and $D > 0.5$, the threshold values of q'_0 turn out to be so large that $\gamma_1 \approx 0$ when $\gamma^2 < 0.8$. One can prove this by performing the necessary numerical calculations. This circumstance essentially simplifies calculation of dependence $q_0(F, D, \gamma)$, which can be produced by the following scheme. First we graphically find dependence $c(D, \gamma)$, where D's are given very small values, typical for the probability of false alarm. Incidentally, with these values we can disregard the second component in (13.4.3), if $\gamma^2 > 0.2$. The q'_0 is determined from the second relationship of (13.4.6)

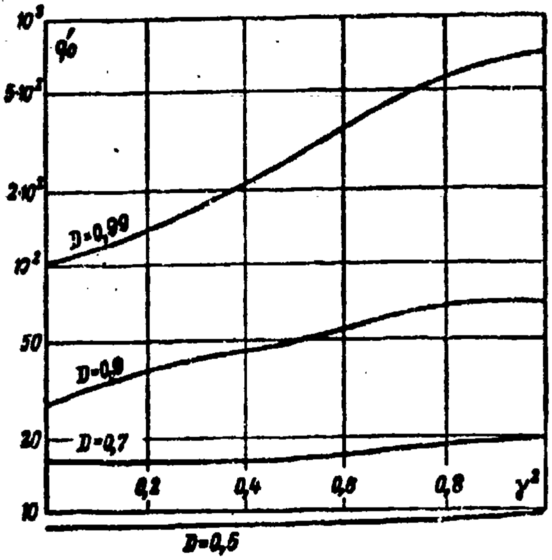

Fig. 13.6. Dependence of the threshold signal-to-interference ratio on the degree of nonorthogonality of the detected signals during optimum processing.

$$q'_0 = \frac{1}{1-\gamma^2}\left\{\frac{c(F,0)}{2c(D,\gamma)}(1-\gamma^2) - 1 + \sqrt{\left[\frac{c(F,0)}{2c(D,\gamma)}(1-\gamma^2)\right]^2 + \gamma^2}\right\}. \tag{13.4.7}$$

Relationship $q'_0(D, \gamma)$ when $F = 10^{-6}$ is shown in Fig. 13.6. As can be seen from the figure, for D's not very close to unity q'_0 weakly depends on parameter γ, slowly increasing with increase of the latter. When $1 - D \ll 1$ this growth is more marked. In § 13.6 the obtained result will be used for comparison of the optimum system and a system with total separation of signals.

§ 13.5. Synthesis of Systems of Resolution In Conditions of Measurement of Coordinates

13.5.1. Formulation of the Problem

During measurement by radar of coordinates of closely located unresolved targets it turns out that due to interaction of signals reflected from these targets the mean values of the measured quantities do not coincide with their true values. With total absence of resolution the radar measures the coordinate of every target. At the same time if for an assigned sounding signal we introduce resolution of

targets with the help of the proper processing of signals in the receiver, in general, the ratio of the signal to noise decreases and fluctuating error increases.

In connection with the shown circumstances it is useful to consider as the criterion of optimality of a resolution system during measurement of coordinates of targets absence of systematic errors of measurement of these coordinates and a minimum of fluctuating errors. The latter, as also earlier, can be estimated by their variances at each given moment of time.

If we apply this criterion for synthesis of the system of measurement of coordinates as a whole, then, as also in the case of one target, with observance of certain not too limiting conditions the system can be divided into two parts. The first of them is the discriminator in the tracking variant or the estimator unit in the nontracking variant of the meter, i.e., a system for processing the r-f signal, producing the current estimate of the measured parameter (set of parameters) λ. The second part of the system are the smoothing circuits.*

In the case of fast fluctuations of the signals the smoothing circuits have less bearing on questions of resolution. At the same time their synthesis in the presence of many targets and many parameters varying randomly in time, subject to measurement, constitutes a very complicated problem. In the present book we will not deal with this problem; we shall limit ourselves to synthesis of the device in which there is concentrated the radio part of the system. Not making any distinctions between tracking and nontracking variants of the system, we shall call this device a discriminator. General properties of optimum discriminators, corresponding to multitarget problems, will be formulated below.

Preliminarily let us note that in accordance with the formulated criterion the optimum discriminator should execute such operations on the received signal which correspond to formation of the current efficient estimate of the measured quantity λ, since the concept of efficiency includes the absence of bias and a minimum variance of the estimate. Practically, considering the approximate nature of fulfillment of optimum operations, it is sufficient to require asymptotic efficiency of the estimate.

Here it is possible to pose two problems. The first, and the simpler of them, arises when to measurement without systematic error with minimum fluctuating error

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*It should be noted that conditions under which an optimum meter indeed can be divided into the shown parts for multitarget problems is far from clear.

we subject one parameter λ_1, the carrier of which is signal $y_1(t, \lambda_1)$, but here there exist other signals $y_p(t, \lambda_p)$, $(p = 2, \ldots, n)$, not orthogonal to $y_1(t)$, parameters of which λ_p are known. For solution of this problem it is necessary to find the efficient (at least asymptotically) estimate of parameter λ_1 for known λ_p. Properties of an optimum discriminator are determined by the known formula for spectral density, corresponding to variance of the efficient estimate

$$S_{\text{опт}1} = \sigma^2_{\text{эф}1} T = -T \frac{1}{\overline{\frac{\partial^2 L(\lambda_1)}{\partial \lambda_1^2}}}, \qquad (13.5.1)$$

where T — time during which parameter λ_1 can be considered constant;

$L(\lambda_1)$ — logarithm of the likelihood function.

A second, more complicated problem corresponds to the case when n signals $y_p(t, \lambda_p)$ $(p = 1, 2, \ldots, n)$ depend on n unknown parameters λ_p. The problem does not change whether it is required to measure with zero systematic and minimum fluctuating errors all n parameters or one of them (for instance, λ_1). In the last case it is necessary all the same to measure all n parameters for optimum realization of compensation for extraneous signals.

For solution of this problem it is necessary to seek an asymptotically jointly-efficient estimate of parameters λ_p and properties of the optimum system obtained as a result of the solution of the problem are determined by variances and cross-correlation moments J_{pq} of the estimates:

$$\|J_{pq}\| = \left\| -\frac{\overline{\partial^2 L(\lambda_1, \ldots, \lambda_n)}}{\partial \lambda_p \partial \lambda_q} \right\|^{-1}, \qquad (13.5.2)$$

where $L(\lambda_1, \ldots, \lambda_n)$ — logarithm of the likelihood function of the parameters.

An ellipsoid with moments of inertia J_{pq} is the minimum among possible ones during measurement of parameters $\lambda_1, \ldots, \lambda_n$ of ellipsoids of dispersion [8, 66]. Questions connected with this problem will be considered in greater detail later.

There exist no direct methods of finding efficient or jointly-efficient estimates. At the same time we know that during measurements of one-dimensional parameters the comparatively easily found maximum likelihood estimates have the property of asymptotic efficiency with small times of correlation of the observed signal [8, 13]. There is every basis to assume that these estimates in the case of measurements of multi-dimensional parameters, too, will possess the property of joint asymptotic

efficiency.

This assumption we shall use as the basis for our further considerations. Here we shall find the maximum likelihood estimates of coordinates of many arbitrarily (frequently closely) located targets and demonstrate their joint asymptotic efficiency for a rapidly fluctuating normal signal.* The practical importance and applicability of this case was repeatedly proven earlier. It should be noted that solution of the first problem (only parameter λ_1 is unknown) will follow as a particular case from solution of the second problem.

For clarification of general properties of optimum discriminators we shall turn first of all to maximum likelihood estimates of multi-dimensional parameters. Then on this base we shall consider particular cases which it is possible to investigate completely and obtain in explicit form circuits and properties of optimum resolution systems. These are cases of n targets with a very large signal-to-noise ratio h and of two targets (n = 2) for any h.

13.5.2. Maximum Likelihood Estimates of Multi-Dimensional Parameters and Their Asymptotic Efficiency. General Properties of an Optimum Discriminator

Let us assume that in field of sight of the radar there are n targets. Then the signal received by the radar can be recorded in the form

$$y(t)=\sum_{p=1}^{n} y_p(t, \lambda_p)+n(t), \tag{13.5.3}$$

where $n(t)$ — normal white noise with spectral density N_0;

$y_p(t, \lambda_p)$ — normal random process with a correlation function depending on the unknown parameter λ_p

$$R_p(t_1, t_2)=\operatorname{Re} P_p u_p(t_1, \lambda_p) u^*_p(t_2, \lambda_p)\rho_p(t_1-t_2)\times \times e^{i\omega_0(t_1-t_2)}. \tag{13.5.4}$$

Here P_p — mean power;

$\rho_p(t)$ — coefficient of correlation of fluctuations;

$u_p(t, \lambda_p)$ — complex law of modulation of the p-th signal:

*Unfortunately, with a slowly fluctuating signal asymptotic efficiency of joint maximum likelihood estimates utilized by us does not take place. As a result, although the structure of the multitarget measuring system itself is found without special difficulties, the question of its optimality remains open. Therefore here we consider only the case of rapid fluctuations of signals reflected from the targets.

$$u_p(t, \lambda_p) = u_{ap}(t, \lambda_p) e^{i\psi_p(t, \lambda_p)}.$$

Index p in these expressions allows for the possible deper- λ_p) and $u_p(t, \lambda_p)$ on other parameters besides λ_p. We shall consid λ_p) are periodic (with period T_r) functions and that the interval of coi of fluctuations τ_{pk} [effective width of function $\rho_p(t)$] is much larger than period T_r. At the same time we shall consider this interval much smaller than time T, during which parameter λ_p is constant:

$$T_r \ll \tau_{pk} \ll T. \tag{13.5.5}$$

Additionally we assume that different reflected signals are statistically independent; therefore functions of their cross-correlation are:

$$\rho_{pq}(t) = P_p \delta_{pq} \rho_p(t),$$

where δ_{pq} — Kronecker delta.

The formulated assumptions are normal and natural. A new element here is the condition of independence of fluctuations of different targets, which is practically always realized.

We introduce designation

$$\frac{1}{T_r}\int_0^{T_r} u^*_p(t, \lambda_p)\, u_q(t, \lambda_q)\, dt = C_{pq}(\lambda_p, \lambda_q), \tag{13.5.6}$$

where $C_{pq}(\lambda_p, \lambda_q)$ — correlation functions of modulations of signals reflected from different targets. These functions with certain small modifications, determined by the essence of the question in connection with which they are introduced into consideration, appeared earlier. Thus, for instance, in Chapter I Vol. I we introduced functions $C_0(\tau, \Omega)$, which are a particular case of functions (13.5.6) when signals from different targets differ in delays and Doppler frequency shifts. Correlation functions of signals were introduced already in § 13.2 of the present chapter [see (13.2.10)]. But for our further purposes most suitable is definition (13.5.6). Let us note that, as a rule, C_{pp} does not depend on the index p and λ_p. We shall consider functions $u_p(t, \lambda_p)$ normalized so that $C_{pp} = 1$.

Let us turn to calculation of maximum likelihood estimates of parameters λ_p and study of their properties. Here in the first place we must obtain an expression for the likelihood functional of parameters λ_p, which for a more general case was found

in § 13.3. We shall repeat briefly calculations of this paragraph in reference to the considered case.

The correlation function of signal y(t) (13.5.3) is equal to

$$\overline{y(t_1)y(t_2)} = R(t_1, t_2) = \mathrm{Re}\, e^{i\omega_0(t_1-t_2)} \times$$
$$\times \sum_{p=1}^{n} u_p(t_1, \lambda_p) u^*_p(t_2, \lambda_p) P_p \rho_p(t_1 - t_2) + N_0 \delta(t_1 - t_2). \tag{13.5.7}$$

We seek function $W(t_1, t_2)$ in the form

$$W(t_1, t_2) = \mathrm{Re}\, e^{i\omega_0(t_1-t_2)} \sum_{p,q=1}^{n} u_p(t, \lambda_p) \times$$
$$\times u^*_q(t_2, \lambda_q) v_{pq}(t_1, t_2) + \frac{1}{N_0} \delta(t_1 - t_2), \tag{13.5.8}$$

where $v_{pq}(t_1, t_2)$ are unknown functions which we consider to vary slowly as compared to $u_p(t, \lambda_p)$.

Substituting expressions (13.5.7) and (13.5.8) in the integral equation for function $W(t_1, t_2)$

$$\int_0^T W(t_1, t_2) R(t_2, t_3)\, dt_2 = \delta(t_1 - t_3) \tag{13.5.9}$$

and considering the rapidly-varying nature of functions $u_p(t, \lambda_p)$ as compared to $\rho_p(t)$ and $v_{pq}(t_1, t_2)$, it is possible for functions $v_{pq}(t_1, t_2)$ to obtain the system of integral equations

$$\rho_{pq}(t_1 - t_3) + N_0^2 v_{pq}(t_1, t_3) + \frac{N_0}{2} \int_0^T \sum_{s=1}^{n} P_p \rho_p(t_1 -$$
$$- t_2) C_{ps}(\lambda_p, \lambda_s) v_{sq}(t_2, t_3)\, dt_2 = 0.$$

In the case of a rapidly fluctuating signal, when the effective duration of peaks of functions $\rho_p(t)$ is considerably less than time T, we may assume that $v_{pq}(t_1, t_3) = v_{pq}(t_1 - t_3)$, and for the Fourier transforms of functions $v_{pq}(t)$ we can obtain system of equations

$$P_p S_p(\omega) \delta_{pq} + N_0^2 V_{pq}(\omega) +$$
$$+ \frac{N_0}{2} \sum_{s=1}^{n} P_p S_p(\omega) C_{ps}(\lambda_p, \lambda_s) V_{sq}(\omega) = 0, \tag{13.5.10}$$

where $S_p(\omega)$ and $V_{pq}(\omega)$ are Fourier transforms of functions $\rho_p(t)$ and $v_{pq}(t)$, respectively.

This system of equations is easily solved. Considering

$$\mathbf{S}(\omega)=\|P_p S_p(\omega)\delta_{pq}\|, \quad \mathbf{V}(\omega)=\|V_{pq}(\omega)\|,$$
$$\mathbf{E}=\|\delta_{pq}\|, \quad \mathbf{C}=\|C_{pq}(\lambda_p, \lambda_q)\|,$$

we rewrite system (13.5.10) in the form

$$\mathbf{S}(\omega)+N_e^2\mathbf{V}(\omega)+\frac{N_0}{2}\mathbf{S}(\omega)\mathbf{C}\mathbf{V}(\omega)=0,$$

from which

$$\mathbf{V}(\omega)=-\frac{1}{N_0^2}\left(\mathbf{E}+\frac{1}{2N_e}\mathbf{S}(\omega)\mathbf{C}\right)^{-1}\mathbf{S}(\omega). \qquad (13.5.11)$$

Thus, basically, we find function $W(t_1, t_2)$. True, matrix inversion in expression (13.5.11) is a very difficult problem in itself.

Knowing function $W(t_1, t_2)$, we can study properties of maximum likelihood estimates of parameters λ_p. The system of maximum likelihood equations has the form

$$\frac{\partial L(\lambda_1, \ldots, \lambda_n)}{\partial\lambda_p}=0. \qquad (13.5.12)$$

Let us designate by $\lambda_1^{(0)}, \ldots, \lambda_n^{(0)}$ the set of values of the parameters which we know are close to the true values of the parameters. In particular, these may be true values of the parameters. Considering that with a sufficiently large time of observation solutions of equations (13.5.12) will be sufficiently close to $\lambda_1^{(0)}, \ldots, \lambda_n^{(0)}$, we can replace equation (13.5.12) by statistically equivalent equations

$$\frac{\partial L(\lambda_1^{(0)}, \ldots, \lambda_n^{(0)})}{\partial\lambda_p}+\sum_{q=1}^{n}\frac{\partial^2 L(\lambda_1^{(0)}, \ldots, \lambda_n^{(0)})}{\partial\lambda_p\partial\lambda_q}(\lambda_q-\lambda_q^{(0)})=0. \qquad (13.5.12')$$

For study of properties of solutions of these equations of paramount importance is matrix

$$\mathbf{A}=\left\|-\frac{\partial^2 L(\lambda_1^{(0)}, \ldots, \lambda_n^{(0)})}{\partial\lambda_q\partial\lambda_p}\right\|.$$

We shall consider $\lambda_1^{(0)}, \ldots, \lambda_n^{(0)}$ true values of the parameters. We find the mean value and variance of this matrix (understanding by mean value of a matrix a matrix composed of mean values of the elements, and by variance of a matrix a matrix

composed of variances of elements). We note here that matrix

$$\bar{\mathbf{A}} = \mathbf{J}^{-1}, \tag{13.5.13}$$

where J is the matrix of variances and cross-correlations of jointly-efficient estimates (13.5.2)

Generalizing slightly the results of Chapter I (Vol. I of this book), we can write the following expression

$$\frac{\partial L(\lambda_1, \ldots, \lambda_n)}{\partial \lambda_p} = -\frac{1}{2}\int_0^T\int_0^T \frac{\partial R(t_1, t_2)}{\partial \lambda_p} W(t_1, t_2)\, dt_1 dt_2 - \\ -\frac{1}{2}\int_0^T\int_0^T \frac{\partial W(t_1, t_2)}{\partial \lambda_p} y(t_1) y(t_2)\, dt_1 dt_2. \tag{13.5.14}$$

From this, after elementary transformations, we obtain

$$\overline{\frac{\partial^2 L(\lambda_1^{(0)}, \ldots, \lambda_n^{(0)})}{\partial \lambda_p \partial \lambda_q}} = -\frac{1}{2}\int_0^T\int_0^T \frac{\partial W(t_1, t_2)}{\partial \lambda_q} \cdot \frac{\partial R(t_1, t_2)}{\partial \lambda_p}\, dt_1 dt_2. \tag{13.5.14'}$$

Substituting in (13.5.14') expressions for $R(t_1, t_2)$ and $W(t_1, t_2)$ and introducing designations

$$C_{pq}^{(10)} = C_{qp}^{(01)*} = \frac{1}{T_r}\int_0^{T_r} \frac{\partial u^*_p(t, \lambda_p^{(0)})}{\partial \lambda_p} u_q(t, \lambda_q^{(0)})\, dt,$$

$$C_{pq}^{(11)} = \frac{1}{T_r}\int_0^{T_r} \frac{\partial u^*_p(t, \lambda_p^{(0)})}{\partial \lambda_p} \cdot \frac{\partial u_q(t, \lambda_q^{(0)})}{\partial \lambda_q}\, dt, \tag{13.5.15}$$

it is easy to obtain the following:

$$\overline{\frac{\partial^2 L(\lambda_1^{(0)}, \ldots, \lambda_n^{(0)})}{\partial \lambda_p \partial \lambda_q}} = \operatorname{Re}\Big\{\sum_{m=1}^{n} (C_{qp}^{(11)} C_{pm} + C_{pm}^{(10)} C_{qp}^{(10)}) \times \int_0^T\int_0^T \rho_p(t_1 - t_2) v_{mq}(t_1 - t_2)\, dt_1 dt_2 + \\ + \sum_{n,\, m=1}^{n} C_{pn}^{(10)} C_{mp} \times \int_0^T\int_0^T \rho_p(t_1 - t_2) \frac{\partial v_{nm}(t_1 - t_2)}{\partial \lambda_q}\, dt_1 dt_2\Big\} \approx T \operatorname{Re}\Big\{\sum_{m=1}^{n} (C_{qp}^{(11)} C_{1m} + \\ + C_{pm}^{(10)} C_{qp}^{(10)}) \int_{-\infty}^{\infty} \rho_p(t) v_{mq}(t)\, dt + \sum_{n,\, m=1}^{n} C_{pn}^{(10)} C_{mp} \int_{-\infty}^{\infty} \rho_p(t) \frac{\partial v_{nm}(t)}{\partial \lambda_q}\, dt\Big\}. \tag{13.5.16}$$

Here we used, first, the rapidly-varying nature of functions $u_p(t, \lambda_p)$ as compared

to $\mu_p(t)$ and $v_{pq}(t)$, which permitted us under signs of integrals to average expressions with functions $u_p(t, \lambda_p)$; second, we used the fact that functions $v_{pq}(t_1 - t_2)$ practically immediately turn into 0 when t_1 goes beyond the limits of interval $(0, T)$, which permitted us to replace integrals over interval $(0, T)$ from functions $v_{pq}(t_1 - t_2)$ by integrals over interval $(-\infty, \infty)$. Very significant is asymptotic dependence of $\overline{\frac{\partial^2 L(\lambda_1^{(0)}, \ldots, \lambda_n^{(0)})}{\partial\lambda_p \partial\lambda_q}}$ on time T, having form

$$\overline{\frac{\partial^2 L(\lambda_1^{(0)}, \ldots, \lambda_n^{(0)})}{\partial\lambda_p \partial\lambda_q}} \sim T.$$

For finding of dispersion $D\left[\frac{\partial^2 L(\lambda_1^{(0)}, \ldots, \lambda_n^{(0)})}{\partial\lambda_p \partial\lambda_q}\right]$ we use relationship

$$D\left[\frac{\partial^2 L(\lambda_1^{(0)}, \ldots, \lambda_n^{(0)})}{\partial\lambda_p \partial\lambda_q}\right] = \frac{1}{2}\int_0^T\int_0^T\int_0^T\int_0^T \frac{\partial^2 W(t_1, t_2)}{\partial\lambda_p \partial\lambda_q}\,\frac{\partial^2 W(t_3, t_4)}{\partial\lambda_p \partial\lambda_q} \times R(t_1, t_3) R(t_2, t_4)\, dt_1 dt_2 dt_3 dt_4$$

(a relationship of this type we derived and used in Chapter X of this book). Exact calculation of this integral is rather cumbersome. For our problem exact knowledge of the magnitude of $D\left[\frac{\partial^2 L(\lambda_1^{(0)}, \ldots, \lambda_n^{(0)})}{\partial\lambda_p \partial\lambda_q}\right]$ is not required; it is necessary only to know the asymptotic dependence of this magnitude on T. It is easy to see that the calculated integral is a sum with finite coefficients of integrals of type

$$\int_0^T\int_0^T\int_0^T\int_0^T v_{kl}(t_1 - t_2)\, v_{mn}(t_3 - t_4)\, \rho_{rs}(t_1 - t_3)\, \rho_{tu}(t_2 - t_4)\, dt_1 \times dt_2 dt_3 dt_4,$$

$$\int_0^T\int_0^T\int_0^T v_{kl}(t_1 - t_2)\, v_{mn}(t_3 - t_2)\, \rho_{rs}(t_1 - t_3)\, dt_1 dt_2 dt_3,$$

$$\int_0^T\int_0^T v_{kl}(t_1 - t_2)\, v_{mn}(t_1 - t_2)\, dt_1 dt_2$$

and their derivatives with respect to λ_p and λ_q up to and including the second order. Coefficients of these integrals are time-averaged values of expressions containing rapidly varying functions $u_p(t, \lambda_p)$. Further it is possible to note that we have equalities

$$\int_0^T\int_0^T\int_0^T\int_0^T v_{kl}(t_1 - t_2)\, v_{mn}(t_3 - t_4)\, \rho_{rs}(t_1 - t_3)\, \rho_{tu}(t_2 - t_4) \times$$
$$\times\, dt_1 dt_2 dt_3 dt_4 \approx T \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_{kl}(t_1)\, v_{mn}(t_2)\, \rho_{rs}(t_1 - t_2 - t_3) \times$$
$$\times\, \rho_{tu}(t_3)\, dt_1 dt_2 dt_3,$$

$$\int_0^T\int_0^T\int_0^T v_{kl}(t_1-t_2)\,v_{mn}(t_3-t_2)\,\rho_{rs}(t_1-t_3)\,dt_1dt_2dt_3 \approx$$
$$\approx T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_{kl}(t_1)\,v_{mn}(t_2)\,\rho_{rs}(t_1-t_2)\,dt_1dt_2,$$
$$\int_0^T\int_0^T v_{kl}(t_1-t_2)\,v_{mn}(t_1-t_2)\,dt_1dt_2 \approx T\int_{-\infty}^{\infty} v_{kl}(t)\,v_{mn}(t)\,dt.$$

From this it follows that

$$D\left[\frac{\partial^2 L(\lambda_1^{(0)},\ldots,\lambda_n^{(0)})}{\partial\lambda_p\partial\lambda_q}\right]\sim T,$$
$$\frac{\sqrt{D\left[\dfrac{\partial^2 L(\lambda_1^{(0)},\ldots,\lambda_n^{(0)})}{\partial\lambda_p\partial\lambda_q}\right]}}{\overline{\dfrac{\partial^2 L(\lambda_1^{(0)},\ldots,\lambda_n^{(0)})}{\partial\lambda_p\partial\lambda_q}}}\sim\frac{1}{\sqrt{T}}.$$

The last relationship shows that for large T quantity $\dfrac{\partial^2 L(\lambda_1^{(0)},\ldots,\lambda_n^{(0)})}{\partial\lambda_p\partial\lambda_q}$ in probability seeks its own mean value and may be replaced by it. Thus, for large T matrix $\mathbf{A}\approx\overline{\mathbf{A}}$. Now we can solve equation (13.5.12'). Using (13.5.13), we obtain

$$\lambda_p-\lambda_p^{(0)} = \sum_{r=1}^{n} J_{pr}\frac{\partial L(\lambda_1^{(0)},\ldots,\lambda_n^{(0)})}{\partial\lambda_r}, \qquad (13.5.17)$$

where J_{pr} – elements of matrix (13.5.2).

Hence

$$\overline{\lambda}_p=\lambda_p^{(0)},$$
$$\overline{(\lambda_p-\lambda_p^{(0)})(\lambda_q-\lambda_q^{(0)})}=$$
$$=\sum_{r,s=1}^{n} J_{pr}J_{qs}\overline{\frac{\partial L(\lambda_1^{(0)},\ldots,\lambda_n^{(0)})}{\partial\lambda_r}\cdot\frac{\partial L(\lambda_1^{(0)},\ldots,\lambda_n^{(0)})}{\partial\lambda_s}}=J_{pq}.$$

From these equalities there ensues, by definition, joint efficiency of maximum likelihood estimates taking place with a rapidly fluctuating signal and a sufficiently large time of observation.*

*Let us note that time of observation T at which we already have approximate efficiency of the considered estimates is greater, the less the coordinates of targets differ.

We note now that all the results presented are easily generalized in the case when we observe, not a scalar, but a vector signal. We meet such a case when the radar has a multiunit antenna. In this case we observe several signals at the output of different elements of the antenna. We shall enumerate briefly the obvious modifications which the above-stated results will obtain in examining this more general case. Signals received by different elements of the antenna will be recorded in the form

$$y_l(t) = \sum_{p=1}^{n} y_p^{(l)}(t, \lambda_p) + n_l(t), \tag{13.5.18}$$

where $y_p^{(l)}(t, \lambda_p)$ – signal from the p-th target at the output of the l-th element of the antenna.

The correlation function of signals $y_p^{(l)}(t, \lambda_p)$ will be recorded in the form

$$R_{l_1 l_2}(t_1, t_2) = \overline{y_p^{(l_1)}(t_1, \lambda_p)\, y_p^{(l_2)}(t_2, \lambda_p)} = \\ = \operatorname{Re} P_p u_p^{(l_1)}(t_1, \lambda_p)\, u_p^{(l_2)*}(t_2, \lambda_p)\, \rho_p(t_1 - t_2)\, e^{i\omega_0 (t_1 - t_2)},$$

where $u_p^{(l)}(t, \lambda_p)$ – complex modulation of the signal reflected from the p-th target and received by the l-th element of the antenna. The correlation function of the whole signal (13.5.18) will have form

$$\overline{y_{l_1}(t_1) y_{l_2}(t_2)} = \operatorname{Re} e^{i\omega_0 (t_1 - t_2)} \sum_{p=1}^{n} P_p \rho_p (t_1 - \\ - t_2)\, u_p^{(l_1)}(t_1, \lambda_p)\, u_p^{(l_2)*}(t_2, \lambda_p) + N_0 \delta_{l_1 l_2} \delta(t_1 - t_2) \tag{13.5.18'}$$

(we consider noises $n_l(t)$ independent). Function $W(t_1, t_2)$ now will be equal to

$$W_{l_1 l_2}(t_1, t_2) = \operatorname{Re} e^{i\omega_0 (t_1 - t_2)} \sum_{p,q=1}^{n} v_{pq}(t_1, t_2)\, u_p^{(l_1)}(t_1, \lambda_p) \times \\ \times u_q^{(l_2)*}(t_2, \lambda_q) + \frac{\delta_{l_1 l_2}}{N_0} \delta(t_1 - t_2), \tag{13.5.18''}$$

where function $v_{pq}(t_1, t_2)$ will be precisely the same as in the case of a scalar signal, only by $C_{pq}(\lambda_p, \lambda_q)$ it is necessary now to understand

$$C_{pq}(\lambda_p, \lambda_q) = \frac{1}{T_r} \int_0^{T_r} \frac{1}{N} \sum_{l=1}^{N} u_p^{(l)*}(t, \lambda_p)\, u_p^{(l)}(t, \lambda_q)\, dt. \tag{13.5.19}$$

Further, relationships (13.5.14) and (13.5.14') are replaced by

$$
\frac{\partial L(\lambda_1,\ldots,\lambda_n)}{\partial\lambda_p}=-\frac{1}{2}\sum_{l_1,l_2=1}^{N}\int_0^T\int_0^T\frac{\partial R_{l_1l_2}(t_1,t_2)}{\partial\lambda_p}\times
$$

$$
\times W_{l_1l_2}(t_1,t_2)\,dt_1dt_2-\frac{1}{2}\sum_{l_1,l_2=1}^{N}\int_0^T\int_0^T\frac{\partial W_{l_1l_2}(t_1,t_2)}{\partial\lambda_p}\times
$$

$$
\times y_{l_1}(t_1)\,y_{l_2}(t_2)\,dt_1dt_2,
$$

$$
\overline{\frac{\partial^2 L(\lambda_1^{(0)},\ldots,\lambda_n^{(0)})}{\partial\lambda_p\partial\lambda_q}}=-\frac{1}{2}\sum_{l_1,l_2=1}^{N}\int_0^T\int_0^T\frac{\partial W_{l_1l_2}(t_1,t_2)}{\partial\lambda_p}\times
$$

$$
\times\frac{\partial R_{l_1l_2}(t_1,t_2)}{\partial\lambda_q}\,dt_1dt_2 \qquad (13.5.20)
$$

(by $\lambda_p^{(0)}$ we again mean the true values of parameters λ_p). Expressions (13.5.19) and (13.5.20) are obtained by generalization of results of Chapter X. Those changes which the vector character of the received signal will entail are limited actually to these.

We proved the asymptotic efficiency of the maximum likelihood estimates. On this basis it is possible to find a system of optimum resolution in the case of parameters of targets which do not vary in time. In the case of slowly varying parameters λ_p, when intervals of approximate constancy of these parameters are great as compared to times of correlation of fluctuations of the signals, the obtained results permit us to find the operations of an optimum discriminator. Actually, let us assume that the output of the p-th channel of a certain meter is $\hat{\lambda}_p(t)$. If this quantity is close to the true value of the measured parameter in every given moment of time, then, considering in (13.5.17) $\lambda_p^{(0)}=\hat{\lambda}_p$, we obtain

$$
\Delta\hat{\lambda}_p=\lambda_p-\hat{\lambda}_p=-\sum_{r=1}^{n}I_{pr}\frac{\partial L(\hat{\lambda}_1,\ldots,\hat{\lambda}_n)}{\partial\lambda_r}, \qquad (13.5.21)
$$

where λ_p is the maximum likelihood estimate obtained during time T.

Considering the unbiased nature of estimate λ_p proved above, we note that the mean $\overline{\Delta\lambda_p}$ is equal to mismatch between the true value of the parameter and the output quantity of the meter. If we now present $\Delta\hat{\lambda}_p$ in the form

$$
\Delta\hat{\lambda}_p=\int_0^T z_p(t)\,dt, \qquad (13.5.22)
$$

$z_p(t)$ is a quantity which is, on the average, equal to the current mismatch between the true and measured values of parameter $\lambda_p(t)$. Considering the very great inertia of

the smoothing circuits, for $z_p(t)$ we can write the statistically equivalent expression

$$z_p(t) = \Delta\lambda_p(t) + \xi_p(t), \tag{13.5.23}$$

where $\Delta\lambda_p(t)$ – true mismatch with respect to the measured coordinate;

$\xi_p(t)$ – white noise. The circumstance that variance of quantity $\Delta\hat{\lambda}_p$ is minimum testifies to minimum spectral density of noise $\xi_p(t)$.

Thus, under the above conditions for any smoothing circuits of a meter optimum processing of the r-f signal is carried out by a discriminator forming the set of quantities $z_p(t)$ $(p = 1, 2, \ldots, n)$. The quality of its operation is characteriz by the matrix of spectral densities $\mathbf{S}_{\text{опт}} = \|S^{(pq)}_{\text{опт}}\|$, where

$$\mathbf{S}_{\text{опт}} = TJ. \tag{13.5.24}$$

In order to relate $S^{(pq)}_{\text{опт}}$ with the equivalent presentation of (13.5.23) we note that noises $\xi_p(t)$ are correlated, in general, among themselves. Their functions of cross-correlation have form

$$\overline{\xi_p(t)\xi_q(s)} = S^{(pq)}_{\text{опт}}\,\delta(t-s). \tag{13.5.25}$$

The problem of synthesis of smoothing circuits of meters of coordinates of closely located targets should be solved separately. In the case of rapid fluctuations it has less bearing on questions of resolution. We note that difficulties connected with matrix inversion in expression (13.5.11) permit us to obtain discernible results only for the case of two targets. But measuring conditions of radar devices usually occur with a large signal-to-noise ratio. Therefore in examining meters of coordinates of many targets it is completely natural and justified to assume a large signal-to-noise ratio. This assumption, as we shall see below, permits us to obtain sufficiently simple and physically intelligible results without limitation on the number of observed targets. The case of an arbitrary signal-to-noise ratio we will consider with the example of a two-target problem.

13.5.3. Optimum Discriminator in the Case of Many Targets with a Large Signal-To-Noise Ratio

The above results [see (13.5.11), (13.5.17)] in principle permit us to determine the general structure of an optimum multitarget discriminator with an arbitrary signal-to-noise ratio. However inversion of the corresponding matrices of high and arbitrary order in this case runs into considerable difficulties, so that the

structure of the optimum discriminator turns out not to be completely discovered. At the same time for measuring problems of special interest is the case of a large signal-to-noise ratio, for which these difficulties can be surmounted.

With a large signal-to-noise ratio elements of matrix $\frac{1}{2N_0}S(\omega)$ with a sufficiently small ω are large, and it is possible to expand (13.5.11) for $V(\omega)$ in a series of negative powers of this matrix. Limiting ourselves to the first two terms of this expansion, we have

$$V(\omega) \approx -\frac{2}{N_0}C^{-1} + 4(CS(\omega)C)^{-1}. \quad (13.5.26)$$

From this for $V_{pq}(\omega)$ we obtain the following expression:

$$V_{pq}(\omega) = -\frac{2}{N_0}C_{pq}^{-1} + 4\sum_{r=1}^{n}\frac{C_{pr}^{-1}C_{rq}^{-1}}{P_r S_r(\omega)}, \quad (13.5.27)$$

where by C_{pq}^{-1} we denote elements of matrix C^{-1}. Subsequently, we use the first term of expansion (13.5.27). Obviously, the condition where it is possible to use such an approximation has the form

$$\left|\frac{2N_0}{C_{pq}^{-1}}\sum_{r=1}^{n}\frac{C_{pr}^{-1}C_{rq}^{-1}}{P_r S_r(\omega)}\right| = \left|\frac{1}{C_{pq}^{-1}}\sum_{r=1}^{n}\frac{1}{h_r}\frac{C_{pr}^{-1}C_{rq}^{-1}}{S_{0r}(\omega)}\right| \ll 1, \quad (13.5.28)$$

where $S_{0r}(\omega) = \Delta f_{cr} S_r(\omega)$ – normalized spectrum of fluctuations of the signal reflected from r-th target;

Δf_{cr} – effective width of the spectrum of fluctuations of this signal;

$h_r = P_r/2N_0\Delta f_{cr}$ – ratio of power of the signal reflected from the r-th target to the power of noise in the band of fluctuations of the signal.

With a sufficiently large signal-to-noise ratio condition (13.5.28) will be carried out. It is necessary, however, to consider that condition (13.5.28) should be realized in an interval of frequencies sufficiently exceeding the width of the spectrum of fluctuations of the signal.

Thus, subsequently we use the first term of expansion (13.5.27). We shall first calculate matrix $\bar{A}$ (13.5.13). Differentiating (13.5.7) and (13.5.8) with respect to parameters λ_p and λ_q, respectively, and substituting the obtained relationships in (13.5.14), we can obtain the following very important result:

$$\overline{A_{pq}} = -\frac{\overline{\partial^2 L(\lambda_1^{(0)},\ldots,\lambda_n^{(0)})}}{\partial\lambda_p\partial\lambda_q} = \frac{TP_p}{N_0}\delta_{qp}\left\{C_{pq}^{(11)}(\lambda_p,\lambda_q) - \right.$$
$$\left. - \sum_{r,s=1}^{n} C_{pr}^{(10)}(\lambda_p,\lambda_r)C_{rs}^{-1}C_{sq}^{(01)}(\lambda_s,\lambda_q)\right\}, \tag{13.5.29}$$

where we use designations (13.5.15).

Thus, matrix $\overline{\mathbf{A}}$ is diagonal, and the jointly-efficient estimates are uncorrelated. Matrix inversion of $\overline{\mathbf{A}}$ causes no difficulties, and for variance of the efficient estimate of parameter λ_p we obtain

$$\sigma^2_{\text{э}\phi\, p} = \frac{1}{\frac{TP_p}{N_0}\left\{C_{pp}^{(11)}(\lambda_p,\lambda_p) - \sum_{r,s=1}^{n} C_{pr}^{(10)}(\lambda_p,\lambda_r)C_{rs}^{-1}C_{sp}^{(01)}(\lambda_s,\lambda_p)\right\}}. \tag{13.5.30}$$

This is a very important formula, characterizing potential accuracy of measurement of coordinates of several targets with a large signal-to-noise ratio. The equivalent spectral density (for this parameter λ_p) is determined from (13.5.30) by multiplication by T.

$$S_{\text{экв}\, p} = \frac{1}{\frac{P_p}{N_0}\left\{C_{pp}^{(11)} - \sum_{r,s=1}^{n} C_{pr}^{(10)}C_{rs}^{-1}C_{sp}^{(01)}\right\}}. \tag{13.5.31}$$

It is important to note that accuracy of measurement of coordinates of the p-th target does not depend on the power of signals reflected from the other targets. This testifies to good compensation of the interfering signals in an optimum meter of coordinates of many targets.

We pass to finding the optimum processing of a signal in a meter of coordinates of many targets. Substituting expressions (13.5.7) and (13.5.8) in (13.5.14) it is easy to prove that the component not containing realizations of signals will be equal to zero. As a result we have

$$\frac{\partial L(\lambda_1,\ldots,\lambda_n)}{\partial\lambda_p} = -\frac{1}{2}\frac{\partial}{\partial\lambda_p}\int_0^T\int_0^T \mathrm{Re}\, e^{i\omega_0(t_1-t_2)} \times \sum_{r,s=1}^{n} u(t_1,\lambda_r)\,u^*(t_2,\lambda_s)\,v_{rs}(t_1-t_2)\,y(t_1)\,y(t_2)\,dt_1dt_2.$$

If the signal-to-noise ratio is great, then within a frequency range sufficiently exceeding the width of the spectrum of fluctuations of the signal we have $V_{rs}(\omega) \approx$ $\approx C_{rs}^{-1}$, i.e., function $v_{rs}(t_1 - t_2)$ can be considered δ-shaped with respect to

fluctuations of the signal. On the other hand, as we know, function $v_{rs}(t)$ is very slow in relation to functions $u_p(t, \lambda_p)$. Now $v_{pq}(t_1 - t_2)$ can be presented thus:

$$v_{pq}(t_1 - t_2) \approx C_{pq}^{-1} v(t_1 - t_2) \approx$$
$$\approx C_{pq}^{-1} \int_0^T h(t - t_1) h(t - t_2) dt,$$

where h(t) – pulse response of the filter integrating modulation of the receiving signal, but passing without change fluctuations of the signal.

Here the derivative of the logarithm of the likelihood functional is rewritten in the form

$$\frac{\partial L(\lambda_1, \ldots, \lambda_n)}{\partial \lambda_p} = -\frac{1}{2} \frac{\partial}{\partial \lambda_p} \int_0^T dt \cdot \mathrm{Re} \int_0^T \int_0^T h(t - t_1) \times$$
$$\times h(t - t_2) e^{i\omega_0(t_1 - t_2)} y(t_1) y(t_2) \sum_{s, r=1}^{n} u_r(t_1, \lambda_r) \times$$
$$\times C_{rs}^{-1} u^*_s(t_2, \lambda_s) dt_1 dt_2. \qquad (13.5.32)$$

The circuit realization of operation (13.5.32) can be carried out by various methods. Here it is important to obtain the simplest circuit. For this we calculate expression

$$\frac{\partial}{\partial \lambda_p} \sum_{r, s=1}^{n} u_r(t_1, \lambda_r) u^*_s(t_2, \lambda_s) C_{rs}^{-1},$$

in (13.5.32). It, obviously, is equal to

$$\frac{\partial u_p(t_1, \lambda_p)}{\partial \lambda_p} \sum_{s=1}^{n} u^*_s(t_2, \lambda_s) C_{sp}^{-1*} + \sum_{s=1}^{n} u_s(t_1, \lambda_s) C_{sp}^{-1} \frac{\partial u^*_p(t_2, \lambda_p)}{\partial \lambda_p} +$$
$$+ \sum_{r, s=1}^{n} u_r(t_1, \lambda_r) u^*_s(t_2, \lambda_s) \frac{dC_{rs}^{-1}}{d\lambda_p}. \qquad (13.5.33)$$

On the other hand, we have

$$\mathbf{C}\mathbf{C}^{-1} = \mathbf{E}, \quad \frac{d\mathbf{C}^{-1}}{d\lambda_p} \mathbf{C} + \mathbf{C}^{-1} \frac{d\mathbf{C}}{d\lambda_p} = 0,$$
$$\frac{d\mathbf{C}^{-1}}{d\lambda_p} = -\mathbf{C}^{-1} \frac{d\mathbf{C}}{d\lambda_p} \mathbf{C}^{-1}.$$

Expanding the last relationship, we obtain

$$\frac{dC_{rs}^{-1}}{d\lambda_p} = -\sum_{i,j=1}^{n} C_{ri}^{-1} \frac{dC_{ij}}{d\lambda_p} C_{js}^{-1} = -C_{rp}^{-1} \sum_{j=1}^{n} C_{pj}^{(10)} C_{js}^{-1} - $$

$$- C_{ps}^{-1} \left(\sum_{j=1}^{n} C_{pj}^{(10)} C_{jr}^{-1} \right)^{*}.$$

Substituting this expression in (13.5.33) and then in (13.5.32), for the derivative of the logarithm of the likelihood functional we finally obtain

$$\frac{\partial L(\lambda_1, \ldots, \lambda_n)}{\partial \lambda_p} = -\frac{1}{2} \int_0^T dt \, \mathrm{Re} \int_0^T h(t-\tau) \left\{ \frac{\partial u_p(\tau, \lambda_p)}{\partial \lambda_p} - \right.$$

$$\left. - \sum_{s=1}^{n} u_s(\tau, \lambda_s) \left[\sum_{j=1}^{n} C_{sj}^{-1} C_{jp}^{(01)} (\lambda_j, \lambda_p) \right] \right\} y(\tau) e^{i\omega_0 \tau} d\tau \times$$

$$\times \left[\int_0^T h(t-\tau) \left(\sum_{s=1}^{n} u_s(\tau, \lambda_s) C_{sp}^{-1} \right) y(\tau) e^{i\omega_0 \tau} d\tau \right]^{*}. \qquad (13.5.34)$$

Omitting the proportionality factor and the integral over t in expression (13.5.34), we have an analytic expression for the operation of the optimum discriminator:

$$z(t) \approx \mathrm{Re} \int_0^T h(t-\tau) \left\{ \frac{\partial u_p(\tau, \lambda_p)}{\partial \lambda_p} - \sum_{s=1}^{n} u_s(\tau, \lambda_s) \times \right.$$

$$\left. \times \left[\sum_{j=1}^{n} C_{sj}^{-1} C_{jp}^{(01)} (\lambda_j, \lambda_p) \right] \right\} y(\tau) e^{i\omega_0 \tau} d\tau \times$$

$$\times \left\{ \int_0^T h(t-\tau) \left[\sum_{s=1}^{n} u_s(\tau, \lambda_s) C_{sp}^{-1} \right] y(\tau) e^{i\omega_0 \tau} d\tau \right\}^{*}_{\lambda_i = \hat{\lambda}_i}. \qquad (13.5.35)$$

The block diagram of an optimum discriminator executing operation (13.5.35) is depicted in Fig. 13.7. It coincides in structure with circuits of discriminators of meters of coordinates of one target; only the heterodyne signals by which we multiply signals in the channels were changed. The physical meaning of introduction of namely such heterodyne signals is rather easily perceived and is very curious.

Signal

$$\sum_{s=1}^{n} u_s(t, \lambda_s) C^{-1}{}_{sp} e^{i\omega_c t} = u(t) e^{i\omega_c t}, \qquad (13.5.36)$$

as it is easy to establish by means of direct calculation, is orthogonal to the interfering signals.

Signal

$$v(t)e^{i\omega_r t}=\left\{\frac{\partial u_p(t,\lambda_p)}{\partial\lambda_p}-\right.$$

$$\left.-\sum_{s=1}^{n}u_s(t,\lambda_s)\left[\sum_{j=1}^{n}C_{sj}^{-1}C_{jp}^{(01)}(\lambda_j,\lambda_p)\right]\right\}e^{i\omega_r t} \qquad (13.5.37)$$

turns out to be orthogonal to all signals without exception. The special form of reference signals at the output of the circuit ensures the existence of a signal only in the presence of mismatch between the input and heterodyne signals with respect to the measured parameter. The output signal is proportional to this mismatch. The presence of a channel with heterodyne signal (13.5.37) is caused by the essence of the measuring problem: we must tune away from interfering signals, even if parameters corresponding to these signals are not known by us exactly.

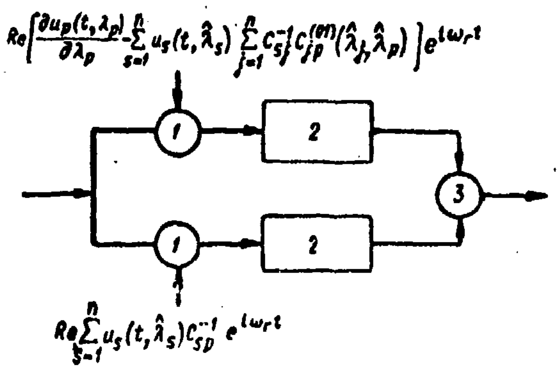


Fig. 13.7. Optimum discriminator of a meter of coordinates of many targets: 1) mixer; 2) filter; 3) multiplier.

The meaning of operations of the synthesized circuit can be grasped more clearly if we analyze this circuit, i.e., directly calculate its characteristics. We shall calculate these characteristics; here we shall not be limited only to the case of a large signal-to-noise ratio, since this calculation is slightly complicated if we consider the signal-to-noise ratio arbitrary.

Furthermore, we consider that filters in the analyzed circuit have a frequency band comparable with the band of fluctuations of the signal. This also does not complicate calculation, but the obtained results will be more valuable from the practical point of view. The fact is that for practical realization of the considered circuit besides knowledge of its optimality for a large signal-to-noise ratio it is very desirable also to know its behavior in the whole range of signal-to-noise ratios; it is also clear that for small signal-to-noise ratios accuracy of the analyzed circuit will essentially depend on the width of the filter passbands, and comparative analysis of all possible situations here is of extraordinary interest.

Thus, we shall consider the circuit depicted in Fig. 13.7. We consider that filters in this circuit have frequency response $H(i\omega)$. Using expression (13.5.35) for the output signal of the considered circuit and introducing designations u(·) and

$v(t)$ of (13.5.36) and (13.5.37) for modulations of the heterodyne signal, we can obtain the following results:

$$\overline{z(t)}=\frac{1}{2}\sum_{k=1}^{n}\operatorname{Re} b_k a^*{}_k\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(i\omega)|^2 S_k(\omega)\,d\omega+N_0\operatorname{Re} C\Delta f_{\Phi}, \tag{13.5.38}$$

where

$$\begin{aligned} a_k&=\frac{1}{T_r}\int_0^{T_r}u(t)\,u^*{}_k(t,\ \lambda_k)\,dt;\\ b_k&=\frac{1}{T_r}\int_0^{T_r}v(t)\,u^*{}_k(t,\ \lambda_k)\,dt;\\ C&=\frac{1}{T_r}\int_0^{T_r}v(t)\,u^*(t)\,dt; \end{aligned} \tag{13.5.39}$$

Δf_{Φ} – effective width of the filter passband.

Considering true values of parameters λ_p to coincide with parameters of the signals of the heterodyne oscillator $u(t)$ (13.5.36) and $v(t)$ (13.5.37), it is easy to find that $b_k = 0$ and systematic error

$$\varepsilon_{\text{сист}}=0.$$

With detuning $\Delta\lambda_p$ between the shown values of the parameters from (13.5.38) and (13.5.39) we obtain

$$\overline{z(t)}=\Delta\lambda_p\,\frac{1}{2}\left(C_{pp}^{(11)}-\sum_{r,s=1}^{n}C_{ps}^{(10)}C_{rs}^{-1}C_{rp}^{(01)}\right)\times$$

$$\times\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(i\omega)|^2 S_p(\omega)\,d\omega, \tag{13.5.40}$$

i.e., the mean value of the signal at the output of the circuit is proportional only to mismatch with respect to the measured parameter and does not depend on interfering signals even in the presence of mistuning (of course, small) between true values of their parameters and the values introduced in the heterodyne signals. Herein lies the meaning of the special and complicated form of these signals. From expression (13.5.40) we find the slope of the discrimination characteristic of the analyzed circuit:

$$K_{\text{д}}=\frac{1}{2}\left(C_{pp}^{(11)}-\sum_{r,s=1}^{n}C_{ps}^{(10)}C_{sr}^{-1}C_{rp}^{(01)}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(i\omega)|^2 S_p(\omega)\,d\omega. \tag{13.5.41}$$

Spectral density at zero frequency of signal z(t) in the absence of mistuning for all parameters is also easily calculated and turns out to be equal to

$$\int_{-\infty}^{\infty} \overline{z(t)\,z(t+\tau)}\,d\tau = \frac{N_0^2}{2}\frac{1}{T_r}\int_0^{T_r}|v(t)|^2\,dt\,\frac{1}{2\pi}\times$$
$$\times\int_{-\infty}^{\infty}|H(i\omega)|^4\left(\frac{S_p(\omega)}{2N_0}+\frac{1}{T_r}\int_0^{T_r}|u(t)|^2\,dt\right)d\omega.$$

From this we obtain the following final expression for equivalent spectral density with respect to parameter λ_p:

$$S_{\text{экв}} = \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(i\omega)|^4\,(h_p S_{\text{э}p}(\omega) + C_{pp}^{-1})\,d\omega}{2h_p^2\left(C_{pp}^{(11)} - \sum_{i,j=1}^{n} C_{pi}^{(10)} C_{ij}^{-1} C_{jp}^{(01)}\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(i\omega)|^2\,S_{\text{э}p}(\omega)\,d\omega\right)^2}, \qquad (13.5.42)$$

where $h_p = P_p/2N_0\Delta f_{cp}$ – ratio of power of the signal reflected from the p-th target to the power of noise in the band of fluctuations of this signal, and

$$S_{\text{э}p}(\omega) = \Delta f_{cp} S_p(\omega)$$

is the normalized spectrum of fluctuations of the signal from this target. It is easy to see that for a large signal-to-noise ratio and widening of the passband of the filter expression (13.5.42) coincides with (13.5.31), i.e.. accuracy of the analyzed circuit in these conditions, naturally, coincides with the potential accuracy. From (13.5.42) it is easy to establish the conditions in which this takes place:

$$h_p\Delta f_{cp} \gg C_{pp}^{-1}\Delta f_{\Phi},$$
$$\Delta f_{\Phi} \gg \Delta f_{cp}.$$

Practically, it is sufficient to have

$$\Delta f_{\Phi} \approx (2 \div 3)\,\Delta f_{cp},$$
$$h_p \gtrsim (10 \div 15)\,C_{pp}^{-1}. \qquad (13.5.43)$$

To produce calculation of accuracies by formulas (13.5.31) and (13.5.42) is fairly difficult in view of the complexity of calculation of elements of matrix C^{-1}. Simple formulas are obtained for two targets. Here

$$\mathbf{C}=\begin{Vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{Vmatrix}=\begin{Vmatrix} 1 & C_{12} \\ C^*_{12} & 1 \end{Vmatrix},$$

$$\mathbf{C}^{-1}=\frac{1}{1-|C_{12}|^2}\begin{Vmatrix} 1 & -C^*_{12} \\ -C_{12} & 1 \end{Vmatrix}.$$

The circuit of the optimum discriminator for measurement of parameter λ_p takes the form depicted in Fig. 13.8. Modulation of heterodyne signals in this circuit can be presented in the form

$$u(t)=\frac{u_1(t,\lambda_1)-u_2(t,\lambda_2)C_{12}}{1-|C_{12}|^2},$$

$$v(t)=\frac{\partial u_1(t,\lambda_1)}{\partial\lambda_1}-u_1(t,\lambda_1)\frac{C_{11}^{(01)}-C^*_{12}C_{21}^{(01)}}{1-|C_{12}|^2}-$$
$$-u_2(t,\lambda_2)\frac{C_{21}^{(01)}-C_{12}C_{11}^{(01)}}{1-|C_{12}|^2},$$

and equivalent spectral density can be presented in the form

$$S_{\text{экв}}=\frac{1}{\frac{2P}{N_0}\left\{C_{11}^{(11)}-\frac{|C_{11}^{(10)}|^2+|C_{12}^{(10)}|^2-2\operatorname{Re}C_{12}^{(10)}C_{11}^{(01)}C_{12}}{1-|C_{12}|^2}\right\}}.$$

The optimum discriminator for the case of two targets will be studied in more detail in the next paragraph. In conclusion we shall briefly consider the case when the signal has a vector character.

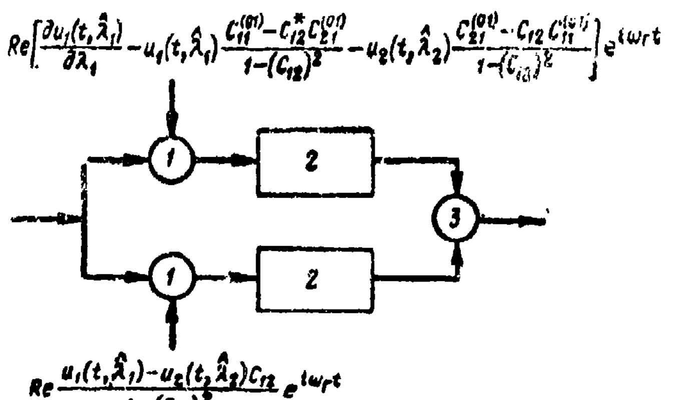


Fig. 13.8. Optimum discriminator of a meter of coordinates of two targets: 1) mixer; 2) filter; 3) multiplier.

In this case, as we have seen, matrix $\mathbf{V}(\omega)$, characterizing the joint functional of the probability density of the observed signals, is expressed by the former formula (13.5.11); only matrix C is modified: its elements are determined now by formula (13.5.19). Consequently, with a large signal-to-noise ratio $\mathbf{V}(\omega)$, as before, is found in the form of (13.5.26) or (13.5.27).

We substitute expression (13.5.18') for correlation functions of the received signals and expression (13.5.18") for functions $W_{l_1 l_2}(t_1, t_2)$ in formula (13.5.2[illegible]). Here it is easy to find that

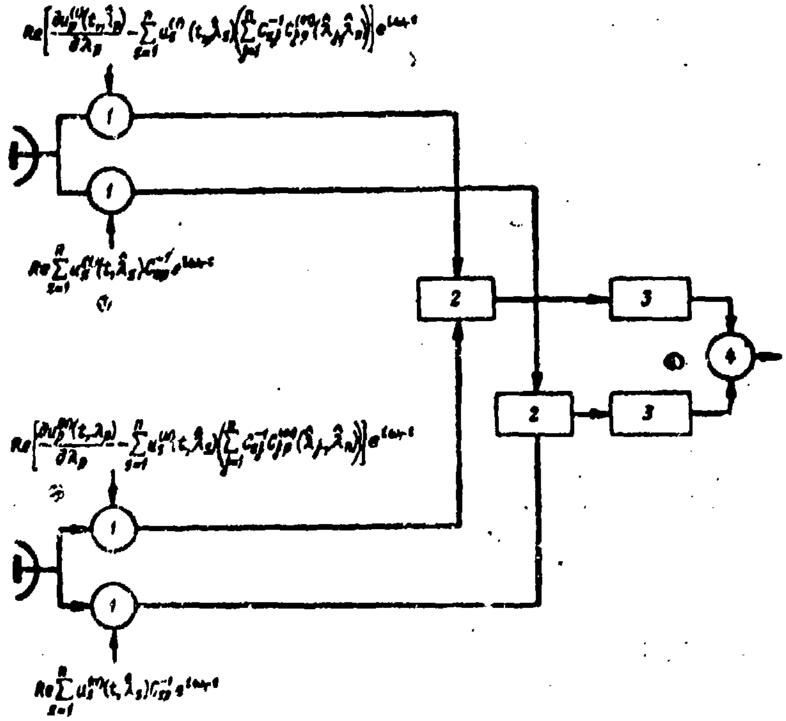
$$\overline{\frac{\partial^2 L(\lambda_1^{(0)}, \ldots, \lambda_n^{(0)})}{\partial\lambda_p \partial\lambda_q}}$$

has the former form (13.5.16), only functions C_{ij} are expressed by formulas (13.5.19) and

$$C_{ij}^{(10)} = \frac{1}{T_r}\int_0^{T_r} \frac{1}{N}\sum_{l=1}^{N} \frac{\partial u_i^{*(l)}(t, \lambda_i)}{\partial \lambda_i} u_j^{(l)}(t, \lambda_j)\, dt,$$
$$C_{ij}^{(11)} = \frac{1}{T_r}\int_0^{T_r} \frac{1}{N}\sum_{l=1}^{N} \frac{\partial u_i^{*(l)}(t, \lambda_i)}{\partial \lambda_i} \frac{\partial u_j^{(l)}(t, \lambda_j)}{\partial \lambda_j}\, dt. \tag{13.5.44}$$

Consequently, $\sigma^2_{\text{эф}\ p}$ and $S_{\text{опт}\ p}$ will also be expressed by the former formulas (13.5.30) and (13.5.31). Then, formula (13.5.20), we can obtain for the optimum operation in the considered case using the following expression:

$$z(t) = \mathrm{Re}\int_0^t h(t-\tau)\sum_{l=1}^{N} y^{(l)}(\tau)\left[\frac{\partial u_p^{(l)}(\tau, \hat{\lambda}_p)}{\partial \lambda_p} - \right.$$
$$\left. - \sum_{s=1}^{n} u_s^{(l)}(\tau, \lambda_s)\left(\sum_{j=1}^{n} C_{sj}^{-1} C_{jp}^{(01)}\right)\right] e^{i\omega_0\tau} d\tau \times$$
$$\times \left\{\int_0^t h(t-\tau)\sum_{l=1}^{N}\left(\sum_{s=1}^{n} u_s^{(l)}(\tau, \hat{\lambda}_s) C^{-1}_{sp}\right) y^{(l)}(\tau)\, e^{i\omega_0\tau} d\tau\right\}^*. \tag{13.5.45}$$



The block diagram of the device executing operation (13.5.45) is depicted in Fig. 13.9. This circuit coincides in structure with analogous circuits of meters of coordinates of one target (see, for instance, §§ 10.2, 10.3, 10.13); however, the form of the heterodyne signal

Fig. 13.9. Optimum discriminator of a meter of angular coordinates of many targets: 1) mixers; 2) adders; 3) filters; 4) multiplier.

is changed. Their structure, as it is possible to see as a result of analysis of operation (13.5.45), obeys again the same principle: to maximally tune away from interfering signals, even if their parameters are not known exactly by us.

13.5.4. Optimum Discriminator in the Case of Two Targets

Let us consider the case of two targets ($n = 2$), located so close to one another that their signals are not orthogonal. The signal-to-noise ratio h we shall consider arbitrary. Let us assume that parameter λ_1, whose carrier is signal $y_1(t, \lambda_1)$, is to be measured. Signal $y_2(t, \lambda_2)$ plays here the role of interference. We first consider that the coordinate of interference λ_2 is known (is measured with great accuracy). Subsequently we shall reject this assumption. Considering coefficients of correlation of the useful and interfering signals identical ($\rho_1(t) = \rho_2(t) = \rho(t)$) and expressing function $W(t, \tau)$, the inverse of the correlation function, in the form

$$W(t, \tau) = \operatorname{Re} v(t, \tau)\, e^{i\omega_0(t-\tau)} + \frac{1}{N_0}\delta(t-\tau), \tag{13.5.46}$$

then, substituting (13.5.46) in equation (13.5.9), we arrive at the following equation for $v(t, \tau)$:

$$\begin{gathered} P_c u_1(t)\int_0^T \rho(t-s)\, u^*_1(s)\, v(s, \tau)\, ds + \\ + P_n u_2(t)\int_0^T \rho(t-s)\, u^*_2(s)\, v(s, \tau)\, ds + \\ + \frac{2P_c}{N_0}\rho(t-\tau)\, u_1(t)\, u^*_1(\tau) + \\ + \frac{2P_n}{N_0}\rho(t-\tau)\, u_2(t)\, u^*_2(\tau) + 2N_0 v(t, \tau) = 0, \end{gathered} \tag{13.5.47}$$

where $P_c = P_1$, $P_n = P_2$ — mean powers of the useful and interfering signals.

Solution of equation (13.5.47) is sought in the form

$$v(t, \tau) = \sum_{k, j=1}^{2} v_{kj}(t-\tau)\, u_k(t)\, u^*_j(\tau). \tag{13.5.48}$$

Substituting (13.5.48) in (13.5.47) and producing approximate calculations allowing for observance of condition $T_r \ll \tau_k \ll T$, for the Fourier transforms of functions $v_{kj}(t - \tau)$, we obtain

$$\left.\begin{aligned}
V_{11}(\omega) &= -\frac{2N_0+P_nS(\omega)}{P_nS(\omega)}F(\omega),\\
V_{22}(\omega) &= -\frac{2N_0+P_cS(\omega)}{P_cS(\omega)}F(\omega),\\
V_{12}(\omega) &= \gamma e^{-i\varphi}F(\omega),\\
V_{21}(\omega) &= \gamma e^{i\varphi}F(\omega),
\end{aligned}\right\} \tag{13.5.49}$$

where

$$\left.\begin{aligned}
S(\omega) &= \int_{-\infty}^{\infty}\rho(\tau)e^{-i\omega\tau}d\tau;\\
F(\omega) &= \frac{2P_cP_n}{N_0}\,\frac{S^2(\omega)}{4N_0^2+2N_0(P_c+P_n)S(\omega)+P_cP_nS^2(\omega)(1-\gamma^2)},
\end{aligned}\right\} \tag{13.5.50}$$

and γ and φ are determined by function C_{21} (13.5.6):

$$C_{21}=C^*_{12}=\gamma e^{i\varphi}=\frac{1}{T_r}\int_0^{T_r}u^*_2(s)u_1(s)\,ds. \tag{13.5.51}$$

We substitute (13.5.46) and (13.5.48) in the likelihood equations (13.5.12), (13.5.14). Here, considering that this equality is valid,

$$\begin{aligned}
-\int_0^T\int_0^T v_{kj}(t-\tau)e^{i\omega_0(t-\tau)}u_k(t)u^*_j(\tau)y(t)y(\tau)dt\,d\tau &=\\
=a_{kj}\int_0^T dt\int_0^t h_{kj}(t-s_1)u_k(s_1)e^{i\omega_0 s_1}y(s_1)\,ds_1 &\times\\
\times\int_0^t h_{kj}(t-s_2)u^*_j(s_2)e^{-i\omega_0 s_2}y(s_2)ds_2&,
\end{aligned} \tag{13.5.52}$$

the term of the likelihood equation containing the received signal can be reduced to the form

$$\begin{aligned}
&-\frac{\partial}{\partial\lambda_1}\int_0^T\int_0^T W(t,\tau)y(t)y(\tau)\,dt\,d\tau=\\
&=\frac{2}{N_0}\frac{\partial}{\partial\lambda_1}\int_0^T dt\left\{\left|\int_0^t h_{11}(t-\tau)u_1(\tau)e^{i\omega_0\tau}y(\tau)\,d\tau\right|^2+\right.\\
&+\left|\int_0^t h_{22}(t-\tau)u_2(\tau)e^{i\omega_0\tau}y(\tau)\,d\tau\right|^2-\\
&-2\gamma\,\mathrm{Re}\int_0^t h_{12}(t-\tau_1)u_1(\tau_1)e^{i\omega_0\tau_1}y(\tau_1)\,d\tau_1\times\\
&\left.\times\int_0^t h_{12}(t-\tau_2)u^*_2(\tau_2)e^{-i(\omega_0\tau_2+\varphi)}y(\tau_2)\,d\tau_2\right\},
\end{aligned}$$

where $h_{kj}(t)$ – pulse responses of filters, squares of moduli of frequency responses of which are defined as

$$|H_{11}(i\omega)|^2 = -\frac{N_0}{2}V_{11}(\omega); \quad |H_{22}(i\omega)|^2 = -\frac{N_0}{2}V_{22}(\omega),$$
$$|H_{12}(i\omega)|^2 = \frac{N_0}{2}\frac{V_{12}(\omega)}{\gamma e^{-i\varphi}} = \frac{N_0}{2}\frac{V_{21}(\omega)}{\gamma e^{i\varphi}}. \tag{13.5.53}$$

The term of the likelihood equation not containing $y(t)$ after substitution of the value of $W(t, \tau)$ and necessary transformations takes the form

$$I = -\int_0^T\int_0^T \frac{\partial R_\gamma(t,\tau)}{\partial\lambda_1} W(t,\tau)\,dt\,d\tau = 4T\gamma\frac{\partial\gamma}{\partial\lambda_1}\frac{1}{2\pi}\times$$
$$\times\int_{-\infty}^{\infty}\frac{P_c P_\pi S^2(\omega)}{4N_0^2 + 2N_0(P_c+P_\pi)S(\omega) + P_c P_\pi S^2(\omega)(1-\gamma^2)}\,d\omega. \tag{13.5.54}$$

With small signal-to-noise ratios

$$I = \frac{P_c P_\pi}{N_0^2}\gamma\frac{\partial\gamma}{\partial\lambda_1}T\frac{1}{2\pi}\int_{-\infty}^{\infty}S^2(\omega)\,d\omega =$$
$$= \frac{4h^4}{q}\gamma\frac{\partial\gamma}{\partial\lambda_1}T\frac{1}{2\pi}\int_{-\infty}^{\infty}S_0^2(\omega)\,d\omega, \tag{13.5.54'}$$

where $h = P_c/2\Delta f_c N_0$ – the signal-to-noise ratio;

$q = P_c/P_\pi$ – the signal-to-interference ratio;

Δf_c – effective width of the spectrum of fluctuations of the signal;

$S_0(\omega) = \Delta f_c S(\omega)$ – normalized spectral density of fluctuations.

For large h, considering $S(\omega)$ a square function, we obtain

$$I \approx T\gamma\frac{\partial\gamma}{\partial\lambda_1}\Delta f_c\frac{4}{1-\gamma^2}. \tag{13.5.54''}$$

Using the preceding equalities, we can write the likelihood equation in the form

$$\frac{\partial}{\partial\lambda_1}\int_0^T dt\left\{\left|\int_0^t h_{11}(t-\tau)u_1(\tau)e^{i\omega_0\tau}y(\tau)\,d\tau\right|^2 + \left|\int_0^t h_{22}(t-\tau)u_2(\tau)e^{i\omega_0\tau}y(\tau)\,d\tau\right|^2 -\right.$$
$$\left.-2\gamma\,\mathrm{Re}\int_0^t h_{12}(t-\tau_1)u_1(\tau_1)e^{i\omega_0\tau_1}y(\tau_1)\,d\tau_1\times\int_0^t h_{12}(t-\tau_2)u^*_2(\tau_2)e^{-i(\omega_0\tau_2+\varphi)}y(\tau_2)\,d\tau_2\right\} + \frac{N_0}{2}I = 0. \tag{13.5.55}$$

If we approximately replace the operation of differentiation by calculation of a finite difference, the optimum circuit, corresponding to (13.5.55), takes the form of Fig. 13.10.* With small detuning of channels $\Delta\lambda_1$ and not too large γ characteristics of filters depend little on detuning. Disregarding this dependence, the circuit can be simplified and reduced to the form of Fig. 13.11. The circuit contains two discriminator channels, intended for working out mismatch with respect to λ_1. These channels of an optimum discriminator when there is no interfering signal $y_2(t)$ only by characteristics of the filters. Both forms of characteristics of filters coincide when $\gamma = 0$. Furthermore, the circuit contains channels of compensation of the interfering signal, and in it there is provided insertion of correction I for compensation of systematic error.

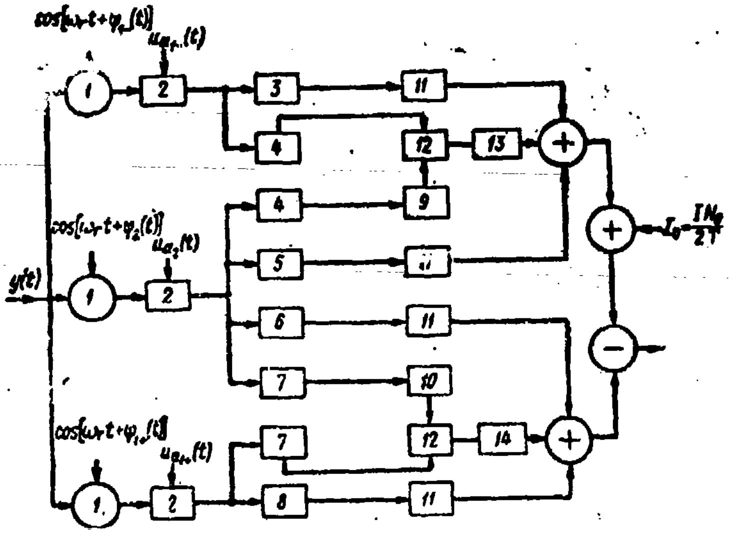


Fig. 13.10. Optimum discriminator for measurement of parameter λ_1 when $n = 2$: 1) mixer; 2) amplifier with gain varying according to the law of amplitude modulation; 3, 4, 5, 6, 7, 8) filters with pulse responses $h_{11-}(t) \cos \omega_{\text{пр}}t$, $h_{12-}(t) \cos \omega_{\text{пр}}t$, $h_{22-}(t) \cos \omega_{\text{пр}}t$, $h_{22+}(t) \cos \omega_{\text{пр}}t$, $h_{12+}(t) \cos \omega_{\text{пр}}t$, $h_{11+}(t) \cos \omega_{\text{пр}}t$, respectively; 9, 10) phase shifters with phase shift φ_- and φ_+, respectively; 11) square-law detector; 12) phase detector; 13, 14) amplifiers with gain $-2\gamma_-$ and $-2\gamma_+$, respectively.

It is possible to imagine other circuits corresponding to operations (13.5.55) produced on signal y(t). In particular, it is possible to create a discriminator circuit with only two channels, occurring in the absence of interfering signals but with reference signals properly modified. The circuit of Fig. 13.11 is expedient for tracking radar meters of coordinates of two targets. In this case as the channel of compensation we can use the channel of measurement of parameter λ_2. Properly transforming the output of this channel, we realize the required compensation. This is simpler than change of the complicated reference signals necessary upon change of distances between targets in the other variant of optimum discriminator mentioned.

*Subscript "-" in the figure corresponds to channels detuned $-\Delta\lambda_1$; and subscript "+" corresponds to detuning $+\Delta\lambda_1$ relative to the measured value of λ_1.

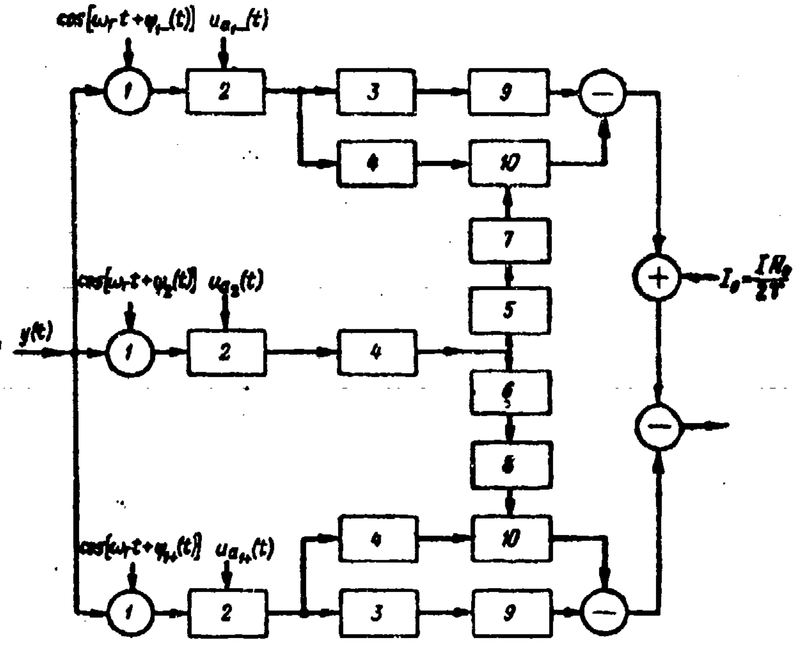


Fig. 13.11. Simplified circuit of an optimum discriminator: 1) mixer; 2) amplifier with gain varying according to the law of amplitude modulation; 3, 4) filters with pulse responses $h_{11}(t)\cos\omega_{\text{пр}}t$ and $h_{12}(t)\cos\omega_{\text{пр}}t$, respectively; 5, 6) phase shifters with phase shift φ_- and φ_+, respectively; 7, 8) amplifiers with gain $2\gamma_-$ and $2\gamma_+$, respectively; 9) square-law detector; 10) phase detector.

In connection with the fact that measuring systems frequently work with a very large signal-to-noise ratio, it is interesting to discuss further simplifications of the considered circuit for large h.

From (13.5.49), (13.5.50), and (13.5.53) it follows that for large h the squares of moduli of frequency responses of all filters become identical and equal in the band of fluctuations of the signal:

$$|H_{11}(i\omega)|^2=|H_{22}(i\omega)|^2=|H_{12}(i\omega)|^2=\frac{1}{1-\gamma^2}\quad(|\omega|\leqslant 2\pi\Delta f_c) \tag{13.5.56}$$

At the same time, in accordance with the hypotheses made in deriving the basic relationships, these filters are integrating for the period of modulation of the signal T_r. It is easy to see that equation (13.5.55) here takes the form

$$\frac{2}{N_0}\int_0^T\frac{\partial}{\partial\lambda_1}\left\{\frac{1}{1-\gamma^2}\left[|Q_1|^2+|Q_2|^2-2\gamma\operatorname{Re}Q_1Q_2^*e^{-i\varphi}\right]\right\}dt+I=0, \tag{13.5.57}$$

where

$$Q_{1,2} = \int_{t-T_r}^{t} u_{1,2}(\tau)\, e^{i\omega_0\tau} y(\tau)\, d\tau. \tag{13.5.58}$$

Equation (13.5.57) will be transformed to

$$\int_0^T \left\{ \frac{\partial}{\partial\lambda_1}\left[\frac{|Q_1 - \gamma e^{i\varphi} Q_2|^2}{1-\gamma^2}\right] + \frac{IN_0}{2T} \right\} dt = 0. \tag{13.5.59}$$

With replacement of differentiation by calculation of a finite difference operations of the discriminator are executed by the circuit of Fig. 13.12. Circuits of compensation in this scheme correspond to total suppression of the interfering signal (see § 13.2). This circuit differs from the circuit optimum in the absence of $y_2(t)$ and with $h \to \infty$, to which there are added only channels of orthogonalization (total suppression) of interference, by the presence of amplifiers with gain factors $\frac{1}{1-\gamma_-^2}$ and $\frac{1}{1-\gamma_+^2}$ and introduction of correction $I_0 = \frac{IN_0}{2T}$. For small γ, very large h and small detunings $\Delta\lambda_1$ the shown differences of circuits are absent.

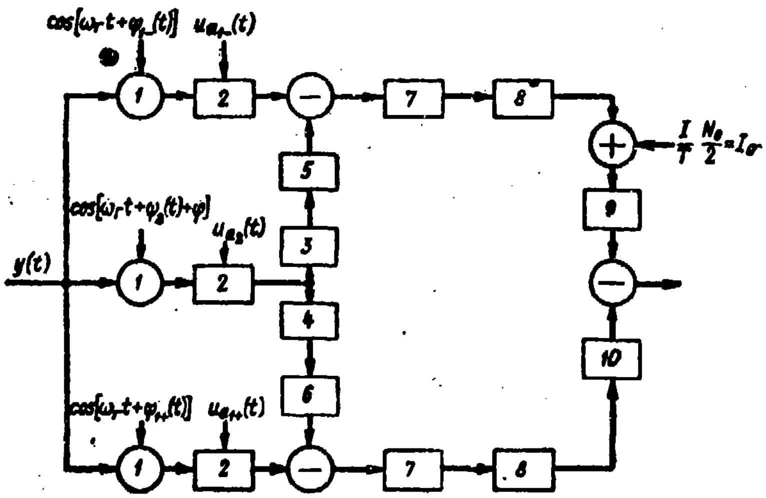


Fig. 13.12. Optimum discriminator for a large signal-to-noise ratio. 1) mixer; 2) amplifier with gain varying according to the law of amplitude modulation; 3, 4) amplifiers with gain γ_- and γ_+, respectively; 5, 6) phase shifters with phase shift $-\Delta\varphi_-$ and $+\Delta_{\varphi+}$, respectively; 7) filter, integrating for period T_r; 8) square-law detector; 9, 10) amplifiers with gain $(1-\gamma_-^2)^{-1}$ and $(1-\gamma_+^2)^{-1}$ respectively.

The optimum circuit ensures absence of systematic error, and the equivalent spectral density of fluctuating error of measurement of parameter λ_1 is defined as $S_{\text{опт}\,1} = \sigma^2_{\text{эф}\,1} T$, where $\sigma^2_{\text{эф}\,1}$ — variance of the efficient estimate.

If $y(t)$ is a normal random process, $S_{\text{опт}\,1}$ is found in accordance with (13.5.14) and (13.5.24) by the formula

$$S^{-1}_{\text{ouri}} = -\frac{1}{2T}\int_0^T\int_0^T \frac{\partial R_y(t,\tau)}{\partial\lambda_1}\frac{\partial W(t,\tau)}{\partial\lambda_1}\,dt\,d\tau. \tag{13.5.60}$$

Substituting in (13.5.60) values of R_y and W and producing approximate calculations, valid under the assumptions made above ($T_r \ll \tau_k \ll T$), we obtain

$$S^{-1}_{\text{ouri}} = \frac{P_c}{2}\Big\{\Big[(1-\gamma^2)A + \frac{2N_0}{P_n}B\Big]C + 2\gamma^2\frac{\partial\varphi}{\partial\lambda_1}AD - \left|\frac{\partial}{\partial\lambda_1}(\gamma e^{i\varphi})\right|^2 A + \frac{2N_0}{P_c}\gamma\frac{\partial\gamma}{\partial\lambda_1}\frac{\partial B}{\partial\lambda_1} - \Big(A + \frac{2N_0}{P_n}B\Big)D^2\Big\}, \tag{13.5.61}$$

where

$$\left.\begin{aligned}
A &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)S(\omega)\,d\omega;\\
B &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,d\omega;\\
C &= C_{11}^{(11)} = \frac{1}{T_r}\int_0^{T_r}\left|\frac{\partial u_1(t)}{\partial\lambda_1}\right|^2 dt;\\
D &= iC_{11}^{(01)} = i\frac{1}{T_r}\int_0^{T_r}\frac{\partial u^*_1(t)}{\partial\lambda_1}u_1(t)\,dt =\\
&= \frac{1}{T_r}\int_0^{T_r}\frac{\partial\psi_1(t)}{\partial\lambda_1}u^2_{a1}(t)\,dt.
\end{aligned}\right\} \tag{13.5.62}$$

For large signal-to-noise ratios h formula (13.5.61) is simplified, taking form

$$S^{-1}_{\text{ouri}} = \frac{h2\Delta f_e}{1-\gamma^2}\Big[(1-\gamma^2)C + 2\gamma^2\frac{\partial\varphi}{\partial\lambda_1}D - \left|\frac{\partial}{\partial\lambda_1}(\gamma e^{i\varphi})\right|^2 - D^2\Big]. \tag{13.5.63}$$

In deriving this formula we assumed that $(1 - \gamma^2)h \gg 1$. It is easy to see that this formula is obtained from (13.5.31) when n = 2.

For small h

$$S^{-1}_{\text{ouri}} = 2h^2(C - D^2)\frac{1}{2\pi}\int_{-\infty}^{\infty} S_e^2(\omega)\,d\omega. \tag{13.5.64}$$

So that the found optimum circuit ensures accuracy of measurement of parameter λ_1, corresponding to formulas (13.5.61)-(13.5.64), we need exact tuning of compensation channels, i.e., unerring knowledge of quantity λ_2. Practically, however, this parameter, corresponding to the coordinate of interference, is also measured with error. From

results of Paragraph 13.5.2 it follows that the best case is when there are systems of the considered form for measurement of parameters λ_1 and λ_2, acting jointly. Tuning of compensation channels in each of these systems is produced in accordance with the output quantity of the other system. To find fluctuating errors of such systems it is necessary to compose matrix of spectral densities (13.5.24). Calculating the element of it which corresponds to error of measurement of parameter λ_1, for the equivalent spectral density we obtain expression

$$S_1 = \frac{S_{\text{опт}1}}{1 - \frac{S_{\text{опт}1} S_{\text{опт}2}}{S_{12}^2}}, \tag{13.5.65}$$

where $S_{\text{опт}2}$ – spectral density of fluctuating error of measurement of parameter λ_2 for known λ_1, which is found from $S_{\text{опт}1}$ if we change the places here of P_c and $P_п$, and also of λ_1 and λ_2;

S_{12} – mutual spectral density, determined according to (13.5.14') as

$$S_{12}^{-1} = -\frac{1}{2T}\int_0^T\int_0^T \frac{\partial R_y(t,\tau)}{\partial\lambda_1}\frac{\partial W(t,\tau)}{\partial\lambda_2}\,dt\,d\tau. \tag{13.5.66}$$

Calculating integral (13.5.66), we have

$$S_{12}^{-1} = -N_0\frac{\partial}{\partial\lambda_2}\left[B\frac{\partial\gamma^2}{\partial\lambda_1}\right]. \tag{13.5.66'}$$

From (13.5.65) it is easy to see that when $\gamma = 0$ equality $S_1 = S_{\text{опт}1}$ is valid, which one should have expected, since here there are no compensation channels, and each parameter is measured independently one from the other. As $h \to \infty$, by calculating S_{12} we prove that this quantity approaches a finite limit, while $S_{\text{опт}1,2} \sim \frac{1}{h}$. Then $S_1 \to S_{\text{опт}1}$, and fluctuating errors for large signal-to-noise ratios are determined by expression (13.5.63).

For small h, using (13.5.64) and calculating S_{12}, we obtain

$$S_1 = \frac{S_{\text{опт}1}}{1 - \left[\frac{\frac{\partial^2(\gamma^2)}{\partial\lambda_1^2}}{C - D^2}\right]^2}, \tag{13.5.67}$$

i.e., there occurs increase of fluctuating error of measurement of parameter λ_1 due to errors of measurement of parameter λ_2.

§ 13.6. Analysis of Systems of Detection and Measurement with Suppression of Disturbing Signals

13.6.1. System of Detection

Output signals of the separation system synthesized in §13.2 can be used for finding a decision about the presence of targets and for measurement of their coordinates exactly as output signals of systems for singling out a signal against a background of noises and interferences, considered in preceding chapters. Such a detector or meter with suppression of disturbing signals differs from the corresponding devices calculated for the presence of only one target only in the form of reference signal or in the response of the shortening filter (in the case of range). As examples, in Fig. 13.13 there are given two variants of a block diagram of a system of detection of targets in a range of distances (with filtration and with multiplication by reference signals shifted in time).

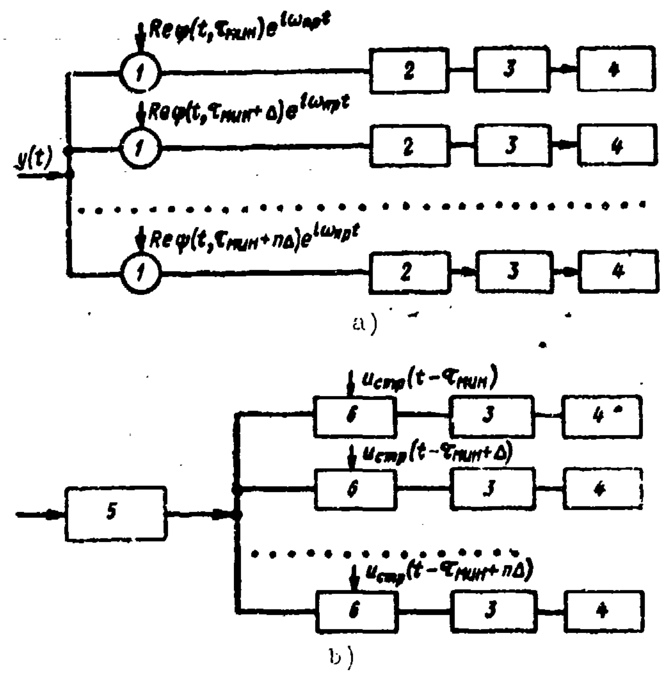


Fig. 13.13. Block diagram of a system of detection with suppression of interfering signals: a) with multiplication by reference signals shifted in time; b) with filtration: 1) mixer, 2) filter; 3) detector; 4) relay; 5) filter matched with the law of modulation; 6) gated amplifier.

We shall consider characteristics of detection for such a system and compare them with the optimum. First of all one should note that the probabilities of false alarm and correct detection of one target with fixed parameters for a system with separation of signals are determined by the very same formulas as for a system without separation of signals. The difference consists only in the powers of the signal and noise obtained after multiplication of the received signal by the reference signal and integration (see Chapter IV, Vol. I).

Thus, comparison of characteristics of detection of one target in systems with suppression and without suppression of interfering signals reduces to comparison of the corresponding signal-to-noise ratios. Such comparison already was conducted in

§ 13.2. The same remark is also valid for an optimum system of detection of a given target against a background of disturbing targets, inasmuch as such a system differs from a system with complete suppression of interfering signals and subsequent processing of the same form as in the absence of interfering signals also basically only in the form of the reference signal. Comparison of the corresponding signal-to-interference ratios for the case of one interfering target was already conducted in 13.2.1.

The difference between characteristics of detection in systems with suppression and without suppression of interfering signals turns out to be more considerable for detection of one or several targets in a certain range of values of parameters if we use a multichannel system of detection. This difference is caused by the fact that noises in detuned channels turn out to be correlated due to the presence in the reference signals of the same components.

Let us consider characteristics of detection for a system of channels in each of which there is produced complete suppression of the signals separated by the remaining channels. Fluctuations of all signals we shall consider slow.

Output voltage of each channel is recorded in the form (see §13.2)

$$r_j = |z_j|^2 = \left|\int_{S_0} y(s)\,\psi(s,\ \lambda_j)\,ds\right|^2 = \left|\sum_{l=1}^{n} w_{jl} Q_l\right|^2. \tag{13.6.1}$$

If the received signal y(s) is a normal random process, the real and imaginary part of complex quantities z_j ($j = 1, \ldots, n$) are distributed by normal law. Therefore joint distribution of these quantities can be recorded in the form

$$p(z_1, \ldots, z_n) = \frac{1}{\pi^n |R_{jk}|} \exp\left\{-\sum_{j,k} W_{jk} z_j z^*_k\right\}, \tag{13.6.2}$$

where $\|W_{jk}\|$ — matrix, the inverse of $\|R_{jk}\|$, and

$$R_{jk} = \overline{z_j z^*_k} = \sum_{l,m} w_{jl} w^*_{km} \overline{Q_l Q^*_m} =$$
$$= \sum_{l,m} w_{jl} w^*_{km} \left[C_{lm} + \sum_{\nu} \frac{E_\nu}{2} C_{l\nu} C_{\nu m}\right] = w_{jk} + \frac{E_k}{2}\delta_{jk}, \tag{13.6.3}$$

where E_ν — energy of the signal to which the ν-th channel is tuned. If there is no such signal, $E_\nu = 0$.

Let us find the joint distribution for quantities r_j. For this it is necessary to pass to polar coordinates, in which $z_j = \sqrt{r_j} e^{i\varphi_j}$, and to integrate the obtained distribution over all φ_j. Introducing for convenience instead of $\|R_{jk}\|$ the matrix of correlation coefficients

$$\rho_{jk} = \frac{R_{jk}}{\sqrt{R_{jj} R_{kk}}}, \qquad (13.6.4)$$

we obtain

$$p(r_1, \ldots, r_n) = \frac{1}{(2\pi)^n |\rho_{jk}| \bar{r}_1 \ldots \bar{r}_n} \times$$

$$\times \int_0^{2\pi} \ldots \int_0^{2\pi} \exp\left[-\sum_{j,k} \theta_{jk} \sqrt{\frac{r_j r_k}{\bar{r}_j \bar{r}_k}} e^{i(\varphi_j - \varphi_k)}\right] d\varphi_1 \ldots d\varphi_n, \qquad (13.6.5)$$

where $\|\theta_{jk}\|$ – matrix, the inverse of $\|\rho_{jk}\|$;

$$\bar{r}_j = \overline{|z_j|^2} = w_{jj} + \frac{E_j}{2} = w_{jj}(1 + q_j), \qquad (13.6.6)$$

and q_j – signal-to-interference ratio at the output of the system of separation for the j-th signal.

We cannot take integral (13.6.5) in general. Therefore we assume that ρ_{jk} is small when $j \neq k$, i.e., that $\|\rho_{jk}\|$ – quasi-diagonal matrix. Inasmuch as

$$\rho_{jk} = \frac{w_{jk}}{\sqrt{w_{jj} w_{kk}}} \frac{1}{\sqrt{(1 + q_j)(1 + q_k)}}, \qquad (13.6.4')$$

the assumption about the quasi-diagonal nature of $\|\rho_{jk}\|$ is satisfied if either matrices $\|w_{jk}\|$ and $\|c_{jk}\|$ are quasi-diagonal, or the signal-to-interference ratios at the output of the system of separation for all targets are great.

If $\rho_{jk} = (1 - \alpha_{jj}) \delta_{jk} + \alpha_{jk}$ and $|\alpha_{jk}| \ll 1$,

then

$$\theta_{jk} \approx (1 + \alpha_{jj}) \delta_{jk} - \alpha_{jk}.$$

Substituting this expression in (13.6.5), expanding the exponential function and the determinant in powers of α_{jk} with an accuracy of the second order, after integration over φ_j we obtain

$$p(r_1, \ldots, r_n) = \frac{1 + \sum\limits_{j \neq k} |\rho_{jk}|^2 \frac{r_j r_k}{\bar{r}_j \bar{r}_k}}{\left(1 + \sum\limits_{j \neq k} |\rho_{jk}|^2\right) \bar{r}_1 \ldots \bar{r}_n} \exp\left(-\sum_j \frac{r_j}{\bar{r}_j}\right). \tag{13.6.7}$$

Using this distribution it is possible to calculate the probability of any combinations of exceedings of thresholds in the channels.

In the case n = 2 integration of (13.6.5) is easily conducted for any $\rho = |\rho_{12}|$. During calculations it is convenient to use the expansion of this distribution in powers of ρ, which has the form

$$p(r_1, r_2) = \frac{1}{\bar{r}_1 \bar{r}_2 (1-\rho)^2} \exp\left[-\left(\frac{r_1}{\bar{r}_1} + \frac{r_2}{\bar{r}_2}\right) \Big/ (1-\rho^2)\right] \times$$
$$\times I_0\left(\frac{2\rho\sqrt{r_1 r_2}}{\sqrt{\bar{r}_1 \bar{r}_2}(1-\rho^2)}\right) = \frac{1}{\bar{r}_1 \bar{r}_2} \sum_{k=0}^{\infty} \left(\frac{\rho^k}{k!}\right)^2 \times$$
$$\times \left(\frac{d^k}{dx^k} x^k e^{-x}\right)_{x=\frac{r_1}{\bar{r}_1}} \left(\frac{d^k}{dy^k} y^k e^{-y}\right)_{y=\frac{r_2}{\bar{r}_2}}. \tag{13.6.8}$$

In order to have the possibility of comparing a system of detection of the considered form with an optimum system of detection of a group of targets, we shall consider the case of two targets appearing simultaneously (see § 13.4). Here, we consider that in the system with separation the simultaneity of appearance of targets is used either by means of summation of the output signals of the corresponding two channels, or by means of making a decision about the presence of both targets when in at least one of these channels we exceed the threshold.

In the first case with the threshold we compare quantity

$$r = r_1 + r_2 = |w_{11}Q_1 + w_{12}Q_2|^2 +$$
$$+ |w_{21}Q_1 + w_{22}Q_2|^2 = \sum_{j,k=1}^{2} v_{jk} Q_j Q^*_k, \tag{13.6.9}$$

where

$$v_{11} = |w_{11}|^2 + |w_{12}|^2; \; v_{22} = |w_{22}|^2 + |w_{12}|^2;$$
$$v_{12} = v^*_{21} = w_{12}(w_{11} + w_{22}).$$

When calculating probabilities of correct detection it is possible to use general formula (13.6.8). However, it is simpler to start from the expression for characteristic function (13.4.2), taking into account (13.6.9). Performing calculations analogous to those which we made in deriving (13.4.3) and (13.4.4) and assuming

$q_1' = q_2' = q_0'$ (see § 13.4), we obtain

$$F(c) = \frac{1+\gamma}{2\gamma} e^{-\frac{c}{1+\gamma}} - \frac{1-\gamma}{2\gamma} e^{-\frac{c}{1-\gamma}}, \qquad (13.6.10)$$

$$D(c) = \frac{1+q'_0(1-\gamma^2)+\gamma}{2\gamma} e^{-\frac{c}{1+q'_0(1-\gamma^2)+\gamma}} -$$

$$- \frac{1+q'_0(1-\gamma^2)-\gamma}{2\gamma} e^{-\frac{c}{1+q'_0(1-\gamma^2)-\gamma}}, \qquad (13.6.11)$$

where c — the threshold quantity, multiplied by $(1 + \gamma^2)/(1 - \gamma^2)$, with which we compare r, and

$$\gamma = |C(\lambda_1, \lambda_2)| / \sqrt{C(\lambda_1, \lambda_1)\, C(\lambda_2, \lambda_2)}.$$

Calculation of relationship $q_0'(D, F, \gamma)$ in this case is conducted almost the same as for the optimum system in § 13.3. Note that

$$D(c, \gamma, q'_0) = F(c_1, \gamma_1), \qquad (13.6.12)$$

where

$$\gamma_1 = \frac{\gamma}{1+q'_0(1-\gamma^2)}; \quad c_1 = \frac{c}{1+q'_0(1-\gamma^2)}. \qquad (13.6.13)$$

Considering that $\gamma_1 \ll 1$ when $\gamma^2 <$ ..., it is possible to calculate q_0' by formula

$$q'_0 = \frac{1}{1-\gamma^2}\left[\frac{c(F, \gamma)}{c(D, 0)} - 1\right]. \qquad (13.6.14)$$

If the decision about the presence of targets is taken when at least one channel exceeds the threshold, the probability of making such a decision is recorded, taking into account (13.6.8), in the form

$$p_1 = 1 - \int_0^c \int_0^c p(r_1, r_2)\, dr_1\, dr_2 = 2e^{-\frac{c}{\overline{r_1}}} - e^{-\frac{2c}{\overline{r_1}}} -$$

$$- \sum_{k=1}^{\infty} \left(\frac{\rho^k}{k!}\right)^2 \left(\frac{d^{k-1}}{dx^{k-1}} x^k e^{-x}\right)^2_{x=\frac{c}{\overline{r_1}}}. \qquad (13.6.15)$$

Substituting in this formula the corresponding values of ρ and $\overline{r_1}$, it is possible to calculate the probability of false alarm F and of correct detection D. For comparison in Fig. 13.14 we show the dependence of the threshold signal-to-interference ratio q_0' (obtained in the absence of separation) on , and on the probability of correct detection of both targets for $F = 10^{-6}$ for different methods of processing.

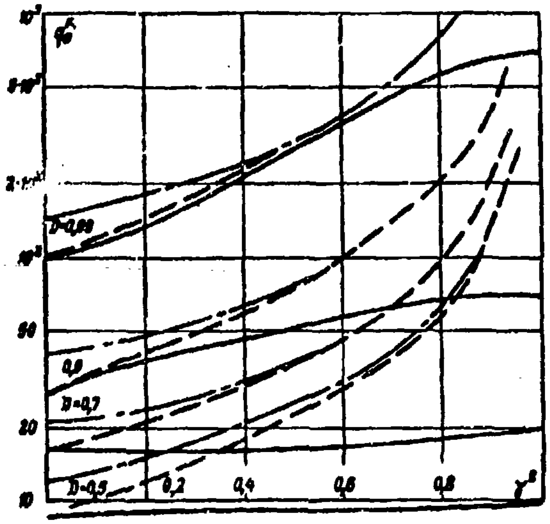


Fig. 13.14. Relationship $q_0'(D, \gamma)$ for $F = 10^{-6}$ for detection of a pair of targets of equal intensity: ——— — optimum processing (13.4.3), (13.4.4); – – – – – summation of signals at the output of channels in a system with separation (13.6.14); – · – · – – independent comparison with the threshold in a system with separation (13.6.15).

From comparison of curves of this figure one may see that loss due to use of a system with separation instead of an optimum one grows with increase of γ and with decrease of the probability of correct detection. For small γ and $\beta = 1 - D$ separation of signals does not lead to essential losses. If, for instance, we consider double loss in the signal-to-interference ratio permissible, it is attained for $\beta = 0.5$ at $\gamma^2 \approx 0.37$; for $\beta = 0.9$ at $\gamma^2 \approx 0.65$; for $\beta = 0.999$ at $\gamma^2 \approx 0.93$. The difference between systems with summation and independent comparison with a threshold of the output signals in the case of two targets is immaterial.

13.6.2. System of Measurement with Complete Suppression of Selected Interfering Signals

Let us consider the discriminator of a tracking meter with total suppression of selected interfering signals. Here, as too in § 13.5, we shall consider fluctuations of the signal to be fast, and noise to be white, and we shall limit ourselves for simplicity to consideration of signals depending only on time. Such consideration embraces problems of measurement of range and measurement of angles with scanning. Generalization of results to the case of space processing is achieved by simple replacement of matrix elements $c_{jk}^{(pq)}(\lambda_j, \lambda_r)$ (for designations see § 13.5) obtained by integration over time by the same elements obtained by integration over time and over space.

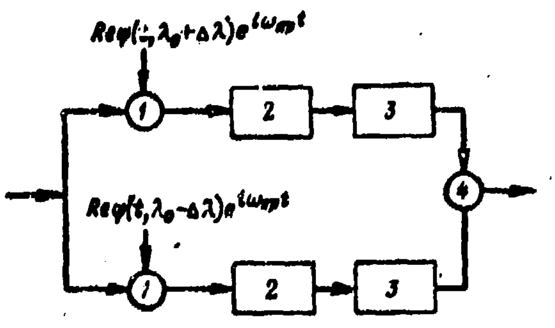


Fig. 13.15. Block diagram of the analyzed discriminator with suppression of interfering signals: 1) mixer; 2) filter; 3) detector; 4) subtractor.

A block diagram of the analyzed discriminator is shown in Fig. 13.15.

Reference signals $\psi(t, \lambda$ and $\psi(t, \lambda_0 - \Delta\lambda)$ are determined by formulas (13.2.9). In order to simplify calculations we consider detuning $\Delta\lambda$ very small, so that

$$z(t)=\left|\int_{-\infty}^{t} h(t-\tau)\psi(\tau, \lambda_0+\Delta\lambda)y(\tau)\,d\tau\right|^2 - \\ -\left|\int_{-\infty}^{t} h(t-\tau)\psi(\tau, \lambda_0-\Delta\lambda)y(\tau)\,d\tau\right|^2 \approx \\ \approx 2\Delta\lambda\frac{\partial}{\partial\lambda_0}\left|\int_{-\infty}^{t} h(t-\tau)\psi(\tau, \lambda_0)y(\tau)\,d\tau\right|^2. \quad (13.6.\ \ )$$

We can hope that the dependence of equivalent spectral density on detuning will in the given case, as for systems without suppression of interfering signals (Chapters VII -XI), be sufficiently weak so that the given assumption immaterially limits the applicability of the results.

Let us consider the gain factor $K_Д$, the equivalent spectral density $S_{ЭКВ}$ and systematic error $\varepsilon_{СИСТ}$ for a system of the considered form. Here we consider that interfering signals, if there are any, are completely suppressed, and we consider only the influence of noises. In general, complete suppression of interfering signals is obtained only for preselected values of their parameters. In certain particular cases (for instance, during phase-code manipulation) suppression of signals at selected points ensures their suppression for all values of parameters beyond the limits of the basic maximum of the function of uncertainty. After very cumbersome transformations, analogous to those used in § 13.5, we obtain

$$K_Д=\left(\frac{\partial\bar{z}}{\partial\lambda_0}\right)_{\hat{\lambda}_0=\lambda_0}=-2\Delta\lambda\frac{P_0}{\Delta f_c}w_{00}\times \\ \times\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(i\omega)|^2 S_0(\omega)\,d\omega\,(b_n-2a_n^2), \quad (13.6.17)$$

$$\varepsilon_{СИСТ}=\frac{\tilde{z}}{K_Д}=\frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(i\omega)|^2\left[1+\frac{h}{w_{00}}S_0(\omega)\right]d\omega}{h(b_n-2a_n^2)\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(i\omega)|^2 S_0(\omega)\,d\omega}a_n\sqrt{w_{00}}, \quad (13.6.18)$$

$$S_{\text{экв}} = \frac{\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} |H(i\omega)|^4 \left\{ b_n + 2a_n^2 + \dfrac{h}{w_{00}} S_0(\omega)(b_n + 4a_n^2) + \dfrac{2h^2}{w_{00}^2} S_0^2(\omega)\, a_n^2 \right\} d\omega}{\dfrac{2h^2}{w_{00}}(b_n - 2a_n^2)^2 \times \left(\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} |H(i\omega)|^2 S_0(\omega)\, d\omega \right)^2}, \tag{13.6.19}$$

where $h = \dfrac{P_0}{2N_0 \Delta f_c}$; $H(i\omega)$ – frequency response of the filter;

$$b_n = C_{00}^{(11)}(\lambda_0, \lambda_0) - \sum_{j,k=0}^{n} C_{0j}^{(10)}(\lambda_0, \lambda_j)\, w_{jk} C_{0k}^{(10)*}(\lambda_0, \lambda_k); \tag{13.6.20}$$

$$a_n = \frac{1}{\sqrt{w_{00}}} \operatorname{Re} \sum_{j=0}^{n} w_{0j} \frac{\partial C^*_{0j}(\lambda_0, \lambda_j)}{\partial \lambda_0}. \tag{13.6.21}$$

In general a discriminator of the considered form gives systematic error which does not approach zero with unlimited increase of the signal-to-interference ratio, which it is necessary to specially compensate. This error disappears with symmetric location of parameters of interfering signals relative to the measured parameter.

In order to prove this we shall consider the case when $C_{jk}(\lambda_j, \lambda_k)$ depends only on difference $\lambda_j - \lambda_k (C_{jk}(\lambda_j, \lambda_k) = C(\lambda_j - \lambda_k))$, and values $\lambda_{-l}, \ldots, \lambda_{-1}, \lambda_1, \ldots, \lambda_l$ of the parameters of the interfering signals are such that $\lambda_j - \lambda_0 = \lambda_0 - \lambda_{-j} (0 \leq j \leq l)$. Here, as it is easy to show, $w_{jk} = w^*_{-j-k} (0 \leq j, k \leq l)$ and $a_n = 0$. Systematic error $\varepsilon_{\text{сист}}$ turns into zero, and the expression for equivalent spectral density takes the following form:

$$S_{\text{экв}} = \frac{\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} |H(i\omega)|^4 \left(1 + \dfrac{h}{w_{00}} S_0(\omega)\right) d\omega}{2\dfrac{h^2}{w_{00}} b_n \left(\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} |H(i\omega)|^2 S_0(\omega)\, d\omega \right)^2} \approx \frac{1}{\dfrac{P_0}{N_0} b_n}. \tag{13.6.22}$$

The last equality in (13.6.22) is valid for a large signal-to-noise ratio at the output of the system of separation of signals ($\frac{h}{w_{00}} \gg 1$) and for $\Delta f_{\Phi} \gg \Delta f_c$ (Δf_{Φ} – width of the passband of the filter).

The obtained expression coincides for large ratios $\frac{h}{w_{00}}$ with the equivalent spectral density of the optimum discriminator and for arbitrary h to the equivalent spectral density of the discriminator analyzed in § 13.5 (Fig. 13.7). This is

completely natural, since with symmetric location of interfering signals $\frac{\partial w_{00}}{\partial \lambda_0} = 0$, and we have equality [see (13.2.13)]

$$\frac{\partial}{\partial \lambda_0} \sum_{j,\,k=0}^{n} w_{jk}^{(n+1)} Q^*_j Q_k = \frac{\partial}{\partial \lambda_0} \sum_{j,\,k=1}^{n} w_{jk}^{(n)} Q^*_j Q_k +$$

$$+ \frac{\partial}{\partial \lambda_0} \sum_{j,k=0}^{n} \frac{w_{0j}^{(n+1)*} w_{0k}^{(n+1)}}{w_{00}^{(n+1)}} Q^*_j Q_k = \frac{1}{w_{00}^{(n+1)}} \frac{\partial}{\partial \lambda_0} \Big| \sum_{j=0}^{n} w_{0j} Q_j \Big|^2,$$

so that for large signal-to-noise ratios processing of a signal in the considered discriminator coincides with the optimum.

It follows from this that a discriminator of the given form can be used with success in systems of measurement if parameters of the suppressed signal are located symmetrically relative to the measured parameter of the target and vary together with this parameter and $C_{jk}(\lambda_j, \lambda_k) = C(\lambda_j - \lambda_k)$. This occurs, for instance, if signals corresponding to spurious maxima of the function of uncertainty are suppressed.

Loss in accuracy connected with suppression of interfering signals is determined for large signal-to-noise ratios by formula

$$\Gamma = \frac{b_0}{b_n} = \frac{C_{00}^{(11)}(\lambda_0, \lambda_0) - |C_{00}^{(10)}(\lambda_0, \lambda_0)|^2}{C_{00}^{(11)}(\lambda_0, \lambda_0) - \sum\limits_{j,k=0}^{n} C_{0j}^{(10)}(\lambda_0, \lambda_j)\, w_{jk} C_{0k}^{(10)*}(\lambda_0, \lambda_k)}. \tag{13.6.23}$$

With an arbitrary signal-to-noise ratio the loss in separation of signals can be accounted for by increasing the spectral density for the system without separation of signals by a factor of Γ/w_{00} and replacing the signal-to-noise ratio h by h/w_{00}.

Note that quantity Γ is equal to one if $C(\lambda)$ = const outside the principal maximum and $C_{00}^{(10)}(\lambda_0, \lambda_0) = 0$. This takes place, for instance, for measurement of range with a signal manipulated in phase by Hoffman's code [see 13.2.2].

The same result is always obtained when values of parameters of suppressed signals coincide with maxima of the function of uncertainty $|C(\lambda - \lambda_0)|^2$, and $C(\lambda - \lambda_0)$ is a real function. Here $C_{0j}^{(10)}(\lambda_0, \lambda_j) = C'(\lambda_0 - \lambda_j) = 0$ at all the considered points. For the shown cases the loss of accuracy for large signal-to-noise ratios is absent.

13.6.3. System of Measurement of Parameters of Motion of Two Targets

In the preceding paragraph we considered the case when we exactly know parameters of the interfering signals. Such a case occurs, for instance, during suppression of signals corresponding to maxima of the function of uncertainty. We turn now to the case of measurement of coordinates of several targets by tracking radar meters. Such systems have a number of outputs equal to the number of measured quantities. Voltages from the outputs and also from intermediate points of the measuring channels can be used for compensation of interfering signals in other channels or for tuning of compensation channels.

Let us consider a tracking meter of coordinates of two targets. Being interested in the meter of coordinate λ_1 of one of them, we assume that in the channel of measurement of quantity λ_1 there is realized compensation of the signal reflected from the second target, where for construction of the compensation channel to maximum extent there is used the channel of coordinate λ_2 of the second target. If we apply here a form of compensation which with exact tuning corresponds to total suppression of the interfering signal, the functional circuit of the discriminator is represented by Fig. 13.16.

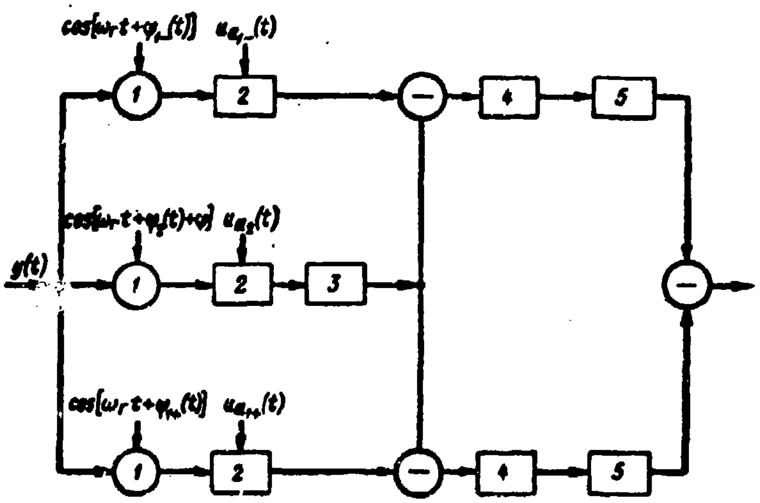


Fig. 13.16. Circuit of a discriminator with suppression of an interfering signal. 1) mixer; 2) amplifier with gain varying according to the law of amplitude modulation; 3) amplifier with gain γ; 4) filter; 5) square-law detector.

The discriminator contains two channels mutually detuned with respect to parameter λ_1 ($\pm\Delta\lambda_1$) intended for measurement of λ_1 in the absence of an interfering signal, and also a channel for compensation of the latter. With proper selection of filters of the basic channels the discriminator is optimum in the absence of an interfering signal or when $\gamma = 0$. For total suppression of the interfering signal, strictly speaking, we would need to select parameters of the compensation channel γ and φ

differently for the two detuned channels. However, if we are interested in tracking meters of both parameters, the channel of compensation is tuned with errors determined by the accuracy of measurement of the parameter λ_2.

For small $\Delta\lambda_1$ errors of measurement of λ_2 (quantities $\Delta\lambda_2$) turn out to be comparable with $\Delta\lambda_1$. Here separate tuning of compensation channels loses meaning and the simplest construction of the compensation channel, corresponding to the considered circuit, is desirable. Let us find systematic error and equivalent spectral density of fluctuating error of the circuit of Fig. 13.16 first on the assumption of absence of error of measurement of $\lambda_2(\Delta\lambda_2 = 0)$, and then taking this error into account.

The signal at the discriminator output can be recorded in the form

$$\Delta u_{\text{д}}(t)=\int_0^t\int_0^t h(t-s_1)\,h(t-s_2)\,\{u_{1-}(s_1)\,u^*_{1-}(s_2)-$$
$$-u_{1+}(s_1)\,u^*_{1+}(s_2)-\gamma\,[(u_{1-}(s_1)-u_{1+}(s_1))\,u^*_2(s_2)\,e^{-i\varphi}+$$
$$+(u^*_{1-}(s_2)-u^*_{1+}(s_2))\,u_2(s_1)\,e^{i\varphi}]\}\times$$
$$\times e^{i\omega_0(s_1-s_2)}\,y(s_1)\,y(s_2)\,ds_1\,ds_2, \tag{13.6.24}$$

where $h(t)$ – pulse response of the low-frequency equivalent of the filter; $u_{1-}(t)$ and $u_{1+}(t)$ – reference laws of modulation in the two detuned channels.

With the same simplifying assumptions as in the preceding paragraph and with coincidence of the true value of λ_1 with the average value of $\hat{\lambda}_1$ among points of tuning of the channels for small detunings $\Delta\lambda_1$ we obtain

$$\overline{\Delta u_{\text{д}}(t)}\big|_{\delta\lambda_1=0}=8N_0B_1\gamma\frac{\partial\gamma}{\partial\lambda_1}\Delta\lambda_1, \tag{13.6.25}$$

where $\delta\lambda_1=\lambda_1-\hat{\lambda}_1$;

$$B_1=\int_0^t h^2(s)\,ds. \tag{13.6.26}$$

The gain factor of the discriminator $K_{\text{д}}$ can be found as

$$K_{\text{д}}=\frac{\overline{d\Delta u_{\text{д}}(t)}}{d\lambda_1}\bigg|_{\delta\lambda_1=0}. \tag{13.6.27}$$

After calculations we have

$$K_{д} = -\frac{4A_1 P_c}{\Delta f_c}\left[C(1-\gamma^2)+\gamma^2\frac{\partial\varphi}{\partial\lambda_1}D - D^2\right]\Delta\lambda_1, \tag{13.6.28}$$

where

$$A_1 = \Delta f_c \int_0^t\int_0^t h(s_1)\,h(s_2)\,\rho(s_1 - s_2)\,ds_1 ds_2. \tag{13.6.29}$$

Systematic error $\varepsilon_{смст}$ is now found as ratio

$$\varepsilon_{смст} = \frac{\overline{\Delta\alpha_{д}(t)}|_{\Delta\lambda_1=0}}{K_{д}} = -\frac{B_1\gamma\frac{\partial\gamma}{\partial\lambda_1}}{A_1 h\left[C(1-\gamma^2)+\gamma^2\frac{\partial\varphi}{\partial\lambda_1}D - D^2\right]}. \tag{13.6.30}$$

Thus, when $\gamma \neq 0$ due to compensation there appears systematic error, inversely proportional to the signal-to-noise ratio h. The presence of such error should have been expected, since in an optimum circuit for compensation of such error there is introduced the corresponding correction, whereas in the given circuit such a measure is not provided. For large h systematic error can be disregarded.

For allowance for error $\Delta\lambda_2$ of tuning of the compensation channel it is necessary in expression (13.6.24) to replace $u_2(t)$ by a certain $u_{20}(t)$. Accordingly γ and φ should be replaced by γ_0 and φ_0. Considering $\Delta\lambda_2$ small and limiting ourselves to the first two terms of the expansion of $u_{20}(t)$ in a Taylor series

$$u_{20}(t) = u_2(t) + \Delta\lambda_2\frac{\partial u_2(t)}{\partial\lambda_2}, \tag{13.6.31}$$

it is easy to find

$$K_{д} = -\frac{4A_1 P_c}{\Delta f_c}\left[C(1-\gamma_0^2)+\gamma_0^2\frac{\partial\varphi_0}{\partial\lambda_1}D - D^2\right]\Delta\lambda_1 \tag{13.6.32}$$

and

$$\varepsilon_{смст} = -\frac{\dfrac{B_1\gamma_0\frac{\partial\gamma_0}{\partial\lambda_1}}{A_1 h} + \dfrac{1}{q}\left[\gamma^2\frac{\partial\varphi}{\partial\lambda_1}\left(D - \frac{\partial\varphi}{\partial\lambda_1}\right) - \left(\frac{\partial\gamma}{\partial\lambda_1}\right)^2\right]\Delta\lambda_2}{C(1-\gamma_0^2)+\gamma_0^2\frac{\partial\varphi_0}{\partial\lambda_1}D - D^2}. \tag{13.6.33}$$

The difference of γ_0 and φ_0 from γ and φ, respectively, for small $\Delta\lambda_2$ can usually be disregarded. However, the presence of a second term in the numerator of (13.6.33) can be accounted for since q and h may differ greatly. For very small q (high power of interfering signals) this term may cause substantial additional error. For sufficiently large q it can be disregarded.

It is necessary to note that for moments t, very remote from the beginning of the process, quantities A_1 and B_1 in (13.6.33) do not depend on t and are determined by formulas

$$\left.\begin{aligned} A_1 &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} S_\varphi(\omega)\,|H(i\omega)|^2\,d\omega, \\ B_1 &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |H(i\omega)|^2\,d\omega, \end{aligned}\right\} \qquad (13.6.34)$$

where $H(i\omega)$ -- frequency response of a filter whose pulse response is $h(t)$.

The equivalent spectral density of fluctuation error is found as

$$S_{\text{экв}} = \frac{1}{K_\text{д}^2}\int\limits_{-\infty}^{\infty} R_\text{д}(\tau)\,d\tau, \qquad (13.6.35)$$

where

$$R_\text{д}(\tau) = \overline{\Delta u_\text{д}(t)\,\Delta u_\text{д}(t+\tau)} - \overline{\Delta u_\text{д}(t)}\;\overline{\Delta u_\text{д}(t+\tau)} \qquad (13.6.36)$$

is the correlation function of the output quantity of the discriminator. If we are interested in steady-state operating conditions (large t), then on the made assumptions ($\tau_1 \ll \tau_K$) this function does not depend on t, and accordingly $S_{\text{экв}}$ is constant.

Substituting (13.6.24) in (13.6.36) and performing calculations corresponding to (13.6.36) and (13.6.35) assuming small $\Delta\lambda_1$ and $\Delta\lambda_2 = 0$, we obtain

$$\begin{aligned} S_{\text{экв}} = {} & \frac{1}{2A_1^2\left[C(1-\gamma^2)+\gamma^2\frac{\partial\varphi}{\partial\lambda_1}D - D^2\right]^2}\times \\ & \times\left\{\frac{1}{q}S_0(1-\gamma^2)^2\left|\frac{\partial}{\partial\lambda_1}(\gamma e^{i\varphi})\right|^2 + \frac{1}{h}S_{\text{сш}}(1-\gamma^2)\left[\frac{M}{2} - D^2 + \right.\right. \\ & \left. + 2\gamma^2\frac{\partial\varphi}{\partial\lambda_1}D + \frac{1}{q}\left|\frac{\partial}{\partial\lambda_1}(\gamma e^{i\varphi})\right|^2\right] + \frac{1}{h^2}S_{\text{ш}}\left[\frac{M}{2}(1-\gamma^2) - D^2 + \right. \\ & \left.\left. + 2\gamma^2\left(\frac{\partial\varphi}{\partial\lambda_1}D + \left(\frac{\partial\gamma}{\partial\lambda_1}\right)^2\right) - \gamma^2\left|\frac{\partial}{\partial\lambda_1}(\gamma e^{i\varphi})\right|^2\right]\right\}, \end{aligned} \qquad (13.6.37)$$

where

$$\left.\begin{aligned}
&S_0=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} S_s^2(\omega)\,|H(i\omega)|^4\,d\omega;\\
&S_{sш}=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} S_s(\omega)\,|H(i\omega)|^4\,d\omega;\\
&S_ш=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |H(i\omega)|^4\,d\omega;\\
&M=C+E;\\
&E=\frac{1}{T_r}\int\limits_0^{T_r}\left[u_{s1}(s)\left(\frac{\partial\psi_1(s)}{\partial\lambda_1}\right)^2-\right.\\
&\left.-\frac{\partial^2 u_{s1}(s)}{\partial\lambda_1^2}\right]u_{s1}(s)\,ds.
\end{aligned}\right\} \qquad (13.6.38)$$

The expression for $S_{экв}$ contains three components. The first of them does not depend on h and is caused by interaction of random components of the useful and interfering signals. The second component, proportional to $\frac{1}{h}$, reflects beats of the signal and interference with noise. Finally, the third component, proportional to $\frac{1}{h^2}$, appeared due to nonlinear transformations of noise.

Thus, in distinction from the optimum circuit, in the given case fluctuating error is different from zero even with infinitely large signal-to-noise ratios h.

Allowance for error in tuning of the compensation channel ($\Delta\lambda_2 \neq 0$) leads, as it is easy to see, to appearance of a correction to formula (13.6.37), proportional to $(\Delta\lambda_2)^2$. If we consider $\Delta\lambda_2$ a small quantity, correction of the order of $(\Delta\lambda_2)^2$ in many cases can be disregarded.

13.6.4. Meter of Range with Two Targets for One Particular Form of Signal Modulation

As an example of application of the results obtained above for systems of measurement of coordinates we shall consider the following particular case. Let us find quantities characterizing errors of a range finder measuring the distance to one of two targets, the distance between which is small. We assume that the sounding signal is a pulse signal with Gaussian pulse shape and, furthermore, we possess intrapulse linear frequency modulation, i.e., the complex law of modulation has the form

$$u(t)=\sqrt{\frac{T_r}{\tau_u}}\,e^{-\frac{\pi t^2}{2\tau_u^2}+i\frac{\beta t^2}{2}}\left(-\frac{T_r}{2}\leqslant t\leqslant\frac{T_r}{2}\right), \qquad (13.6.39)$$

where $\beta = \frac{d\omega}{dt}$ — rate of change of frequency,

$\tau_{и}$ — effective duration of a pulse, equal to

$$\tau_{и} = \frac{1}{u^2_{a\,макс}} \int\limits_{-\frac{T_r}{2}}^{\frac{T_r}{2}} u^2_a(t)\,dt \qquad (\tau_{и} \ll T_r).$$

We consider that the correlation function of fluctuations of the reflected signal is exponential, i.e.,

$$\rho(t) = e^{-\alpha|t|},$$

or

$$S_0(\omega) = \frac{\alpha^2}{\alpha^2 + \omega^2}. \qquad (13.6.40)$$

Let us consider accuracy of measurement of range when using discriminators built by the optimum scheme (Figs. 13.10 and 13.11), and also by schemes of Fig. 13.9 and Fig. 13.16, which in certain conditions, stated above, should give results close to the best ones, peculiar to the optimum circuit.

As the basic characteristic we shall use equivalent spectral density $S_{экв}$. In the same cases, when there are systematic errors, we shall also estimate them.

Properties of the optimum discriminator are completely characterized by $S_{опт}$, determined by expression (13.5.61).

Calculating the quantities in this formula, we have

$$\left.\begin{aligned}
&C = \frac{\pi}{\varepsilon\tau_{и}^2} + \frac{\beta^2\tau_{и}^2}{2\pi},\\
&D = 0\\
&\gamma = e^{-\frac{\pi\delta^2}{4\tau_{и}^2}\left(1+\frac{\beta^2\tau_{и}^4}{\pi^2}\right)},\\
&\varphi = \frac{\beta\delta m}{2},\\
&A = \frac{\pi\alpha}{2\mu N_0^2}\left(\frac{a_1 - a_2}{\alpha} - \right.\\
&\qquad \left. - \frac{a_1}{\sqrt{\alpha^2 + \xi - \mu}} + \frac{a_2}{\sqrt{\alpha^2 + \xi + \mu}}\right),\\
&B = \frac{\pi}{4\mu N_0^2}\left(\frac{1}{\sqrt{\alpha^2 + \xi - \mu}} - \frac{1}{\sqrt{\alpha^2 + \xi + \mu}}\right),
\end{aligned}\right\} \qquad (13.6.41)$$

where

$$\delta = \tau_1 - \tau_2,$$

τ_1, τ_2 — delays of modu' .tion of signals reflected from the two targets,

$$m = \tau_1 + \tau_2,$$
$$\xi = \frac{(P_c + P_п)a}{2N_0},$$
$$\varkappa = \frac{2P_c P_п a^2}{N_0},$$
$$\mu^2 = \xi^2 - \frac{P_c P_п a^2}{N_0^2}(1 - \gamma^2),$$
$$a_1 = \frac{1}{\xi - \mu},$$
$$a_2 = \frac{1}{\xi + \mu}.$$

With intrapulse frequency modulation deviation of frequency is usually selected large, so that $\beta\tau_И^2 \gg 1$. Fulfillment of this condition permits results of calculations (both for the optimum and also for other circuits) to approximately coincide for cases of the presence and absence of frequency modulation if only we introduce in formulas quantity x, determined as $x = \tau_И$ without frequency modulation and $x = \frac{\pi}{\beta\tau_И}$ with it.

Designating $y = \sqrt{\frac{\pi}{2}}\frac{\delta}{x}$ — relative difference of delays of signals reflected from two targets,* we have

$$C = \frac{\pi}{2x^2};\ \gamma = e^{-\frac{y^2}{2}},$$

and formula (13.5.61) takes form

$$S_{опт}^{-1} = \frac{P_c}{2}\left\{\left[(1 - \gamma^2)A + \frac{2N_0}{P_п}B\right]C + \frac{2N_0}{P_c}\gamma\gamma' B' - \gamma'^2 A\right\}, \qquad (13.6.42)$$

where the stroke signifies differentiation with respect to τ_1.

Results of calculations of $S_{опт}$ for case $q = \frac{P_c}{P_П} = 1$ (identical powers of the useful and interfering signals) are presented in Fig. 13.17.

In this same figure there is depicted the asymptotic dependence of $S_{опт}$ on y

$$S_{опт} = \frac{2x^2(1 - e^{-y^2})}{\pi a h(1 - e^{-y^2} - y^2 e^{-y^2})}, \qquad (13.6.43)$$

taking place for very large $h(1 - e^{-y^2})$.

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*It is also possible to call this quantity the relative distance between targets.

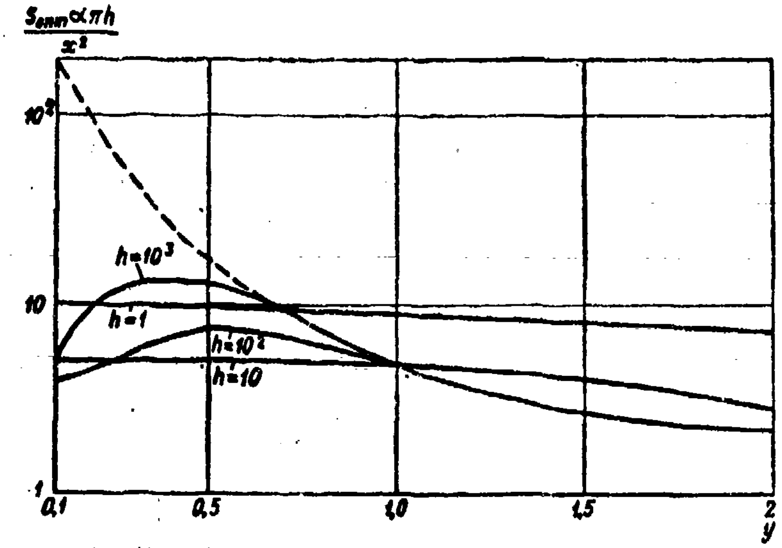

Fig. 13.17. Dependence of equivalent spectral density of an optimum discriminator on the relative distance between targets y: — — — by the formula (13.6.43).

With decrease of quantity y, proportional to the distance between targets, spectral density of fluctuating error grows, which is explained by two factors. First, there take place power losses connected with compensation of the interfering signal. Second, compensation of the interfering signal is inaccurate due to errors of measurement. This inaccuracy leads to maximum errors in the region of the greatest slope of the function of uncertainty of the signal. With greater nearness of the targets the distance between them corresponds to the flat part of the curve of the function of uncertainty, due to which equivalent spectral density, due to the second of the shown factors, decreases. With small signal-to-noise ratios h quantity $S_{экв}$ depends little on the distance between targets, which was clear from general formula (13.6.64).

Equivalent spectral density for a range discriminator built by the scheme of Fig. 13.[illegible] is determined by formula (13.5.42), which for the considered case takes the form

$$S_{экв} = \frac{hS_{см} + \frac{S_{ш}}{1-\gamma^2}}{2h^2\left(E - \frac{\gamma'^2}{1-\gamma^2}\right)A_1^2}. \quad (13.6.44)$$

Quantity A_1 is determined by expression (13.6.34), and quantities $S_{ш}$, $S_{см}$

and E — by expressions (13.6.38) We consider that the frequency response of the filters in these expressions has the form

$$|H(i\omega)|^2 = \frac{hS_0(\omega)}{1+hS_0(\omega)}, \tag{13.6.45}$$

i.e., filters are optimum in conditions of resolved targets.

Producing calculations, we obtain

$$\left.\begin{aligned} A_1 &= \frac{1}{2}\frac{ah}{\sqrt{1+h}\,(1+\sqrt{1+h})}, \\ S_{\text{ш}} &= \frac{h^2 a}{4(1+h)^{3/2}}, \\ S_{\text{ош}} &= \frac{h^2 a\,(2\sqrt{1+h}+1)}{4(1+h)^{3/2}(1+\sqrt{1+h})^2}, \\ E &= C = \frac{\pi}{2x^2}. \end{aligned}\right\} \tag{13.6.46}$$

For large h formula (13.6.44) after substitution of values of coefficients takes the simple form

$$S_{\text{экв}} = \frac{x^2}{\pi a h}\cdot\frac{2(1-e^{-y^2})+\frac{1}{\sqrt{h}}}{1-e^{-y^2}-y^2e^{-y^2}}. \tag{13.6.47}$$

From comparison of (13.6.43) and (13.6.47) we see that for large h and not too small y the considered circuit possesses the same properties as the optimum. The difference between this circuit and the optimum appears at small h and small y.

For the discriminator of Fig. 13.16 equivalent spectral density is determined by formula (13.6.37). Finding from (13.6.38) coefficients S_c and M in this formula, we obtain

$$\begin{gathered} S_c = \frac{h^2 a\,[(1+\sqrt{1+h})^2+\sqrt{1+h}]}{4(1+h)^{3/2}(1+\sqrt{1+h})^2}, \\ M = E + C = \frac{\pi}{x^2}. \end{gathered} \tag{13.6.48}$$

In Fig. 13.18 are curves of the dependences of $S_{\text{экв}}$ on the relative distance between targets y for different values of the signal-to-noise ratio h for all three considered circuits. For large h and relatively small y, at which application of circuits of compensation of signals still has meaning, the circuit of Fig. 13.? turns out to be close to optimum. For small h its properties differ from properties of the optimum circuit in a certain range of values of y more than properties of the circuit

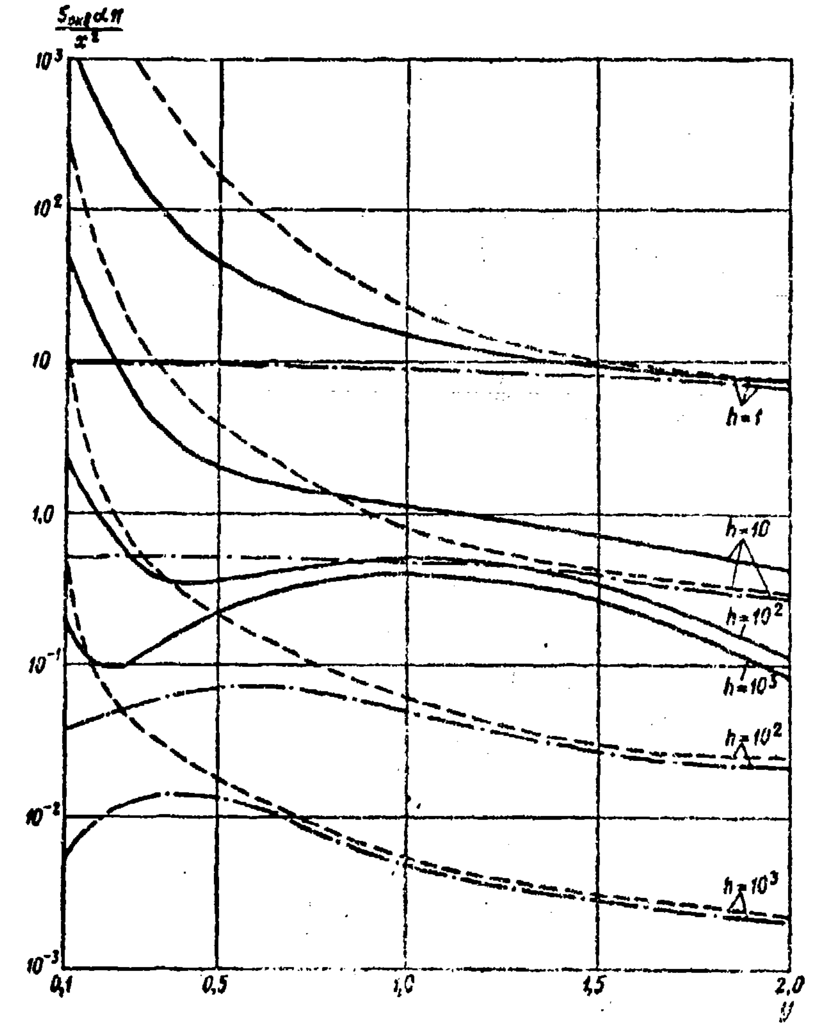

Fig. 13.18. Dependence of equivalent spectral density of the three considered forms of discriminators on the relative distance between targets y: ——— for the circuit of Fig. 13.16, – – – – for the circuit of Fig. 13.?. – · – · – for the optimum circuit.

of Fig. 13.16.

At large h the last circuit possesses considerably greater fluctuating error than the two preceding ones.

Furthermore, in distinction from the above considered circuit it possesses systematic error, determined by formula (13.?.3?). Substituting in this formula values of coefficients, we find that

$$\left.\begin{aligned} \varepsilon_{\text{сист}} &= \varepsilon_{\text{сист}\,1} + \varepsilon_{\text{сист}\,2}, \\ \varepsilon_{\text{сист}\,1} &= \sqrt{\frac{2}{\pi}}\, x \frac{1+\sqrt{1+h}}{h} \frac{ye^{-y^2}}{1-e^{-y^2}} \\ \varepsilon_{\text{сист}\,2} &= \Delta\tau_2 \frac{y^2 e^{-y^2}}{1-e^{-y^2}}, \end{aligned}\right\} \quad (13.6.49)$$

where

and

$\Delta\tau_2$ – error of measurement of delay of the signal reflected from the second target.

The curve of the dependence of $\varepsilon_{\text{сист}\,1}$ on y is depicted in Fig. 13.19.

The ratio of the two components of systematic error is defined as

$$z = \frac{\varepsilon_{\text{сист}\,1}}{\varepsilon_{\text{сист}\,2}} = \frac{2x^2}{\pi} \frac{1+\sqrt{1+h}}{h} \frac{1}{\delta \cdot \Delta\tau_2}. \quad (13.6.50)$$

From this we can find the limiting value of error $\Delta\tau_2$ at which error $\varepsilon_{\text{сист}\,2}$ can be disregarded as compared to $\varepsilon_{\text{сист}\,1}$ in the range of δ interesting us.

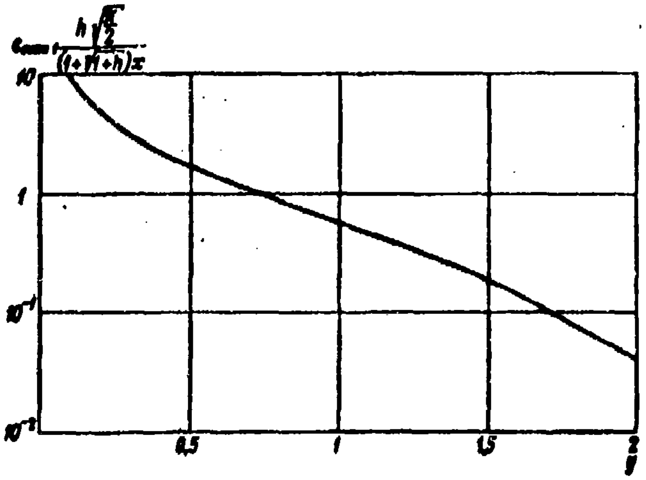

Fig. 13.19. The dependence of systematic error of the discriminator (Fig 13.16) on the relative distance between targets y.

The given relationships illustrate application of the general formulas obtained above and explicitly show the dependence of basic characteristics of discriminators on different physical parameters.

§ 13.7. Conclusion

Besides selection of the form of the sounding signal or the form and dimensions of the aperture of the antenna of great value for increase of resolving power of

radar systems is synthesis of optimum methods of processing signals. This synthesis can be conducted in two ways. The first of them consists in finding operations of the receiver produced on the r-f signal for which there is provided the best power relationships between the separated signal, interference and possible interfering signals at the output of the reciever. These operations can be found with various additional conditions, connected with the specific nature of resolution. In particular they are found on the condition of total suppression of interfering signals.

The second method consists in statistical synthesis of an optimum radar, exec... in the very best manner various functions in the presence of several closely loca... targets. Here, as a matter of fact, it would be possible, in general, not to introduce the concept of resolution, since it is not necessary to turn to this idea for solution of the corresponding problems. However, considering, on the one hand, the extent of the idea of resolving power, and on the other hand, interest in comparison of these methods of synthesis, we treated a series of solved statistical problems as problems of optimum resolution. From comparison of results obtained by various methods it is clear that in most cases with not too close coincidence of the signals reflected from various targets and with not too small a signal-to-noise ratio, the basis for operations optimum for resolution of targets is general. It consists of compensation of signals from interfering targets, considered in § ... This result is interesting in connection with its simplicity and the possibility of application for increase of resolving power of systems, optimum without interfering signals and found in the preceding chapters.

At the same time, by turning to the theory of statistical solutions we obtain ... systems, optimum in conditions of detection and measurement of coordinates, carrying out not only processing of r-f signals in receivers, but also subsequent nonlinear transformations. The found solutions, besides confirming the correctness of the above mentioned general basis of resolution, are also of independent interest. Optimum nonlinear transformations of signals have in the presence of disturbing targets specific peculiarities which cannot be found by investigation of separate singling out of signals alone and sometimes render substantial influence on resolving power. Characteristics of the corresponding optimum and quasi-optimum systems determine possibilities of resolution of targets.

Multi-target problems are often complicated and diverse. The given solutions are, obviously, only an insignificant part of the solutions of practical interest in this area. We shall show several problems whose solution, in the opinion of the authors, are necessary in connection with the problem of resolution of targets.

For meters of coordinates it would be of interest to find optimum (in the sense of absence of systematic and a minimum of fluctuating errors) systems in the case of slow fluctuations of the signal, when the maximum likelihood estimates cease to be efficient, and then it is necessary to look for other methods of construction of efficient estimates.

Both for problems of measurement and also for problems of detection it is expedient to find optimum systems for non-Gaussian fluctuating signals.

If the form of discriminators of systems of optimum resolution is found in a number of interesting cases, the form of smoothing circuits was not discussed in general. This problem deserves serious attention.

In examing questions of detection of targets we assumed that their number and possible location are known to some extent. Very important is solution of the problems of detection with an unknown number and position of targets. Furthermore, one should solve the problem of optimum search in the presence of many unresolved targets with finding of the optimum law of change of resolving power in the course of searching.

In connection with these problems there is also the question of suppression of interfering signals, not at assigned points, but in a certain range of values of parameters.

Similar problems were solved for definite forms of sounding signals. The obtained solutions give us the possibility of carrying out selection of sounding signals which ensure the best characteristics with an optimum method of their processing. Such a selection of signals and also solution on the basis of the obtained general results of different particular problems are practically very important.

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